Lecture 12
topological sort,
BFS

CS 161 Design and Analysis of Algorithms

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DFS Algorithm from a Vertex

Algorithm DFS(G, v):

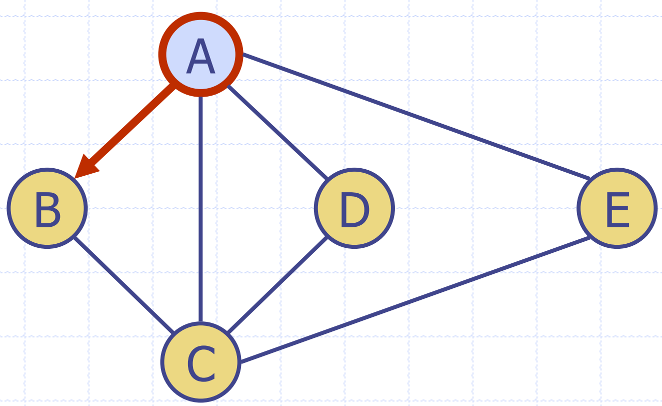
Input: A graph G and a vertex v in G

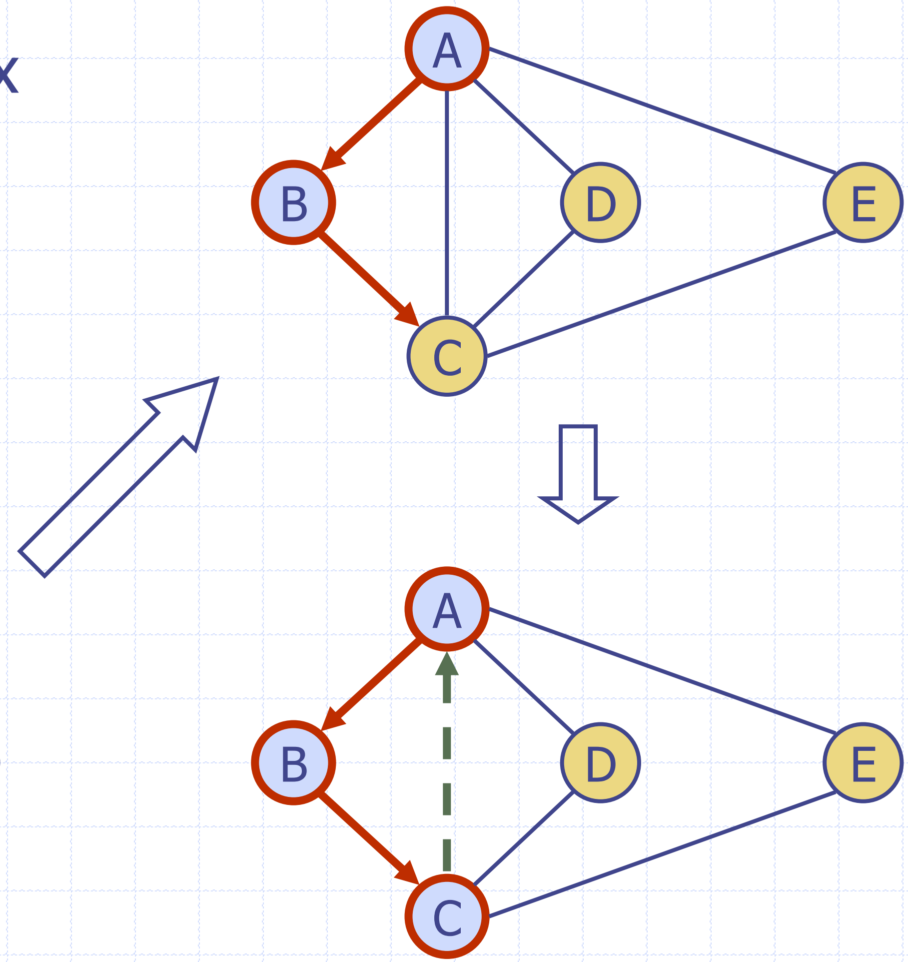
Output: A labeling of the edges in the connected component of v as discovery edges and back edges, and the vertices in the connected component of v as explored

```
Label v as explored
for each edge, e, that is incident to v in G do
    if  e is unexplored  then
        Let w be the end vertex of e opposite from v
        if  w is unexplored  then
            Label e as a discovery edge
            DFS(G, w)
        else
            Label e as a back edge
```

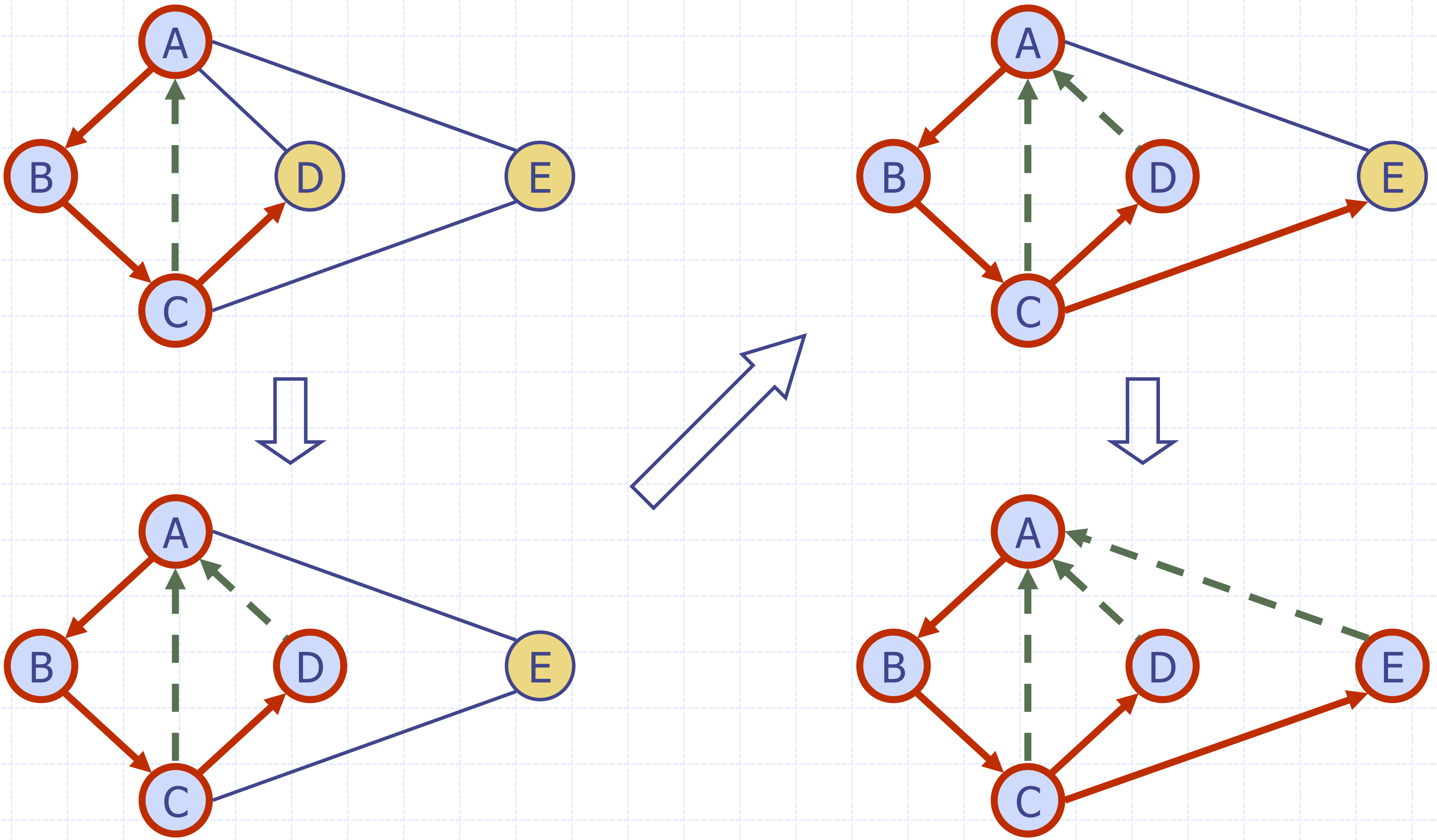
Example

- (A) unexplored vertex
- (A) visited vertex
- unexplored edge
- discovery edge
- back edge

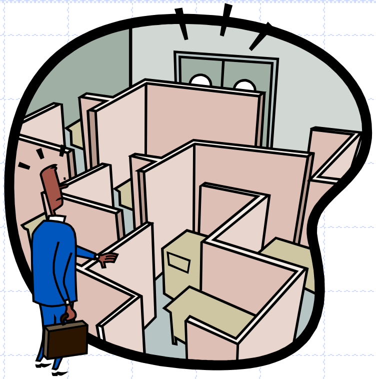




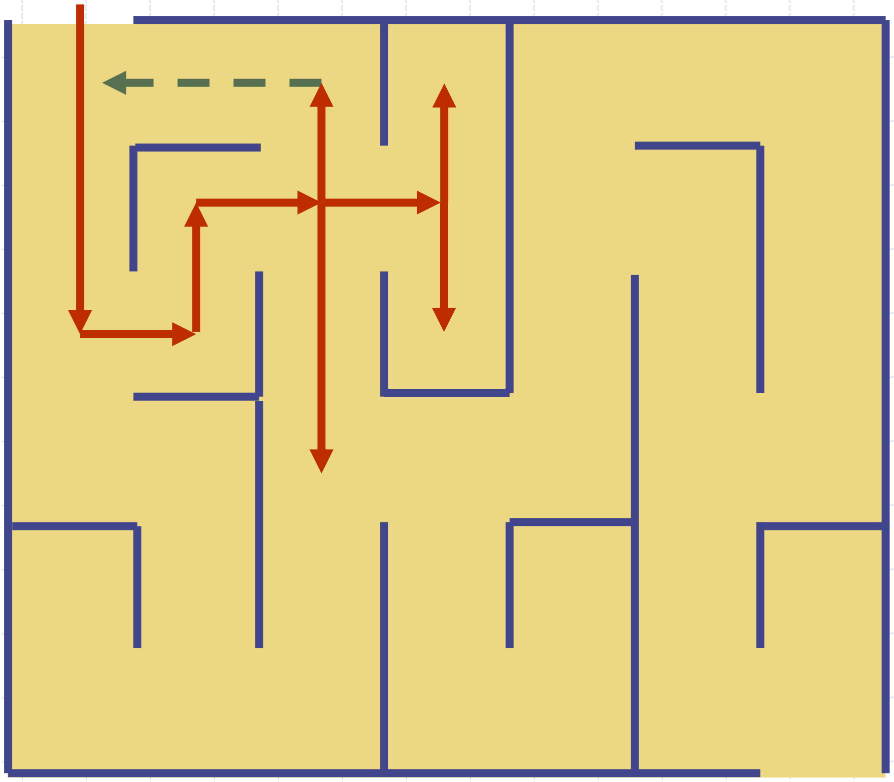
Example (cont.)



