Opinion Dynamics in Networks: Convergence, Stability and Lack of Explosion

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Outline

Our model

Results and Proof sketch

Two examples

Previous and Future work

How do people form opinions?

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Big question in social sciences.

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- *Influence:* Tendency of people to become similar to those with whom they interact.
- *Selection:* Get more influence from some people and less from others (according to some trait).

A lot of mathematical models have been introduced in physics/computer science etc.

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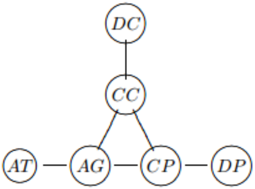
- Kempe et al. flow $f_{u\to v} = x_u x_v \left(\frac{1}{(\alpha-1)x_u+1} - \frac{1}{(\alpha-1)x_v+1}\right) = x_u x_v \left[(x_v - x_u)\frac{(a-1)}{((a-1)x_u+1)((a-1)x_v+1)}\right]$, where $\alpha \geq 1$ constant.

The dynamics are given by the update rule

$$x_u^{t+1} = x_u^t + \sum_{v\in N(u)} f_{v\to u}^t.$$

Kempe et al. model (cont.)

Picture from Kempe et al.



A small graph showing religions, with the possible transitions between agnostics, atheists, casual protestants, devout protestants, casual catholics, and devout catholics.

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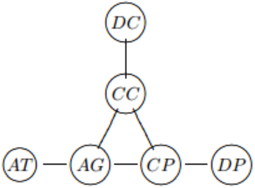

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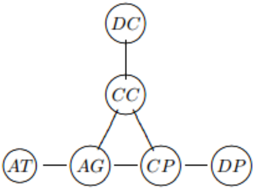

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- They also show a fixed point is Lyapunov-stable if and only if the active nodes (i.e., those that have positive population mass) form an independent set.

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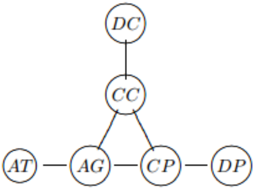

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- They show that the dynamics converges to fixed points.
- They also show a fixed point is Lyapunov-stable if and only if the active nodes (i.e., those that have positive population mass) form an independent set.
- Question: "Predict the equilibrium to which the system converges starting from a given initial mass vector".

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Add new features: Births and deaths of opinions

- Birth: with probability p, a new type v is created and takes mass from the existing types.
- Death: a type with mass smaller than ϵ dies out and move its mass to the existing types.

Main Theorems

Assumptions on F_{uv}:

- continuously differentiable,
- F_{uv} is odd and increasing,
- $F_{uv}(0) = 0$.

Theorem (**Main 1**)

Suppose that $\max_{z\in[-1,1]} |F_{uv}(z)| < 1/2$ *for all* $uv \in E(G)$. *If the initial mass vector* $\mathbf{x}^{(0)}$ *is chosen from a (continuous) atomless distribution, then the dynamics converges point-wise with probability* 1 *to a point whose active types form an independent set in* G.

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Remark: This theorem holds assuming no births and deaths.

Main Theorems (cont.)

- Set $\alpha_{\min} = \min_z F'_{uv}(z)$ and $\alpha_{\max} = \max_z F'_{uv}(z)$.
- Dynamics is (T, d)-stable if and only if $\forall T \le t \le T + d$, no population mass moves at step t.

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Theorem (Main 2a)

Assume that $\alpha_{\min} > 0$. Let $p < \min\left(\frac{\epsilon^4 \alpha_{\min}}{3}, \frac{2}{3}\right)$ and $t > \frac{1}{\epsilon^4 \alpha_{\min} - 3p}$. With probability at least $1 - e^{-tp/6}$, the dynamics is $\left(T, \frac{1}{3p}\right)$-stable for some $T \le t$ ("stable for long enough").

