Multiplicative Weights Update with Constant Step-Size in Congestion Games
Convergence, Limit Cycles & Chaos

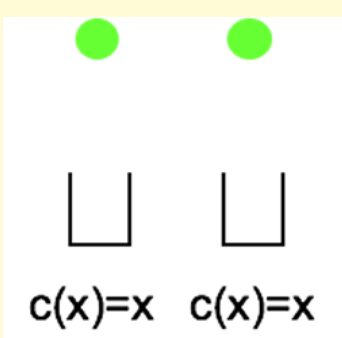
Gerasimos Palaiopanos[1] Ioannis Panageas[2] Georgios Piliouras[1]

[1] Singapore University of Technology and Design, [2] MIT

Question: How does MWU behave in Congestion Games?

Congestion Games

Rosenthal, R.W. '73 $(\mathcal{N}; E; (S_i)_{i\in\mathcal{N}}; (c_e)_{e\in E})$ where $\mathcal{N}$: *agents*, E: a set of resources with positive cost functions c_e. Each player i has a strategy set S_i of subsets of E ($S_i \subseteq 2^E$). E.g.,



$\Phi(\mathbf{s}) = \sum_{e\in E}\sum_{j=1}^{\ell_e(\mathbf{s})} c_e(j)$: the potential function.

Randomization: Each player i has a probability distribution over s_i.

- $p_{i\gamma}$: probability i chooses strategy γ and $c_{i\gamma}$: the expected cost of player i given that he chooses strategy γ.

MWU$_e$: Exponential variant (Hedge)

The update rule (function) for MWU$_e$:

$$p_{i\gamma}(t+1) = p_{i\gamma}(t)\frac{(1-\epsilon_i)^{c_{i\gamma}(t)}}{\sum_{\gamma'\in S_i} p_{i\gamma'}(t)(1-\epsilon_i)^{c_{i\gamma'}(t)}}, \tag{1}$$

$\forall i \in \mathcal{N}, \forall \gamma \in S_i$, where $\epsilon_i < 1$.

MWU$_\ell$: Linear version of MWU

The update rule (function) for MWU$_\ell$:

$$p_{i\gamma}(t+1) = p_{i\gamma}(t)\frac{1-\epsilon_i c_{i\gamma}(t)}{1-\epsilon_i \sum_{\gamma'\in S_i} p_{i\gamma'}(t) c_{i\gamma'}(t)} \tag{2}$$

$\forall i \in \mathcal{N}, \forall \gamma \in S_i$, where ϵ_i is a small constant.

Discrete Time Dynamical Systems and Chaos

Let $f : X \to X$ continuous on a compact set $X \subset \mathbb{R}$.

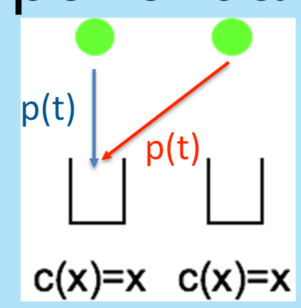
Limit Cycle: A periodic orbit (i.e., $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ with $\mathbf{z}_{i+1} = f(\mathbf{z}_i)$ for $1 \le i \le k-1$ and $f(\mathbf{z}_k) = \mathbf{z}_1$) that some initial conditions converge to.

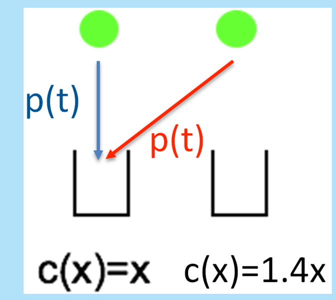
Li and Yorke Chaos: For each $k \in \mathbb{Z}^+$, there exists a periodic point $p \in X$ of period k and there is an uncountably infinite set $S \subseteq X$ that is *"scrambled"*.