DFS and Maze Traversal



- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



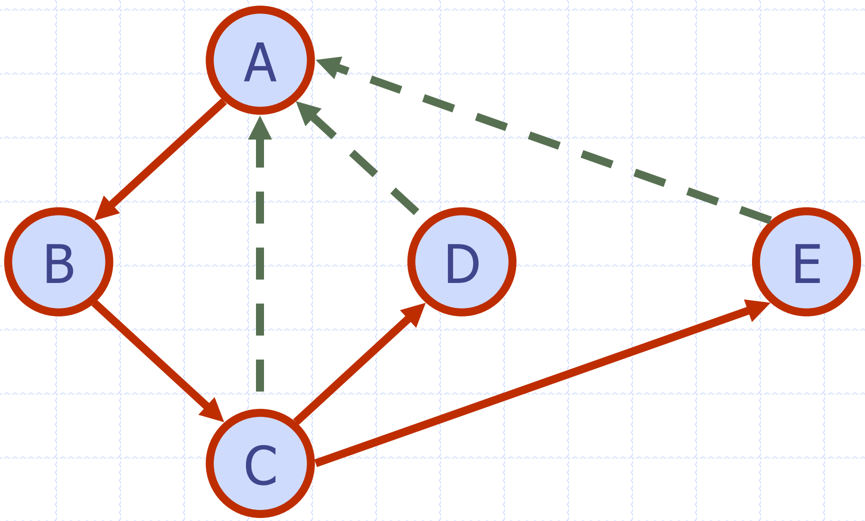
Properties of DFS

Property 1

DFS(*G*, *v*) visits all the vertices and edges in the connected component of *v*

Property 2

The discovery edges labeled by *DFS*(*G*, *v*) form a spanning tree of the connected component of *v*



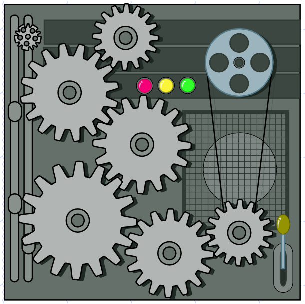
The General DFS Algorithm

- Perform a DFS from each unexplored vertex:

Algorithm DFS(G):
  Input: A graph G
  Output: A labeling of the vertices in each connected component of G as explored

  Initially label each vertex in v as unexplored
  for each vertex, v, in G **do**
    if v is unexplored **then**
      DFS(G, v)

Analysis of DFS



- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

Cycle detection

- Graph G has a cycle iff DFS **has a back edge**

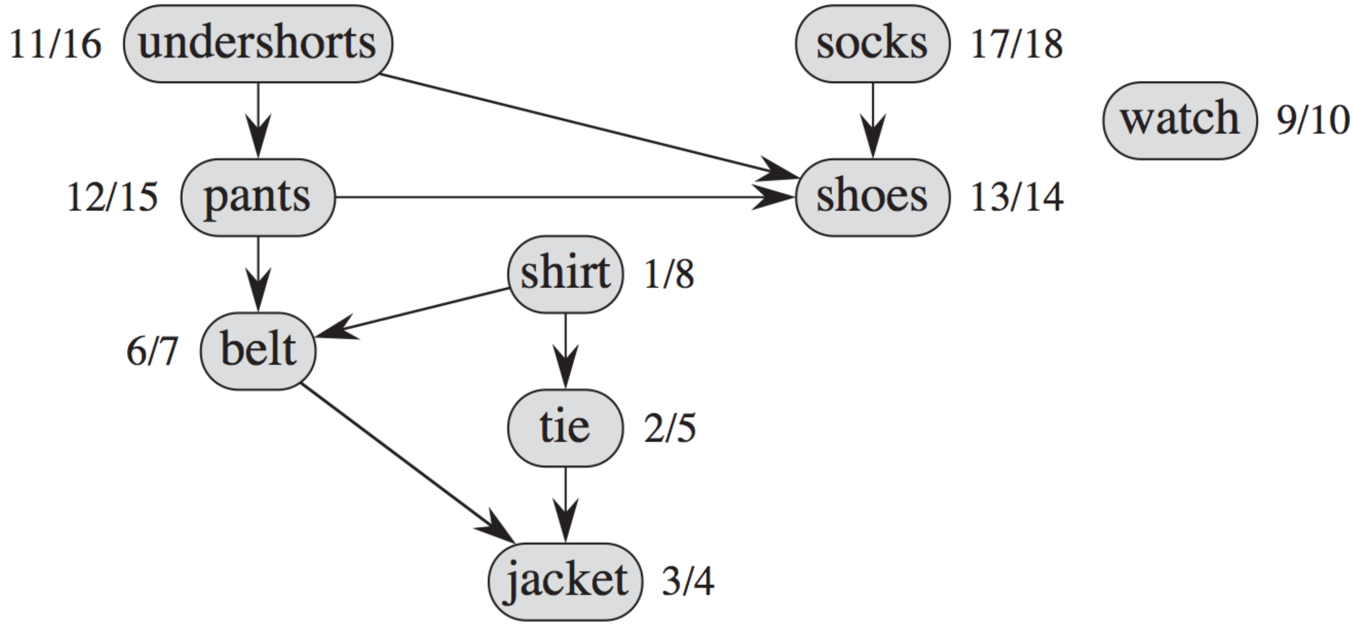
Directed Acyclic Graph = DAG

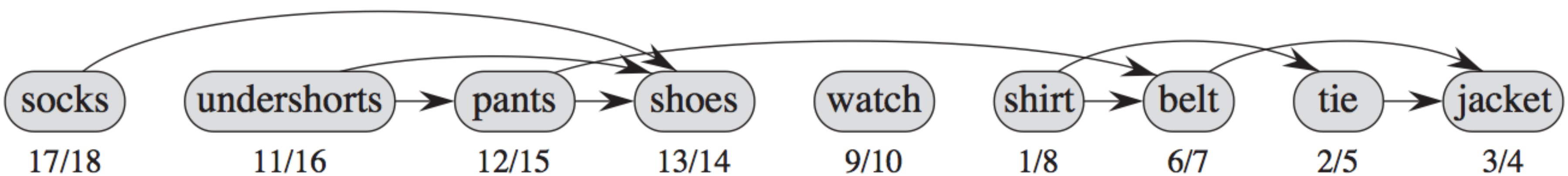
Topological sort

Topological sort of a DAG $G=(V,E)$

1. Run DFS(G), compute finishing times of nodes
2. Output the nodes in decreasing order of finishing times

The Graph – relationship between clothing procedures

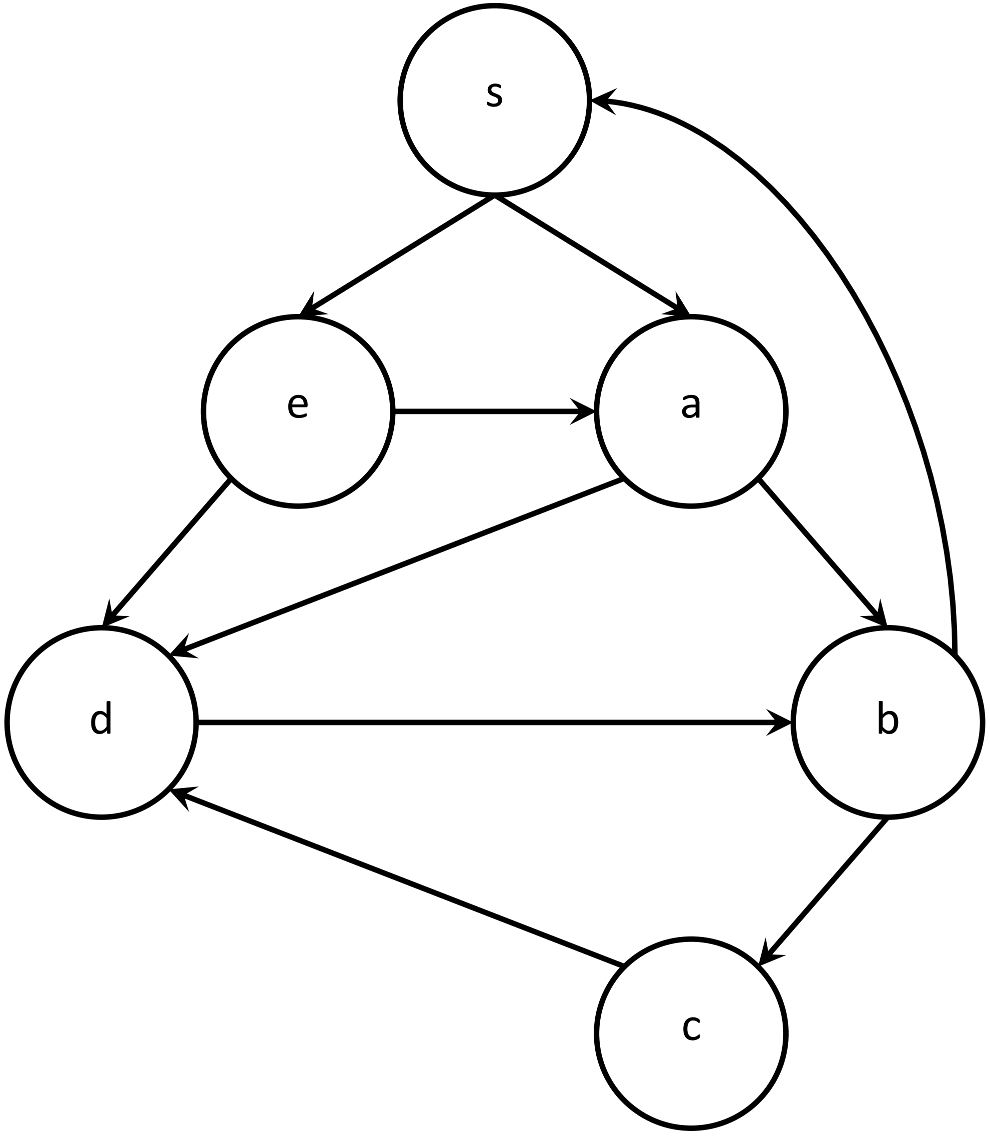




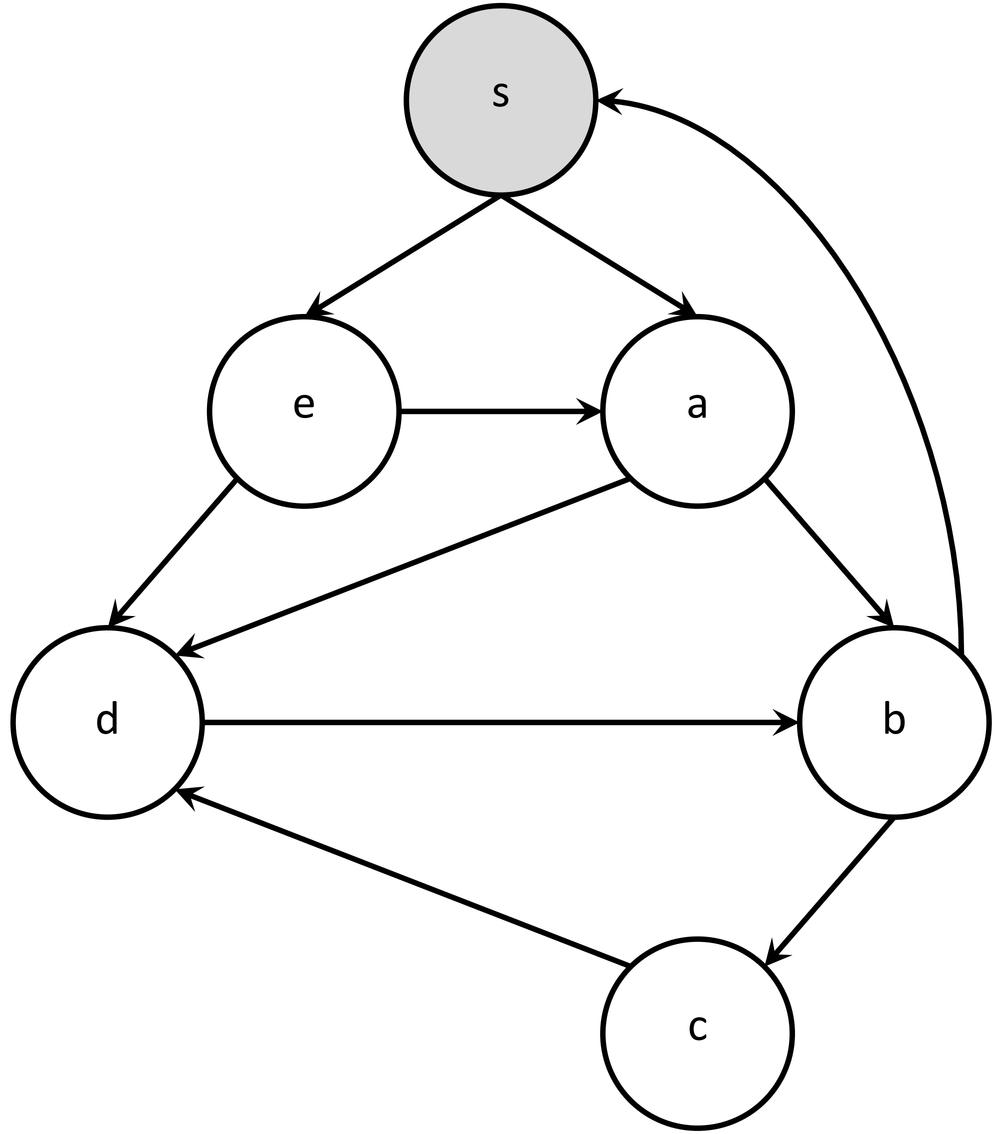
The Topological sort – a workable sequence of clothing

