Problem 1 - Short questions [5 points]

Answer the following short questions. No general proofs are needed.

(i) A bipartite graph has left vertices a, b, right vertices $1, 2$, and edges $(a, 1), (a, 2), (b, 1)$. Is $M = \{(a, 1)\}$ a maximal matching? Is it a maximum matching? Give a maximum matching. [1 point]

(ii) Is $g(t) = t^4$ strongly convex on all of $\mathbb{R}$ for some $\mu > 0$? Answer yes or no and compute $g''(0)$. [1 point]

(iii) Minimize $f(t) = (t-3)^2$ over $C = [-1, 2]$. Starting at $t_0 = 1$, compute one projected gradient step with $\eta = 1$: $t_1 = \Pi_C(t_0 - \eta f'(t_0))$. Give the point before projection and t_1. [1 point]

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be L-smooth and bounded below, but not necessarily convex. For gradient descent with $\eta = 1/L$, which is guaranteed to tend to zero as $K \to \infty$: (A) $\min_{0 \le k < K} |f'(t_k)|$, (B) $f(t_K) - \inf_{t \in \mathbb{R}} f(t)$, or (C) both? [1 point]

(v) Let $F(t) = \frac{1}{2}\big(f_1(t) + f_2(t)\big)$, where $f_1(t) = \frac{1}{2}(t-1)^2$ and $f_2(t) = \frac{1}{2}(t+1)^2$. At $t = 0$, sample i uniformly from $\{1, 2\}$ and use $f_i'(0)$ as a stochastic derivative. Give its two possible values and its expectation. [1 point]

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Problem 2 - Gradient descent and strong convexity [7 points]

Consider

$$f(x_1, x_2) = x_1^2 + 2x_2^2, \qquad x_0 = (2, 1).$$

(i) Compute the gradient and Hessian, find the global minimizer x^*, and state the largest strong-convexity constant μ and the smallest gradient-Lipschitz constant L. [2 points]

(ii) Use gradient descent $x_{k+1} = x_k - \eta \nabla f(x_k)$ with $\eta = 1/L$. Compute $x_1, x_2, f(x_1)$, and $f(x_2)$. [3 points]

(iii) Using your μ and L, evaluate the following upper bound at $k = 2$. Compare it with the actual value of $f(x_2) - f(x^*)$. [2 points]

$$f(x_k) - f(x^*) \le \left(1 - \frac{\mu}{L}\right)^k \big(f(x_0) - f(x^*)\big).$$

You may use the displayed convergence bound without proving it. Show your calculations; exact fractions are acceptable.
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Problem 3 - A short strong-convexity proof [4 points]

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and μ-strongly convex, where $\mu > 0$. Suppose x^* is an unconstrained global minimizer of f. Prove that, for every $x \in \mathbb{R}$,

$$f(x) - f(x^*) \geq \frac{\mu}{2}(x - x^*)^2.$$

Deduce that x^* is the unique global minimizer. [4 points]
Hint: Use $f'(x^*) = 0$ and the definition of strong convexity:

$$f(y) \geq f(z) + f'(z)(y - z) + \frac{\mu}{2}(y - z)^2 \qquad \text{for every } y, z \in \mathbb{R}.$$

A short proof is sufficient.
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Problem 4 - Independent set and integer programming [9 points]

An **independent set** of a finite simple undirected graph $G = (V, E)$ is a subset $S \subseteq V$ containing no two vertices joined by an edge. The **maximum independent set problem** asks for an independent set with as many vertices as possible.

(i) Introduce one variable x_v for each vertex v, with $x_v = 1$ meaning that v is selected. **Write an integer linear program** for maximum independent set, using one exclusion constraint per edge and variable bounds. Include the objective and variable domains. Then **write its LP relaxation**, retaining all variable bounds. Your formulation must work for every such graph, including graphs with isolated vertices. [4 points]

(ii) Is every vertex (extreme point) of the LP relaxation integral? Give a small counterexample. Draw your graph, specify a fractional feasible point, and show that it is a vertex by identifying tight constraints that uniquely determine it. For your graph, also give the maximum independent set size and the optimal LP value, with a brief justification. [5 points]

Hints for part (ii):

- Try a triangle (three vertices, with an edge between every pair), and give all variables the same fractional value.
- You may use this fact without proof: a feasible point is a vertex if some of its tight constraints, treated as equalities, uniquely determine all its coordinates. A constraint is *tight* when equality holds.
- Summing the edge constraints can help you bound the LP objective.

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