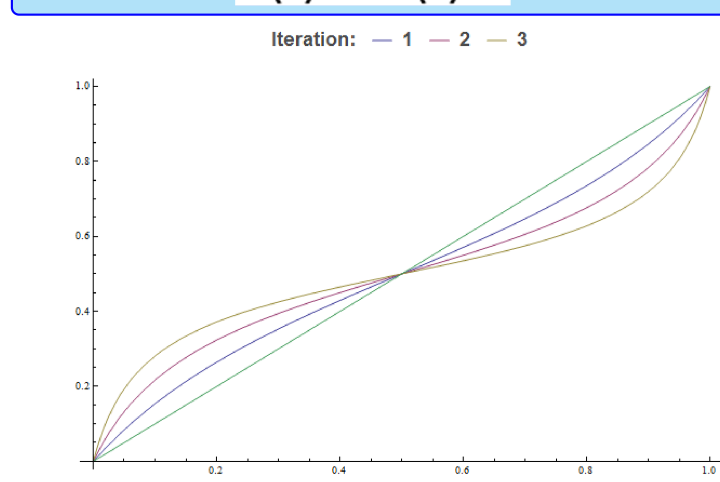
Congestion Games: Examples

Symmetric, 2 agents, 2 edges, 2 strategies

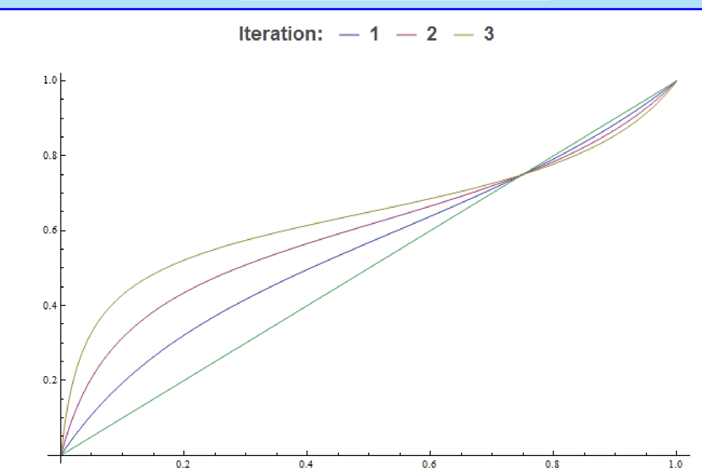
We plot the evolution of the (common) mixed strategy of both agents under linear/exponential MWU with the same ϵ.