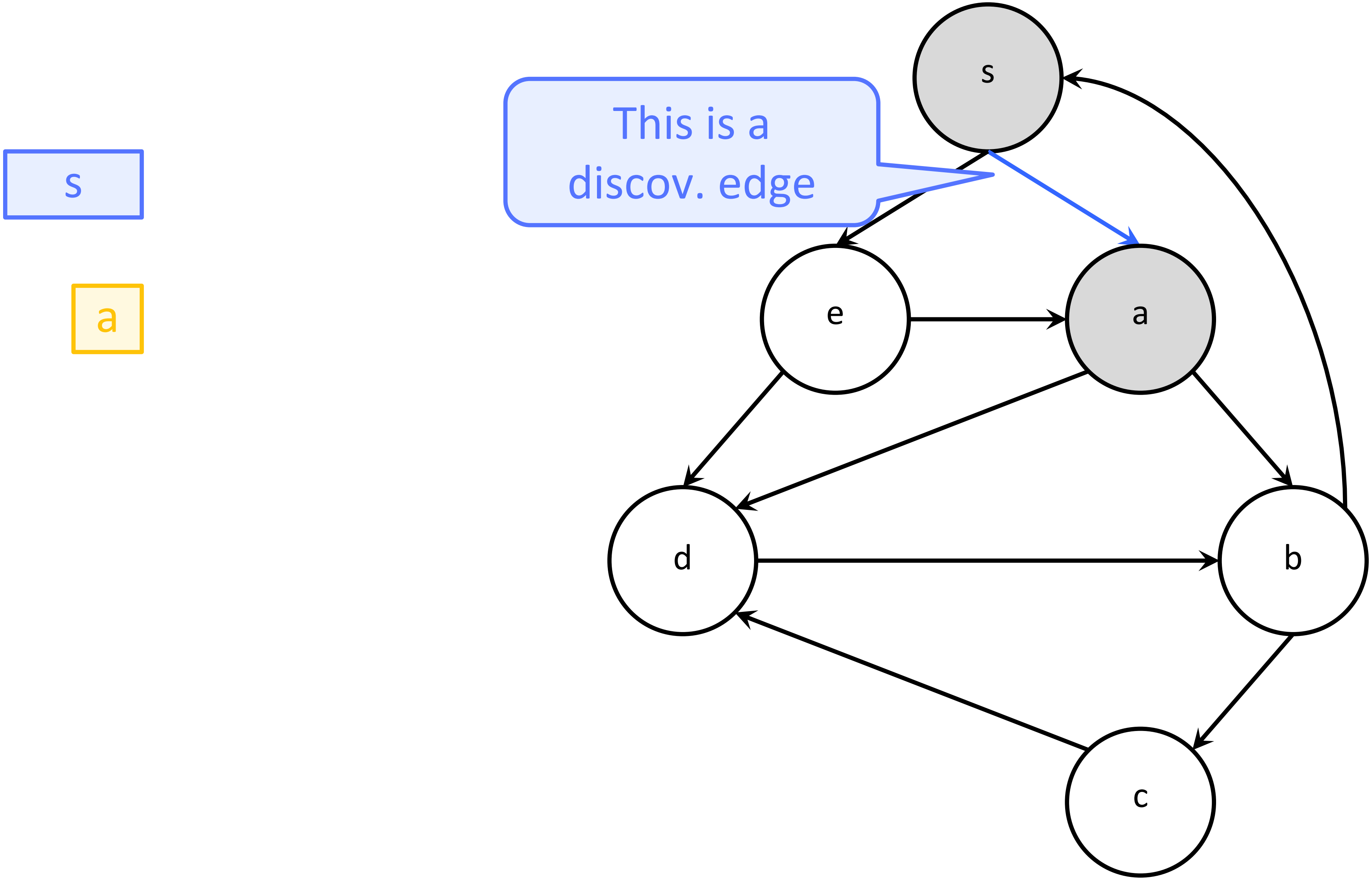
TOPOLOGICAL

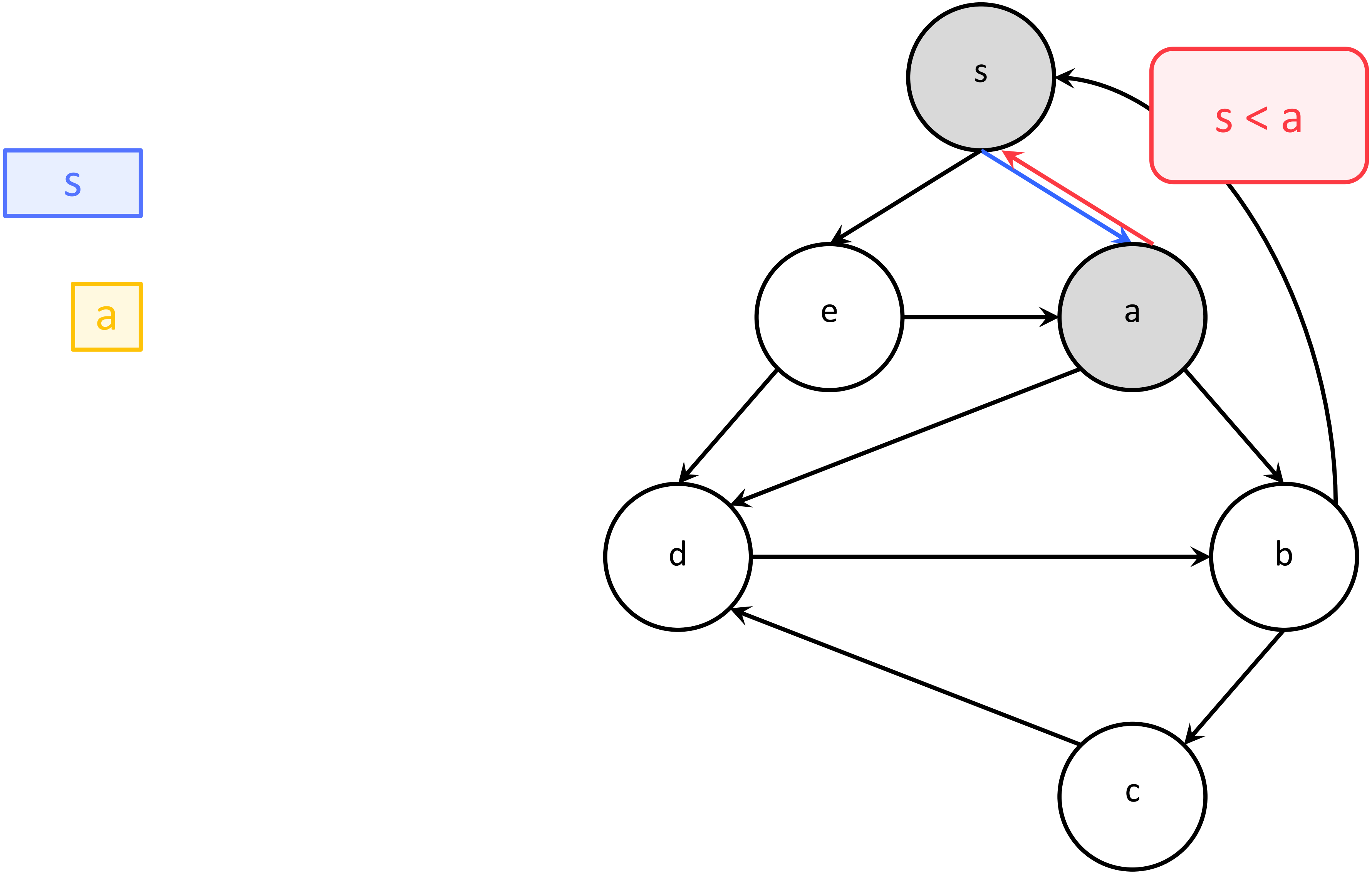
SORT

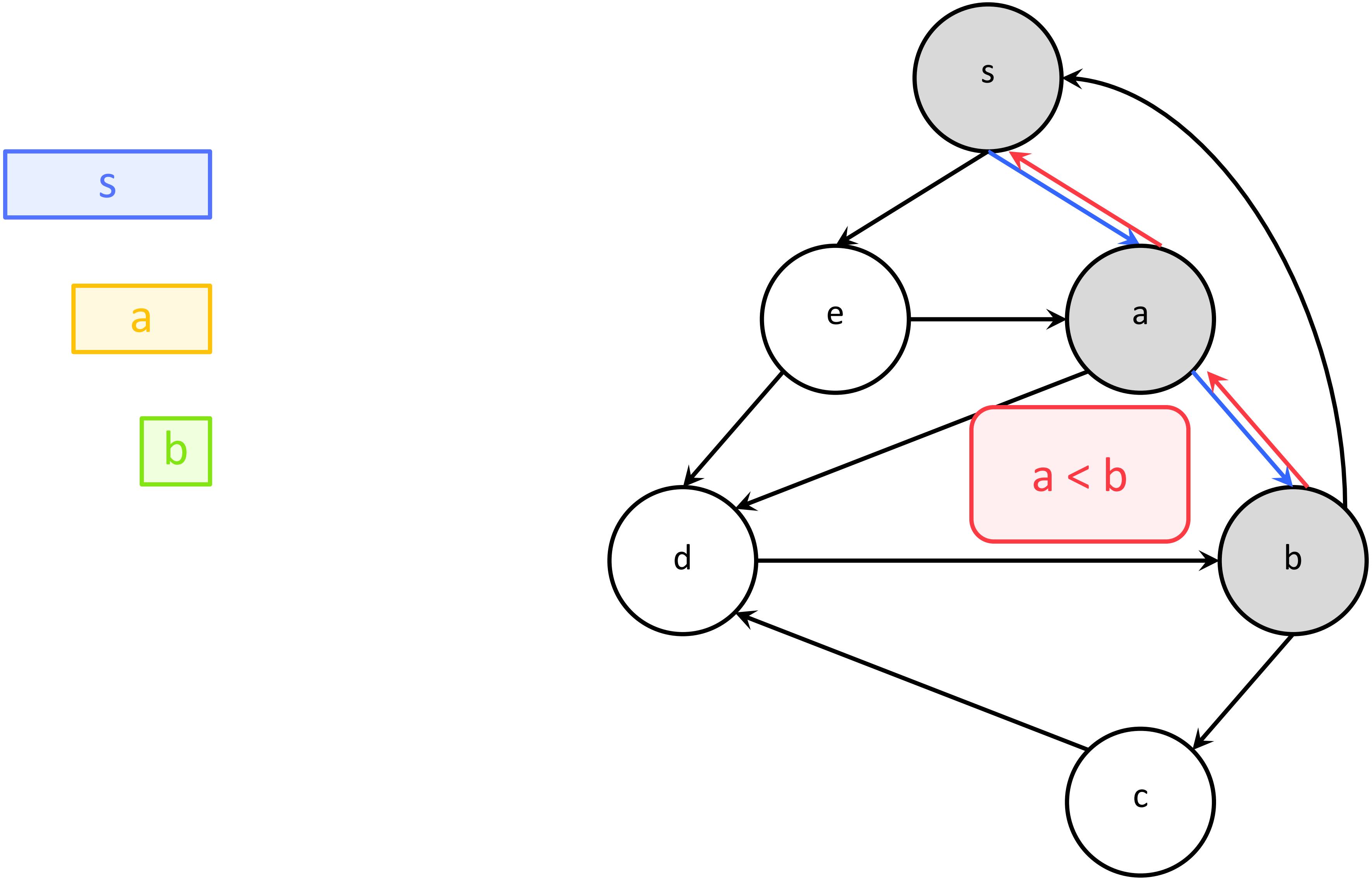


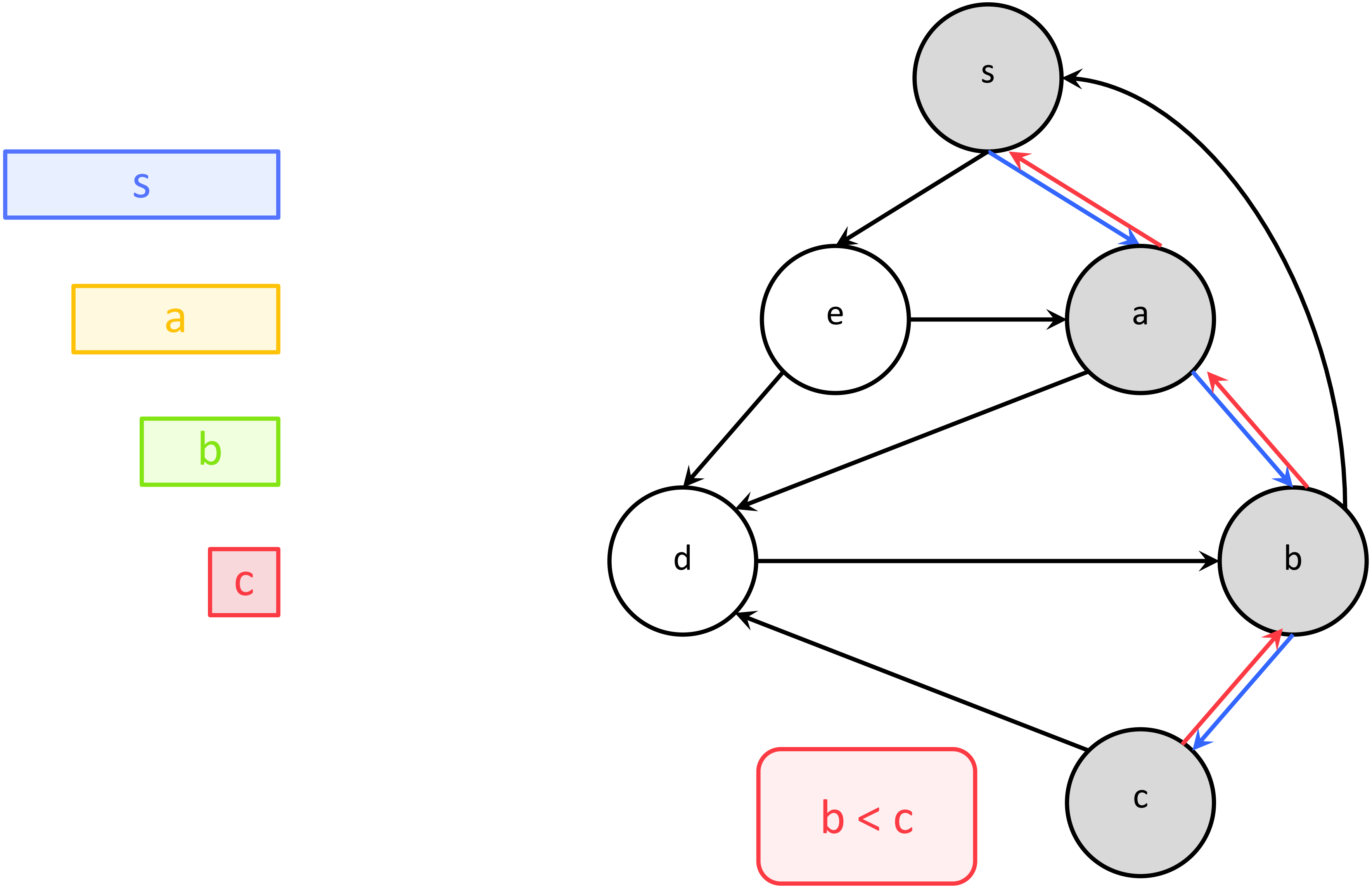
S

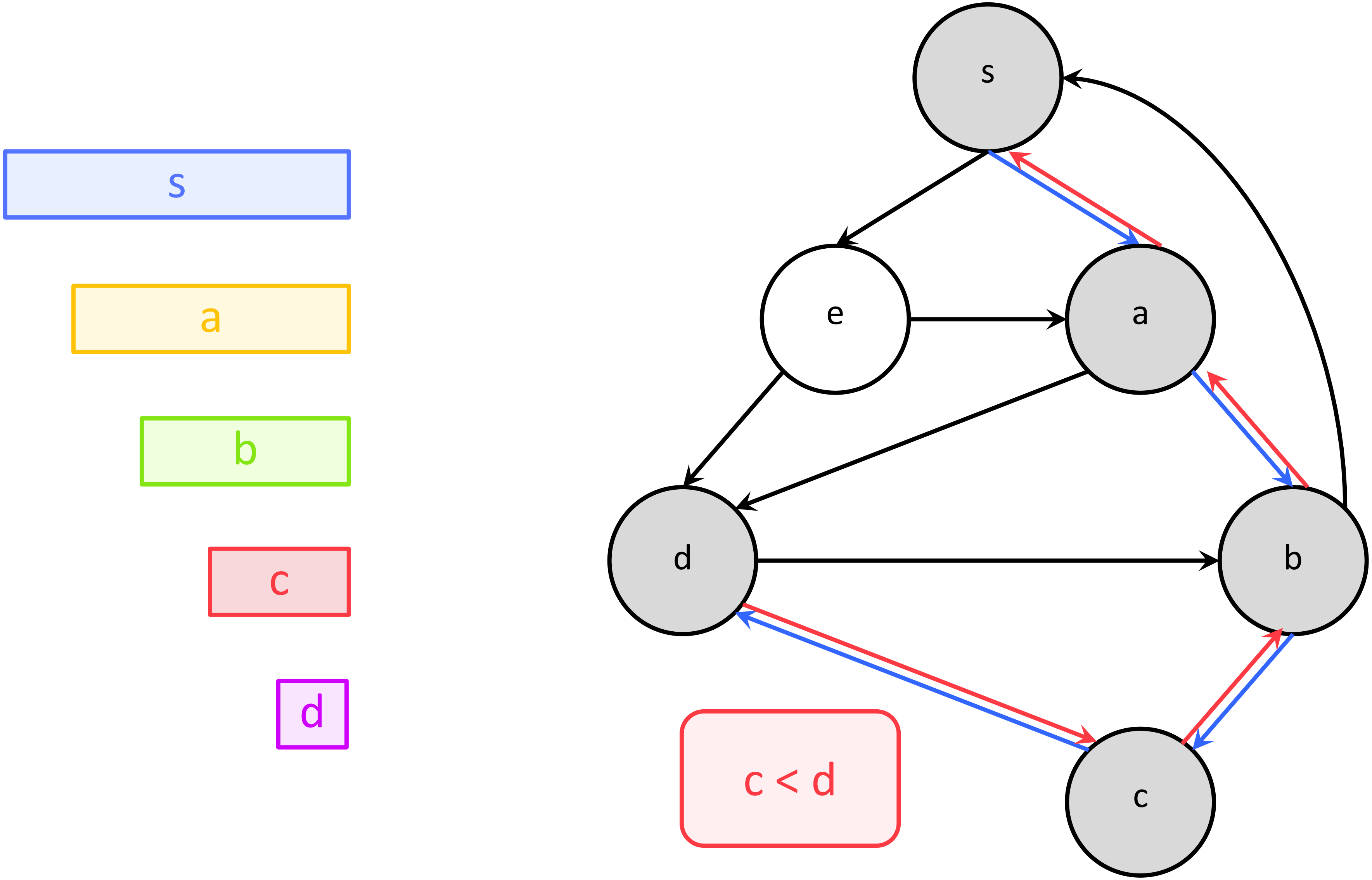



s
a
This is a