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Theorem (Main 2b)

Let $\alpha_{\max} \le p/512$ and $t \ge (16/p)\log^2(1/\epsilon)$. The dynamics at step t has at most $72\log(1/\epsilon)$ types with probability at least $1 - 3\epsilon$ (no explosion).

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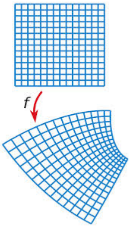
Proof steps of Main 1

- 1. Convergence: Show that opinion dynamics converges pointwise. Find a Lyapunov (potential function) that strictly increases (unless dynamics reaches a fixed point)
- 2. Diffeomorphism: Prove that the function of the update rule is a diffeomorphism in $\mathcal{S}$ (eigenvalue analysis, show Jacobian is invertible).
- 3. Measure zero: Use center-stable manifold theorem.

Why are these technicalities important?

- Manifold: Topological space that "looks like" Euclidean space near each point.
- **Diffeomorphism**
  A diffeomorphism is a map between manifolds which is continuously differentiable and has a continuously differentiable inverse.



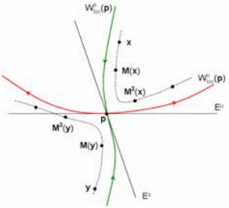
This is a useful technical smoothness condition that allows us to apply standard theorems about dynamical systems. (e.g., Center-Stable Manifold theorem).

Center-Stable Manifold theorem

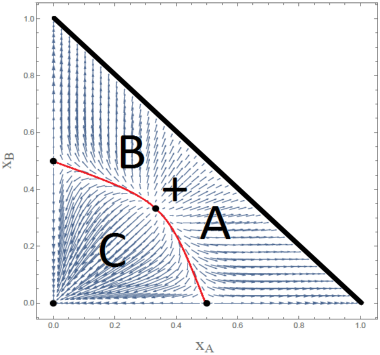
Center-Stable Manifold theorem (informally)

If the rule of the dynamics is a diffeomorphism then

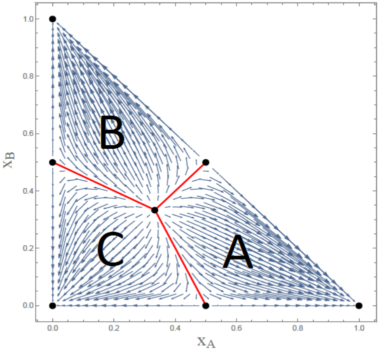
- For every fixed point $\mathbf{p}$, there exists an open ball $B_{\mathbf{p}}$ so that if trajectory $q(n)$ is inside $B_{\mathbf{p}}$ for all $n \geq 0$ then $p(0)$ belongs to a (local) center stable manifold $W_{sc}(\mathbf{p})$ which has dimension equal to the dimension of the space spanned by eigenvectors of the Jacobian (at $\mathbf{p}$) with eigenvalues of absolute value ≤ 1.



Examples - Phase portrait



3-path



Triangle

Previous work

- D. Kempe, J. M. Kleinberg, S. Oren, and A. Slivkins. Selection and influence in cultural dynamics.
- R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence: models, analysis and simulation.
- A. Montanari and A. Saberi. The spread of innovations in social networks.
- E. Mossel, J. Neeman, and O. Tamuz. Majority dynamics and aggregation of information in social networks.
- Many more...

Future work

- **Rate of convergence (without births and deaths):** How fast does our migration dynamics converge point-wise to fixed points for different choices of functions F_{uv}? How does the structure of G influence the time needed for convergence?
- **Average case analysis:** Which independent sets are more likely to occur if we start at random in the simplex. Assuming $F_{uv}(z) = a_{uv}z$ (linear functions) or $F_{uv}(z) = a_{uv}z^3$ etc (cubic), do the values of α_{uv}'s affect the likelihood of the linearly stable fixed points[1]?
- **Continuous time** version (our theorems still hold).

[1] this likelihood of a fixed point is called region of attraction.

Thank you!