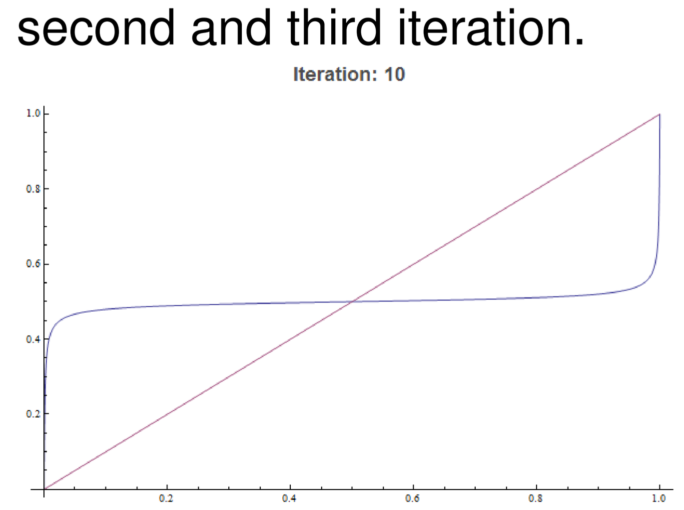




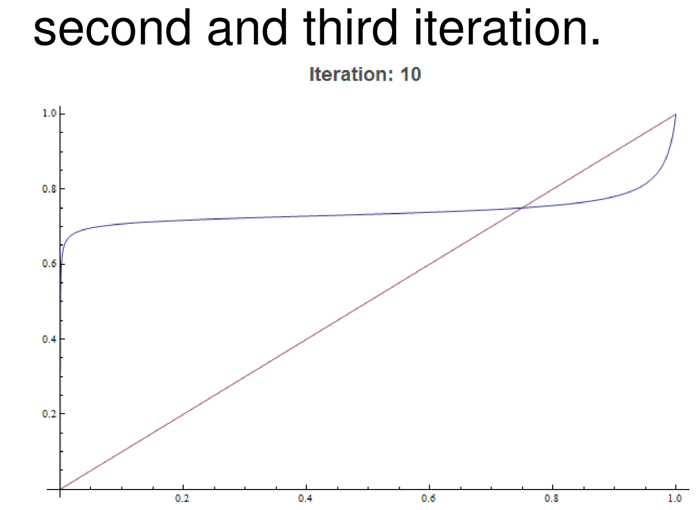
(a) Linear MWU$_\ell$: Plot of first, second and third iteration.



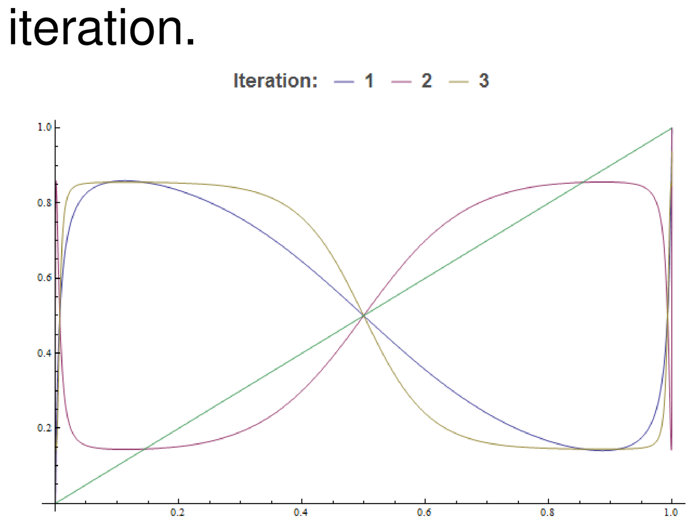
(b) Linear MWU$_\ell$: Plot of first, second and third iteration.



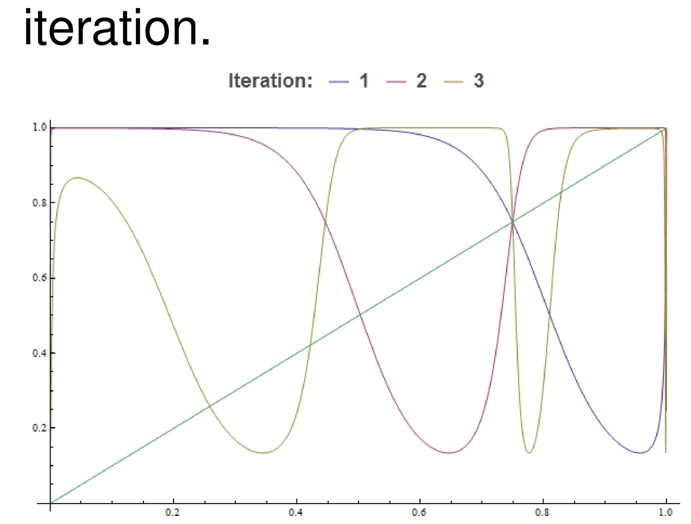
(c) Linear MWU$_\ell$: Plot of tenth iteration.



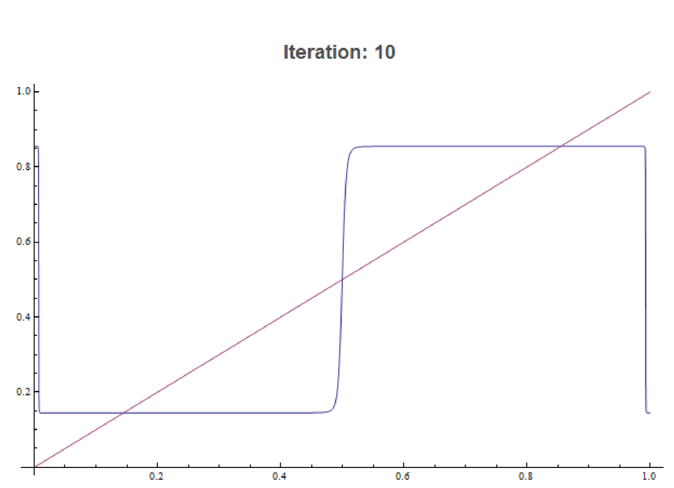
(d) Linear MWU$_\ell$: Plot of tenth iteration.