discov. edge
s
e
a
d
b
c

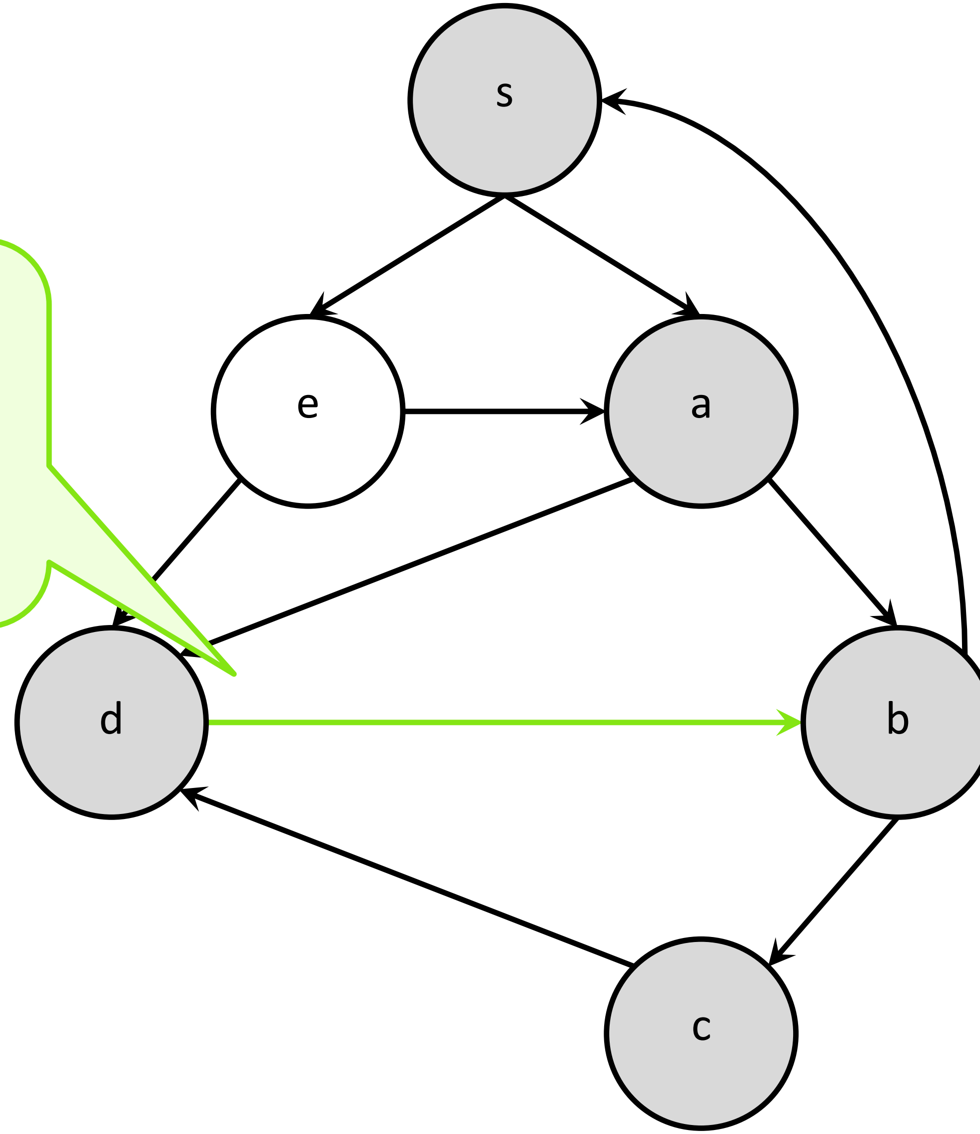

s
a
s
s < a
e
a
d
b
c


s
a
b
s
e
a
d
b
c
a < b

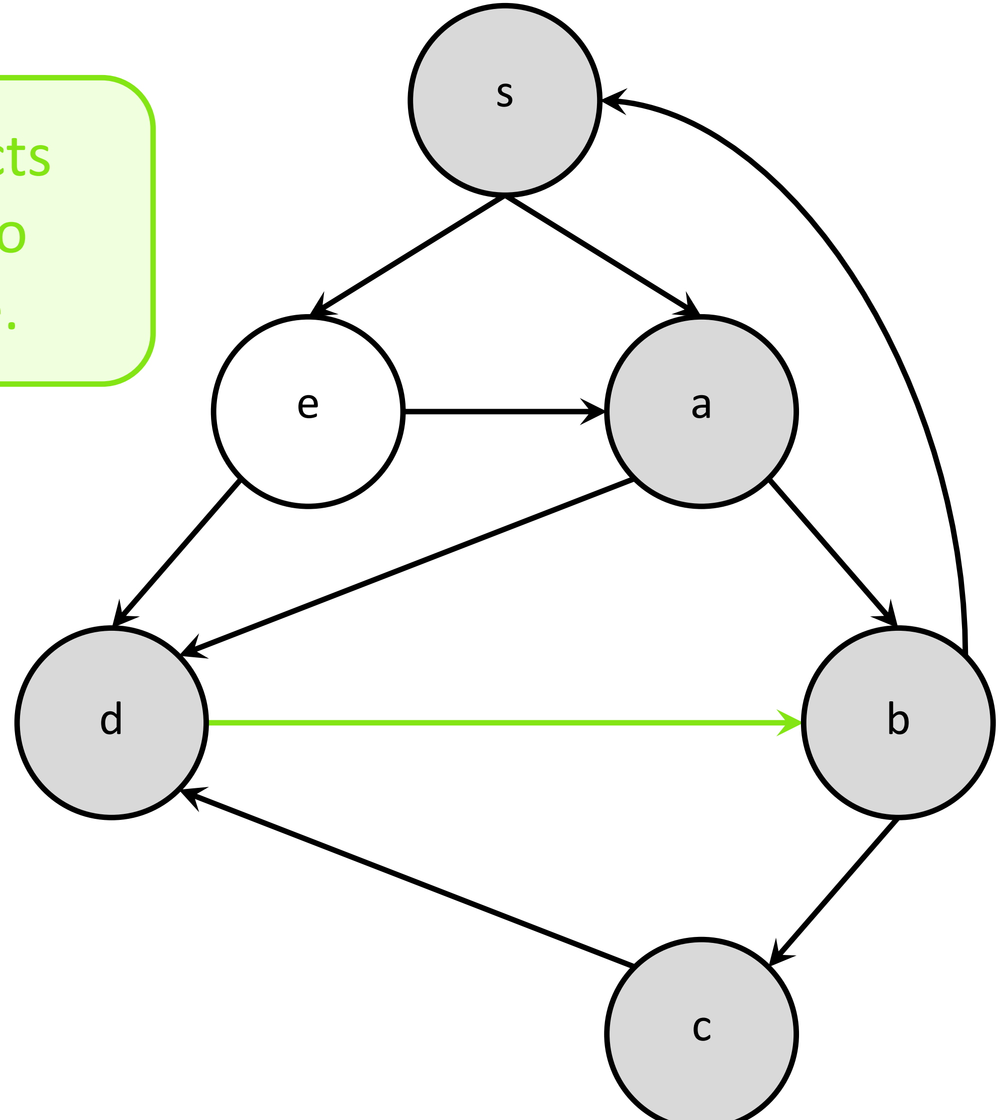

s
a
b
c
s
e
a
d
b
c
b < c

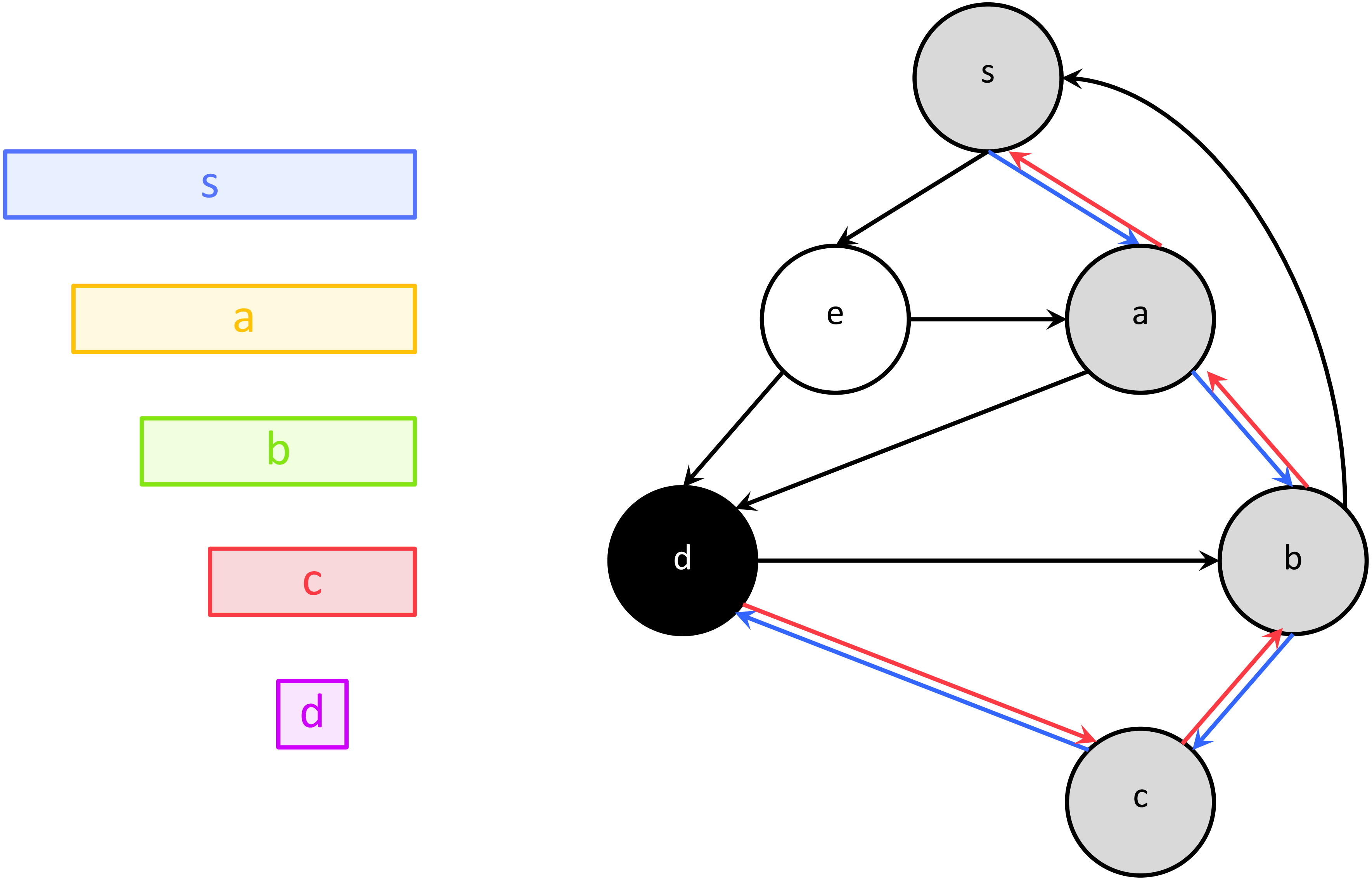

s
a
b
c
d
s
e
a
d
b
c
c < d

This is a *back edge*.
We don't follow it
because the grey node **b**
is on the stack.

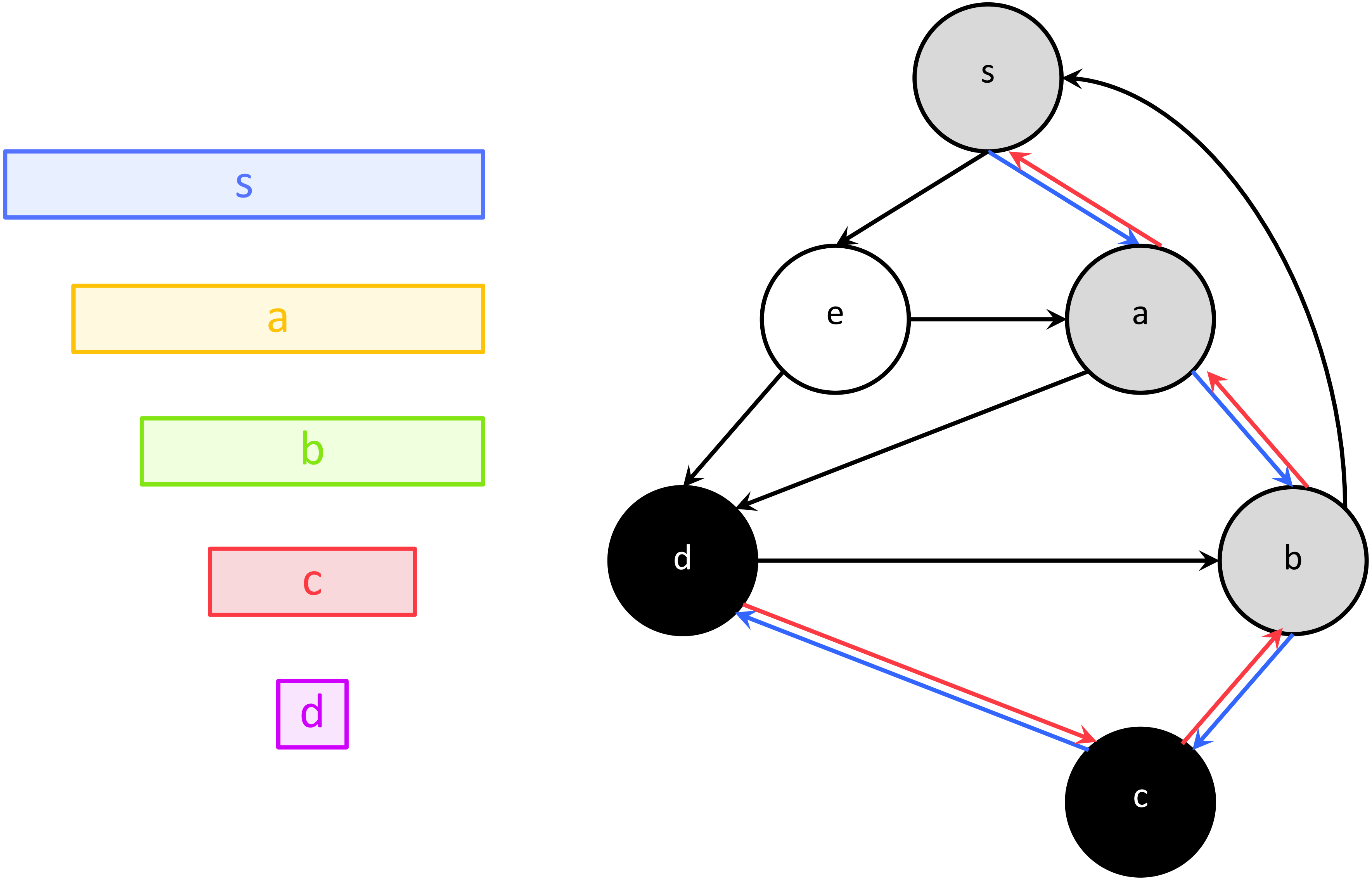


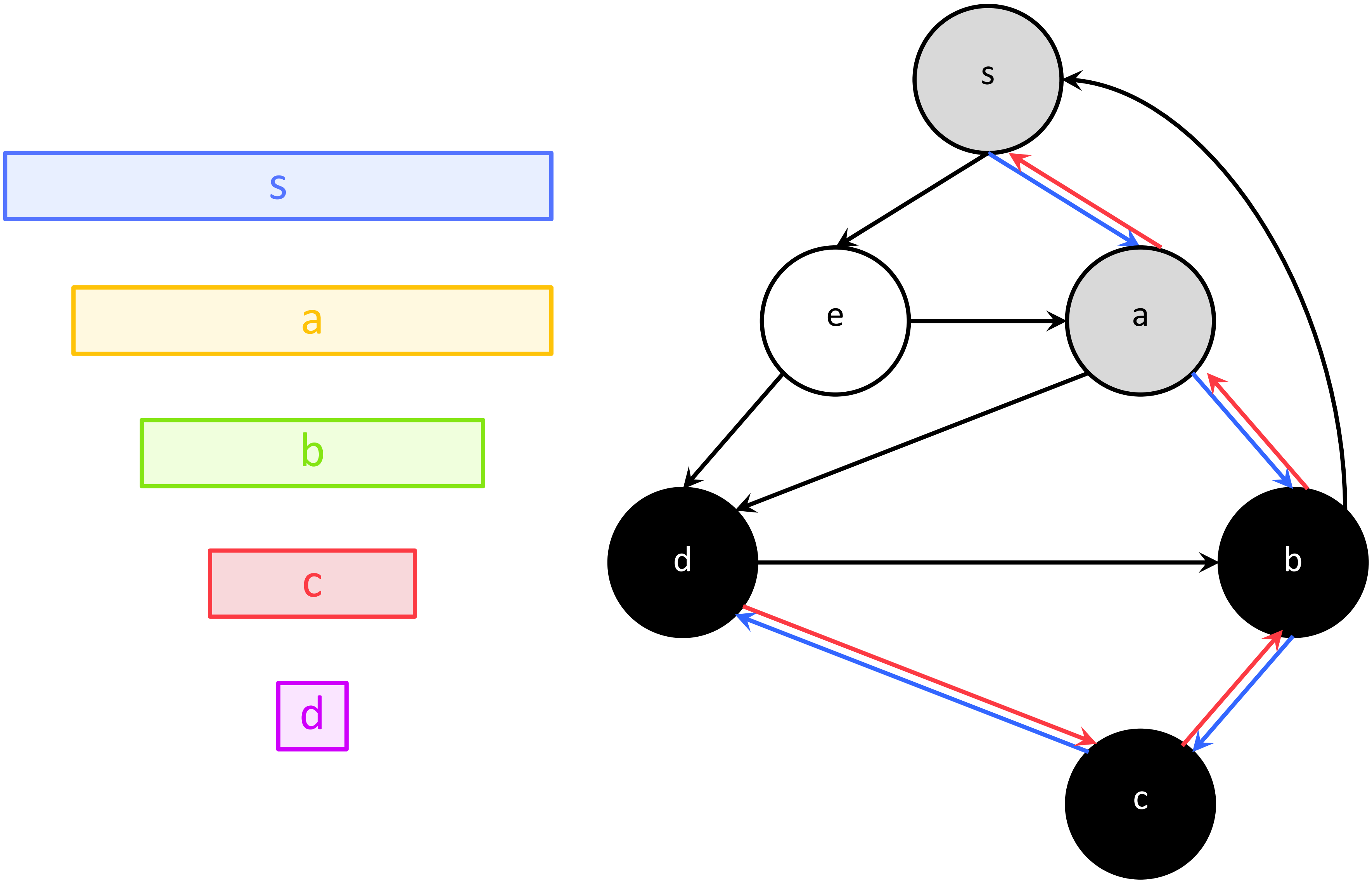
A *back edge* connects from a grey node to another grey node.