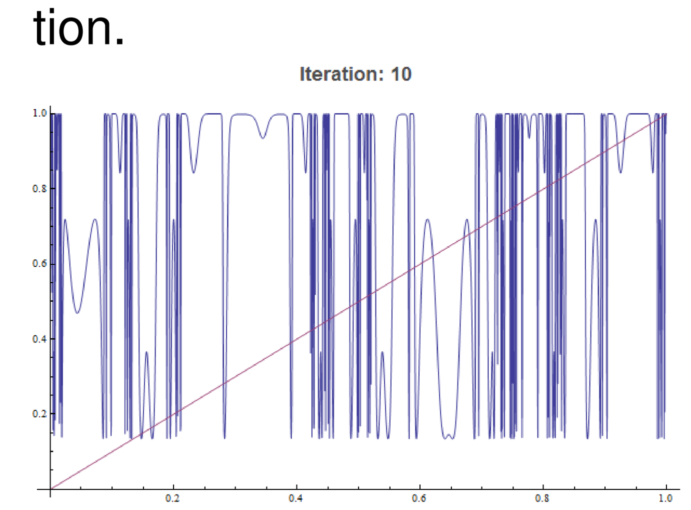
(e) Exponential MWU$_e$: Plot of first, second and third iteration.



(f) Exponential MWU$_e$: Plot of first, second and third iteration.



(g) Exponential MWU$_e$: Plot of tenth iteration.



(h) Exponential MWU$_e$: Plot of tenth iteration.

Convergence of MWU$_\ell$ to Nash Equilibria

Theorem

*Function $\Psi \stackrel{\text{def}}{=} \mathbb{E}_{\mathbf{s}\sim\mathbf{p}}[\Phi(\mathbf{s})]$ is decreasing w.r.t. time, i.e. $\Psi(\mathbf{p}(t+1)) \le \Psi(\mathbf{p}(t))$ where equality $\Psi(\mathbf{p}(t+1)) = \Psi(\mathbf{p}(t))$ holds **only** at fixed points.*

Theorem

Assume that the fixed points of MWU$_\ell$ are isolated. MWU$_\ell$ dynamics converges to Nash equilibrium.

Theorem [Li & Yorke, '75]: Period three implies chaos

Let J be an interval and let $F : J \to J$ be continuous. Assume there is a point $a \in J$ for which the points $b = F(a), c = F^2(a)$ and $d = F^3(a)$, satisfy

$$d \le a < b < c \text{ (or } d \ge a > b > c).$$

Then

1. For every $k = 1, 2, \ldots$ there is a periodic point in J having period k.
2. There is an uncountable *"scrambled"* set $S \subset J$ (containing no periodic points), which satisfies the following conditions:
   - For every $p, q \in S$ with $p \ne q$,
     $$\lim_{n\to\infty}\sup |F^n(p) - F^n(q)| > 0 \text{ and } \lim_{n\to\infty}\inf |F^n(p) - F^n(q)| = 0.$$
   - For every point $p \in S$ and periodic point $q \in J$,
     $$\lim_{n\to\infty}\sup |F^n(p) - F^n(q)| > 0.$$

Non-Convergence of MWU$_e$: Limit Cycle and Chaos

Theorem

There exist two player two strategy symmetric congestion games such that MWU_e exhibits Li-Yorke chaos. There also exist such games where MWU_e has an uncountably infinite set of initial conditions converging to a limit cycle.

Corollary

For any $1 > \epsilon > 0$ and n, there exists a n-player congestion game $G(\epsilon)$ (depending on ϵ) so that MWU_e dynamics exhibits Li-Yorke chaos.

ARXIV

- https://arxiv.org/abs/1703.01138