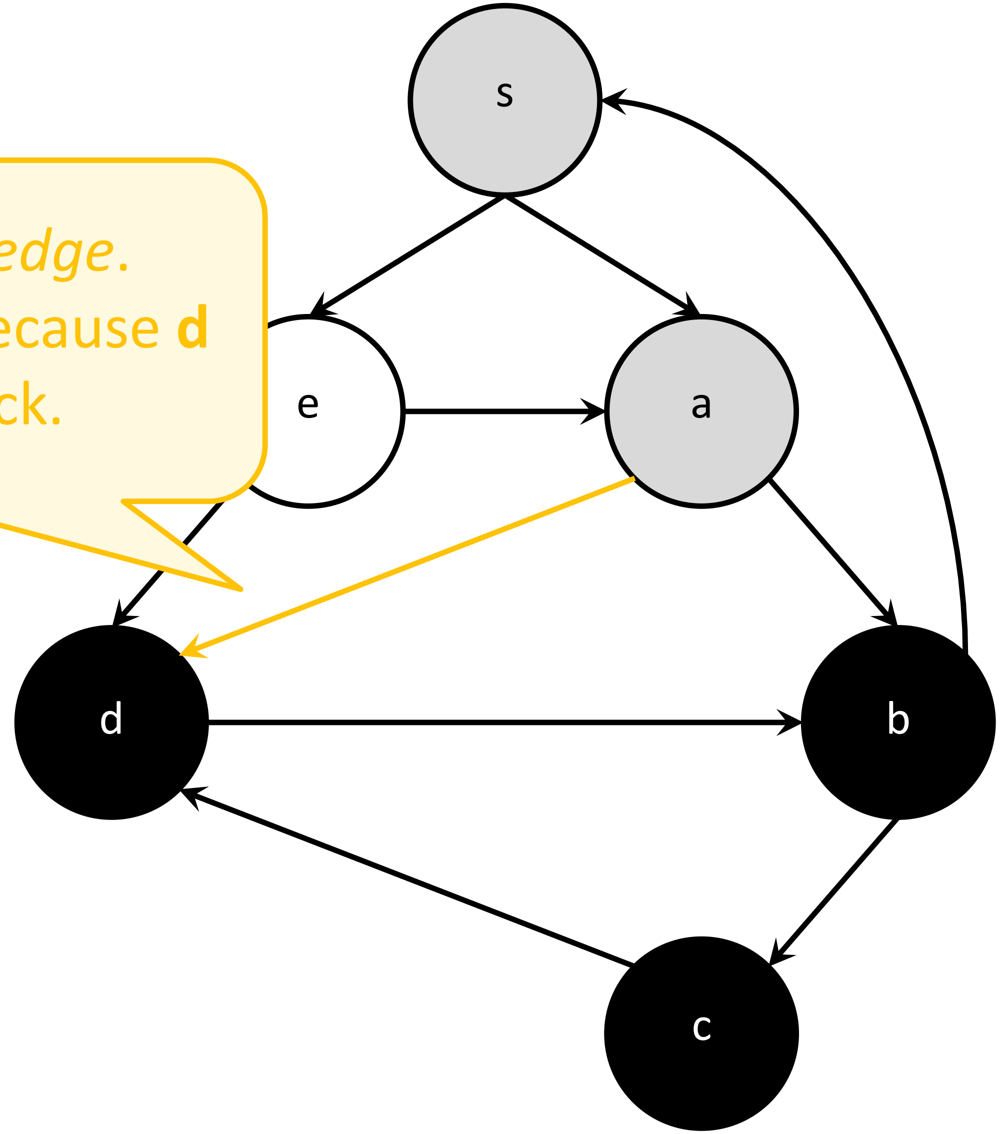


s
a
b
c
d
s
e
a
d
b
c

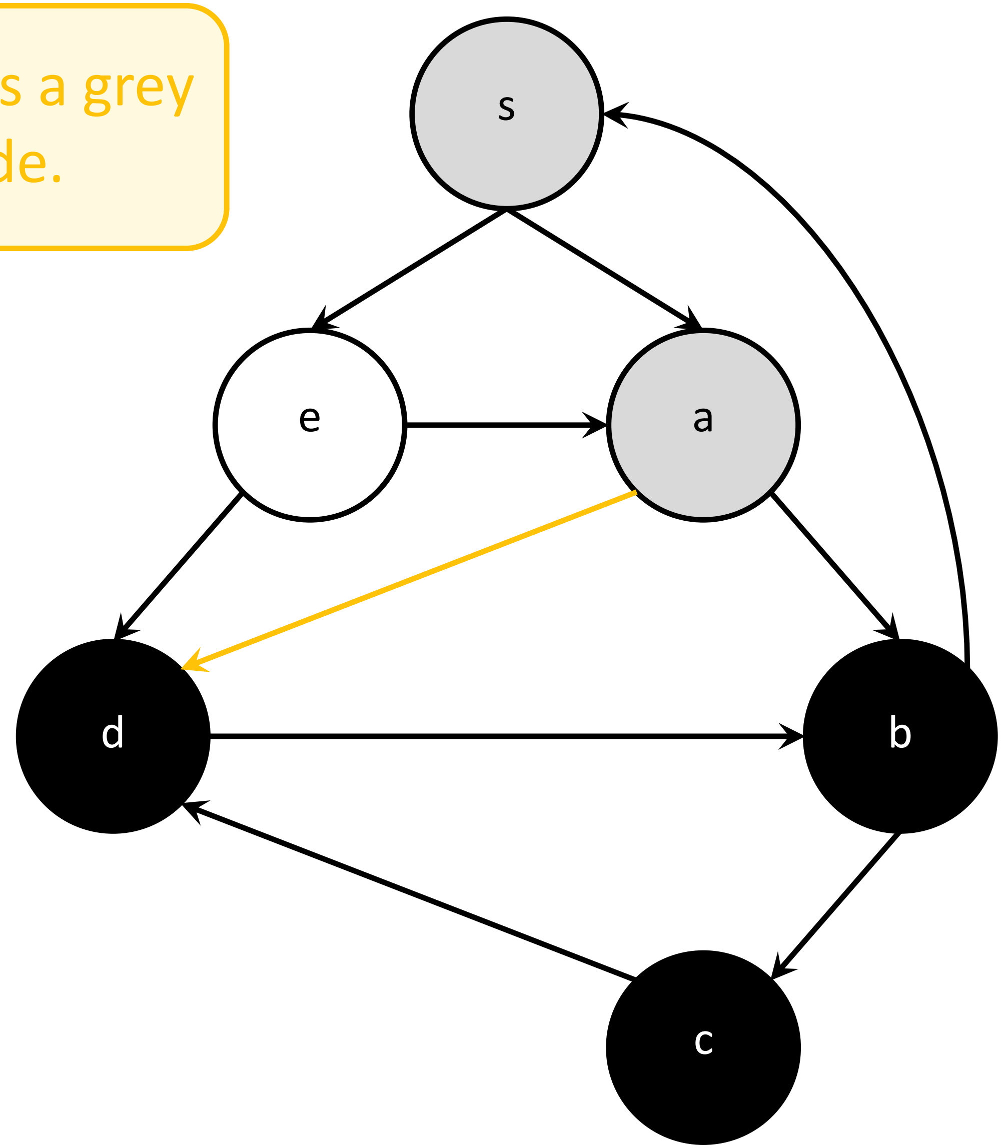

s
a
b
c
d
s
e
a
d
b
c

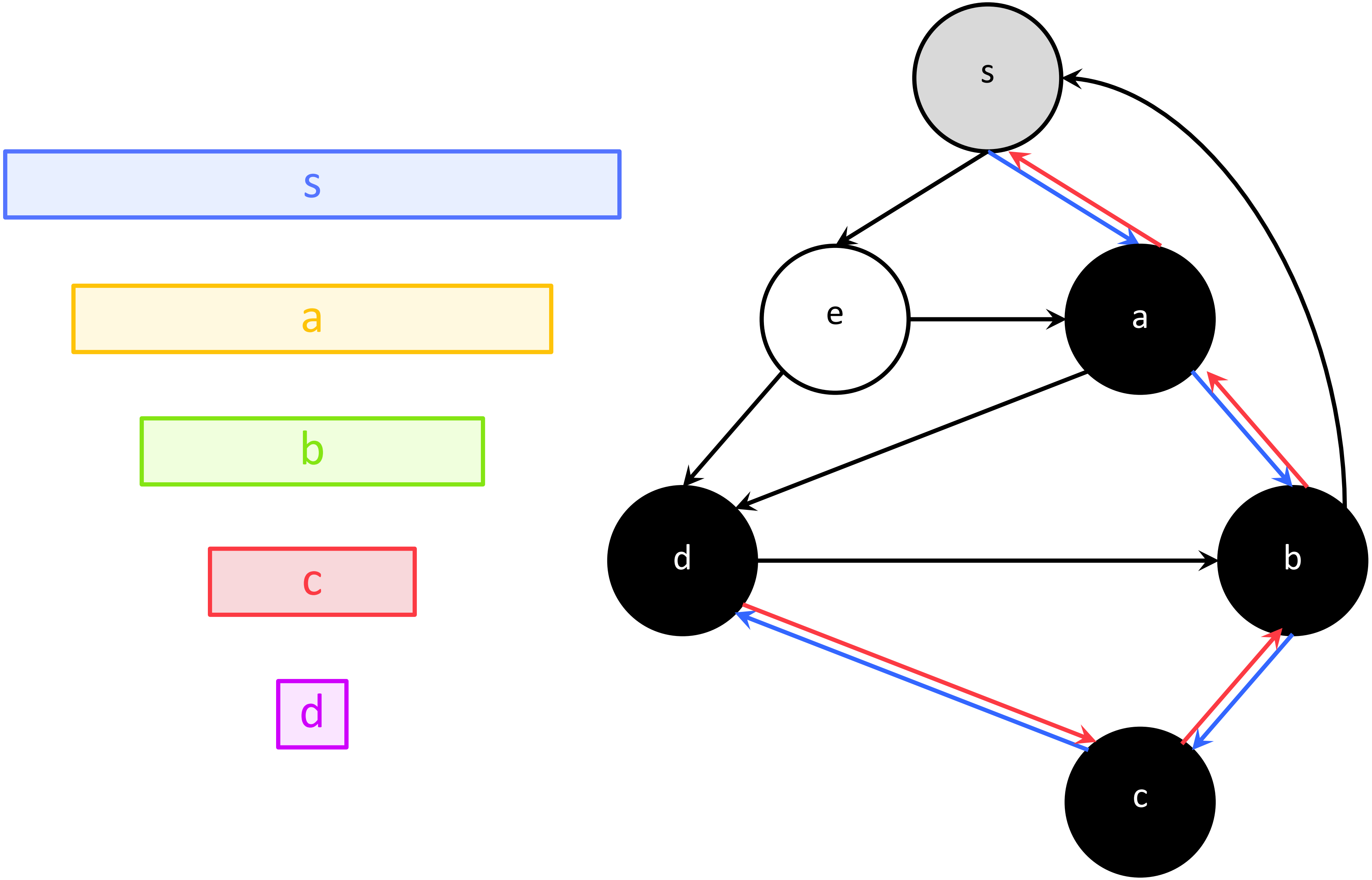

s
a
b
c
d
s
e
a
d
b
c

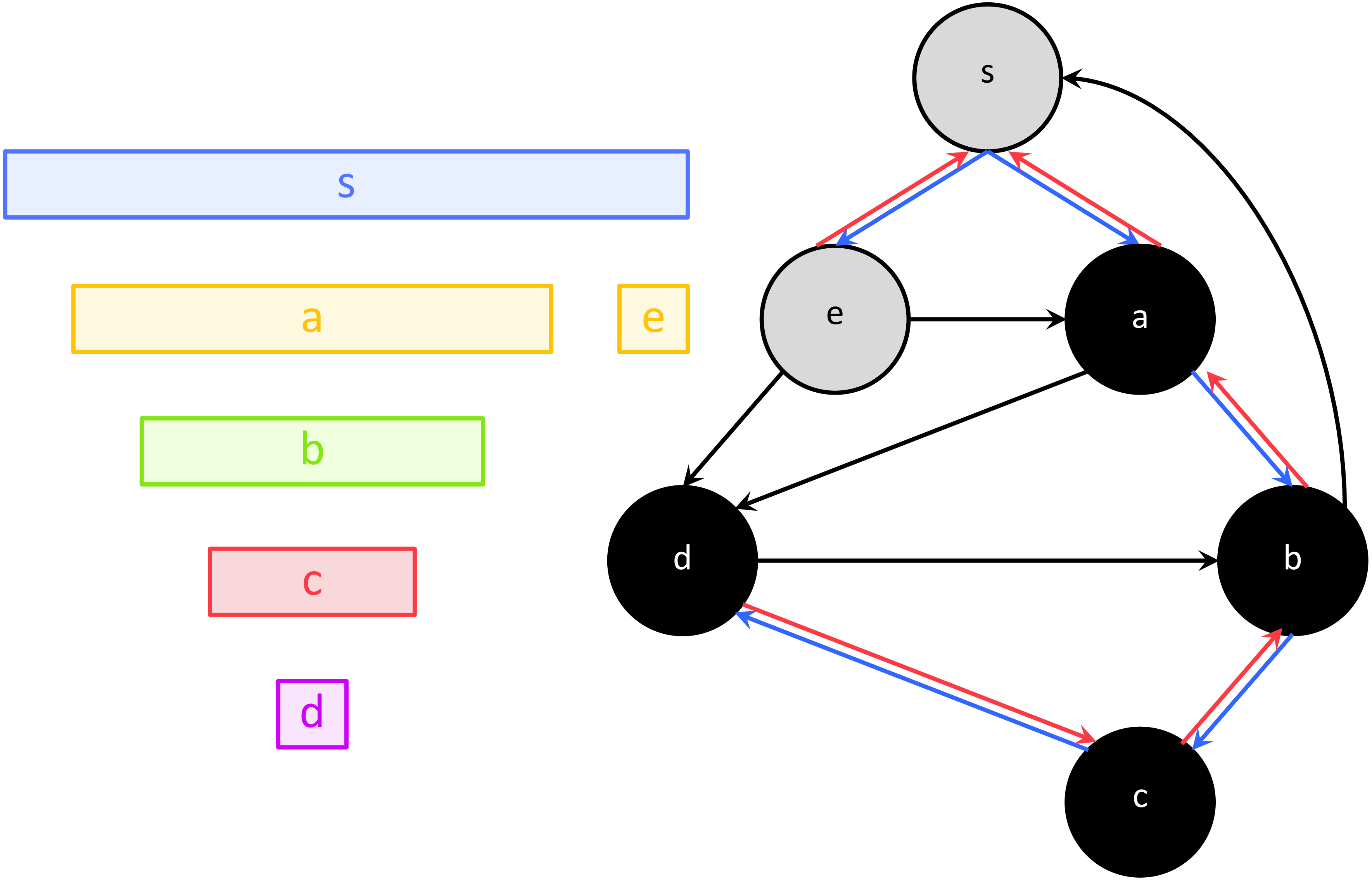
This is a *forward edge*.
We don't follow it because **d** is coloured black.



A *forward edge* connects a grey node to a black node.

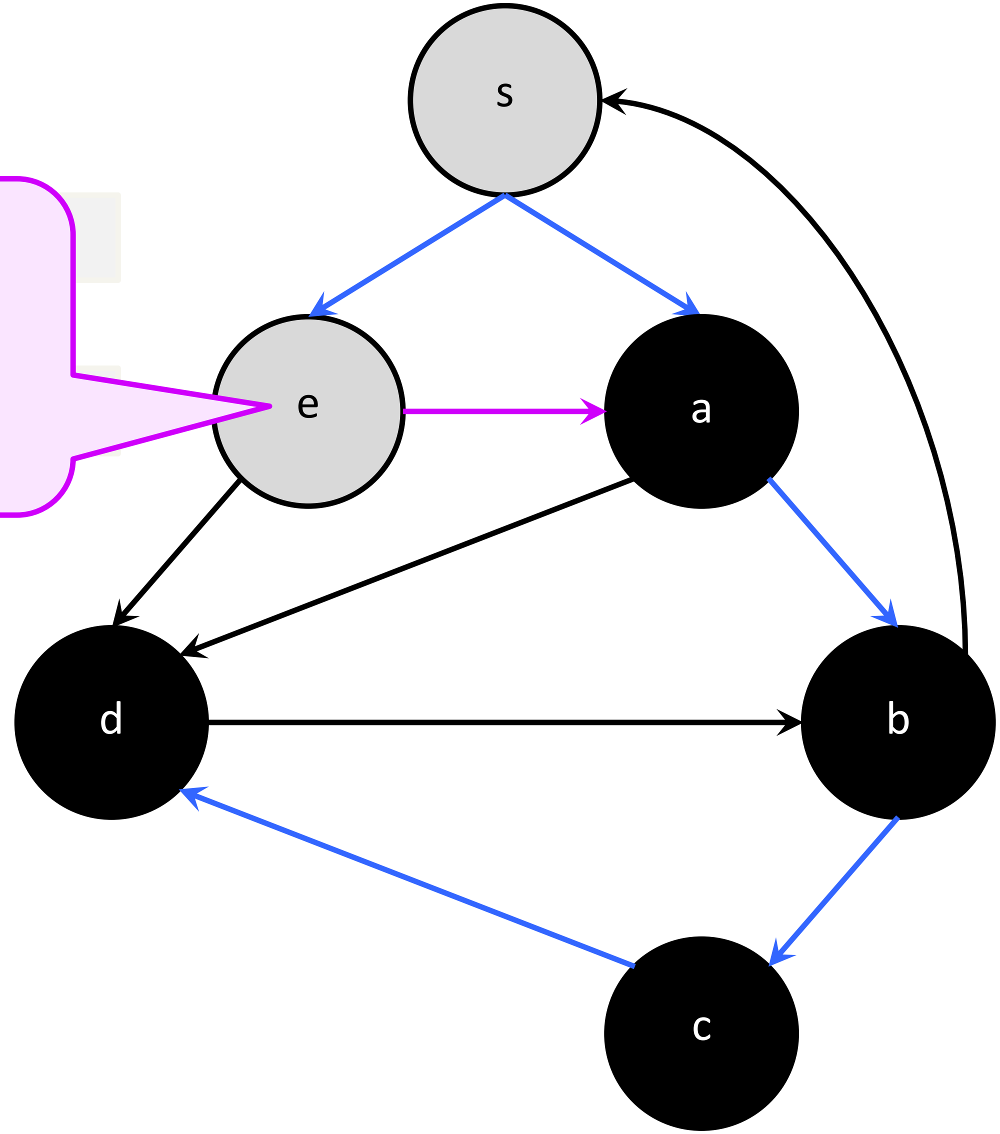



s
a
b
c
d
s
e
a
d
b
c

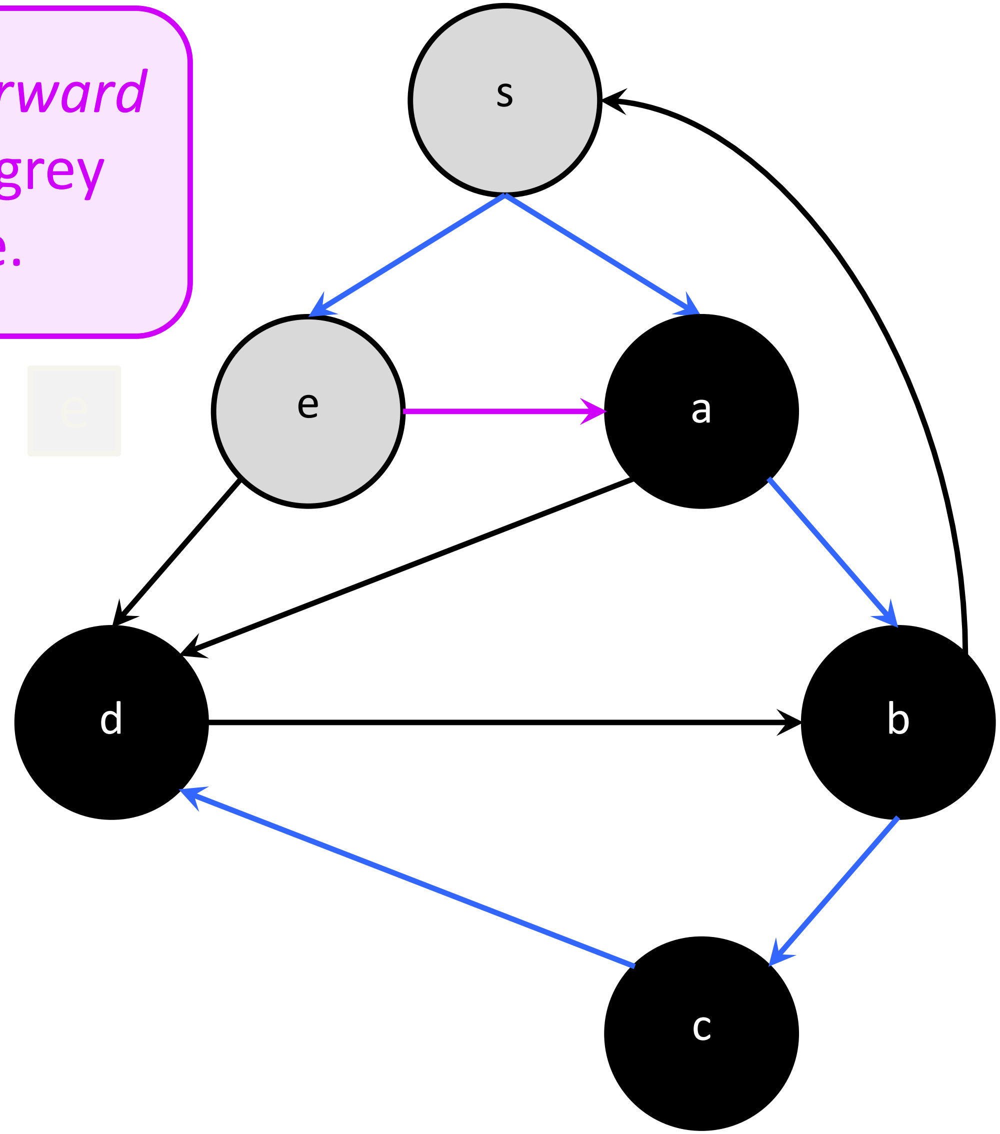


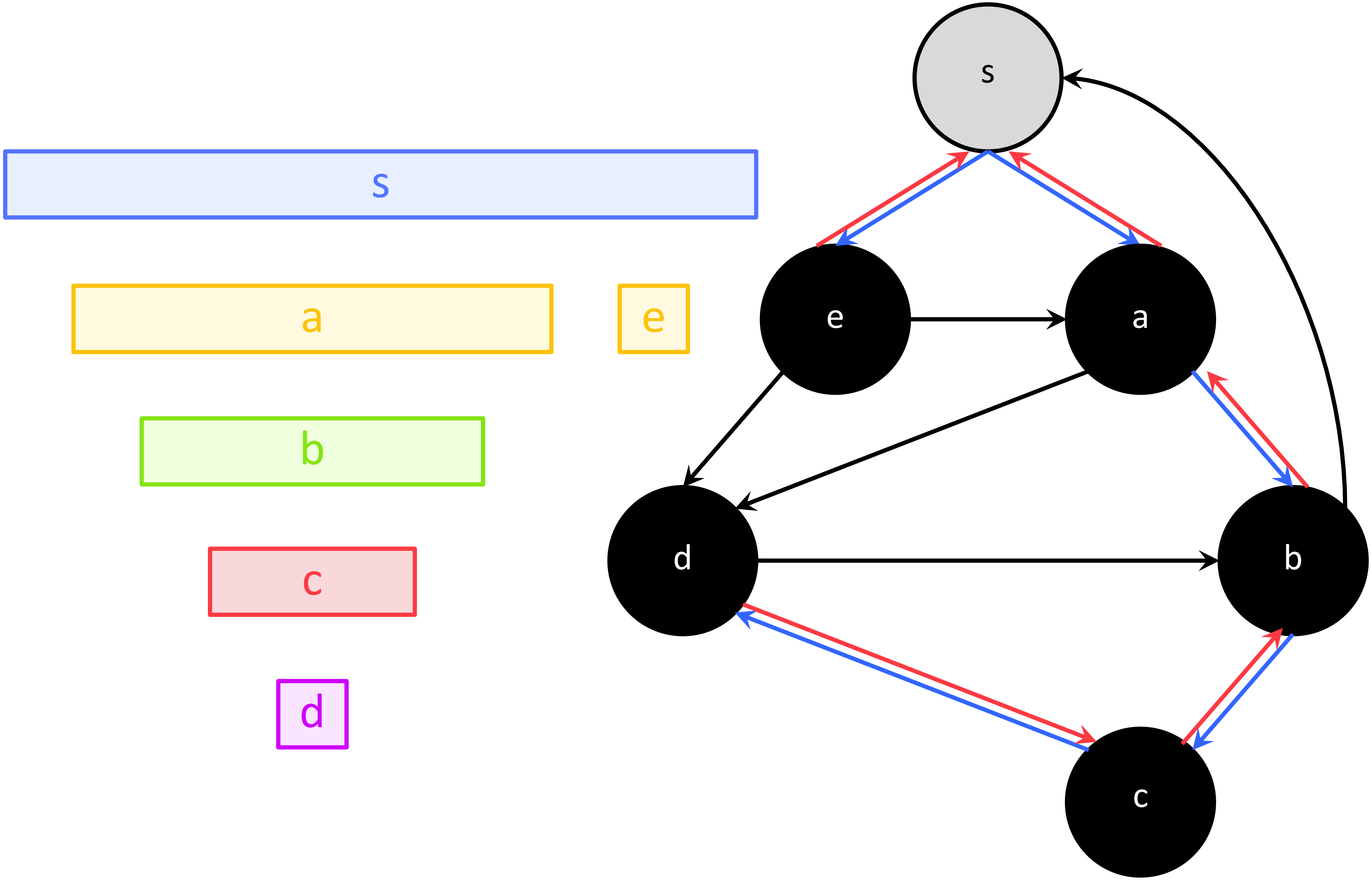
s
a
e
b
c
d
s
e
a
d
b
c

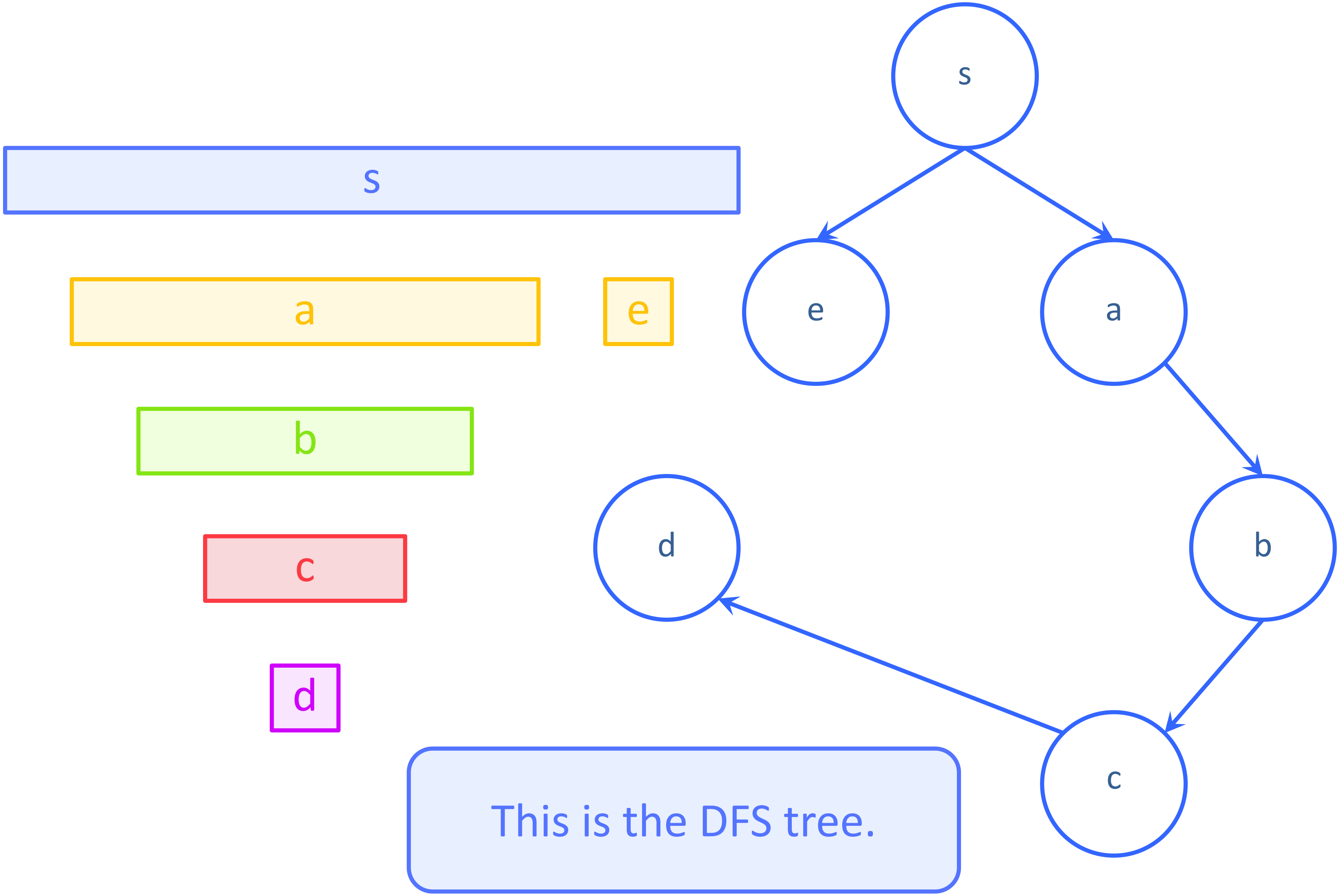
This is a *cross edge*.
It connects between two
different subtrees.

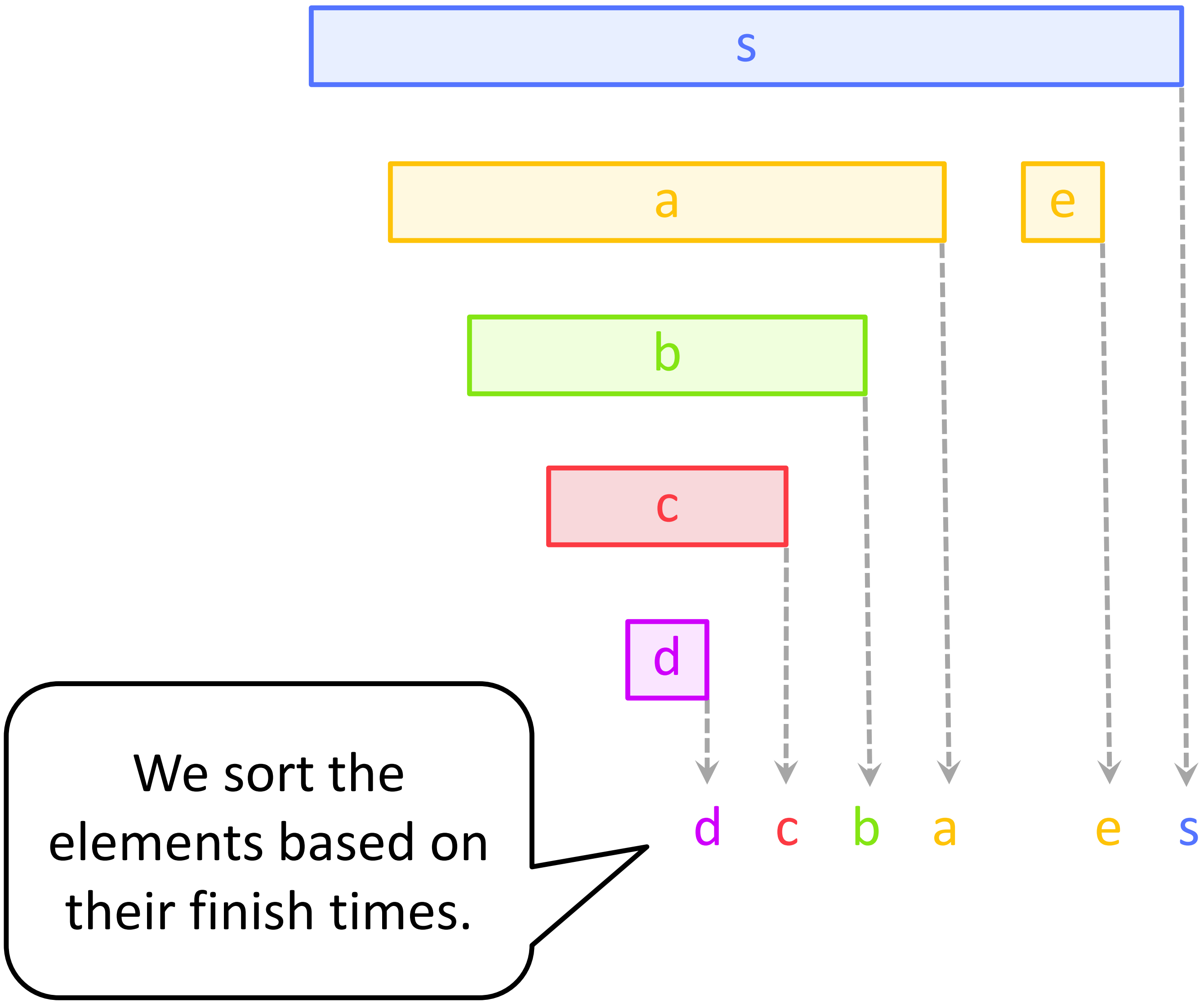


Both *cross edges* and *forward edges* connect from a grey node to a black one.



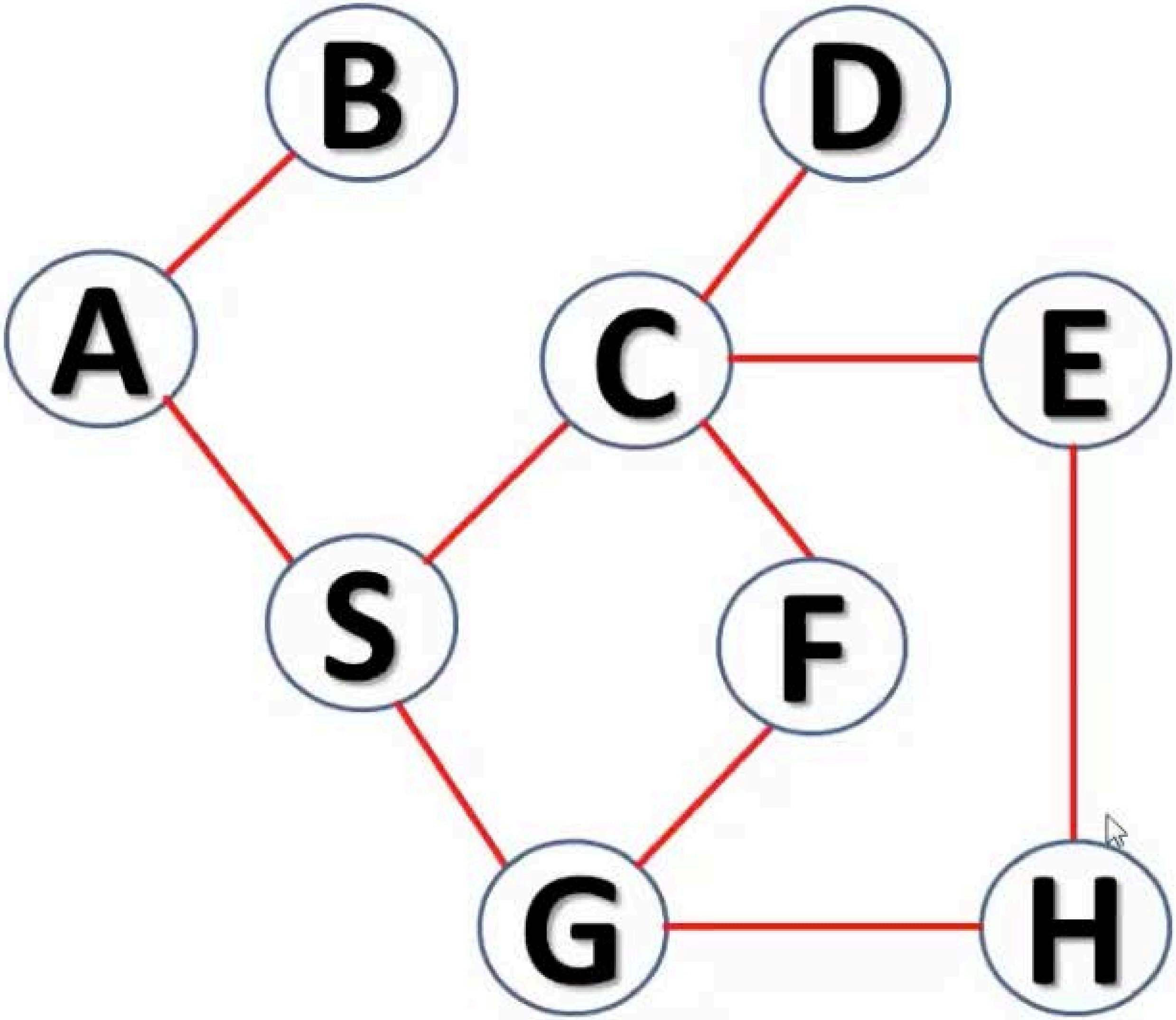

s
a
e
b
c
d
s
e
a
d
b
c


s
a
e
b
c
d
This is the DFS tree.
s
e
a
b
d
c

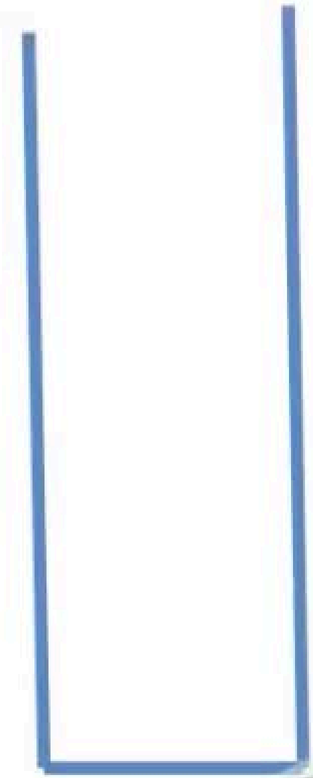

s
a
e
b
c
d
We sort the elements based on their finish times.
d c b a e s

1. DFS WITH STACK

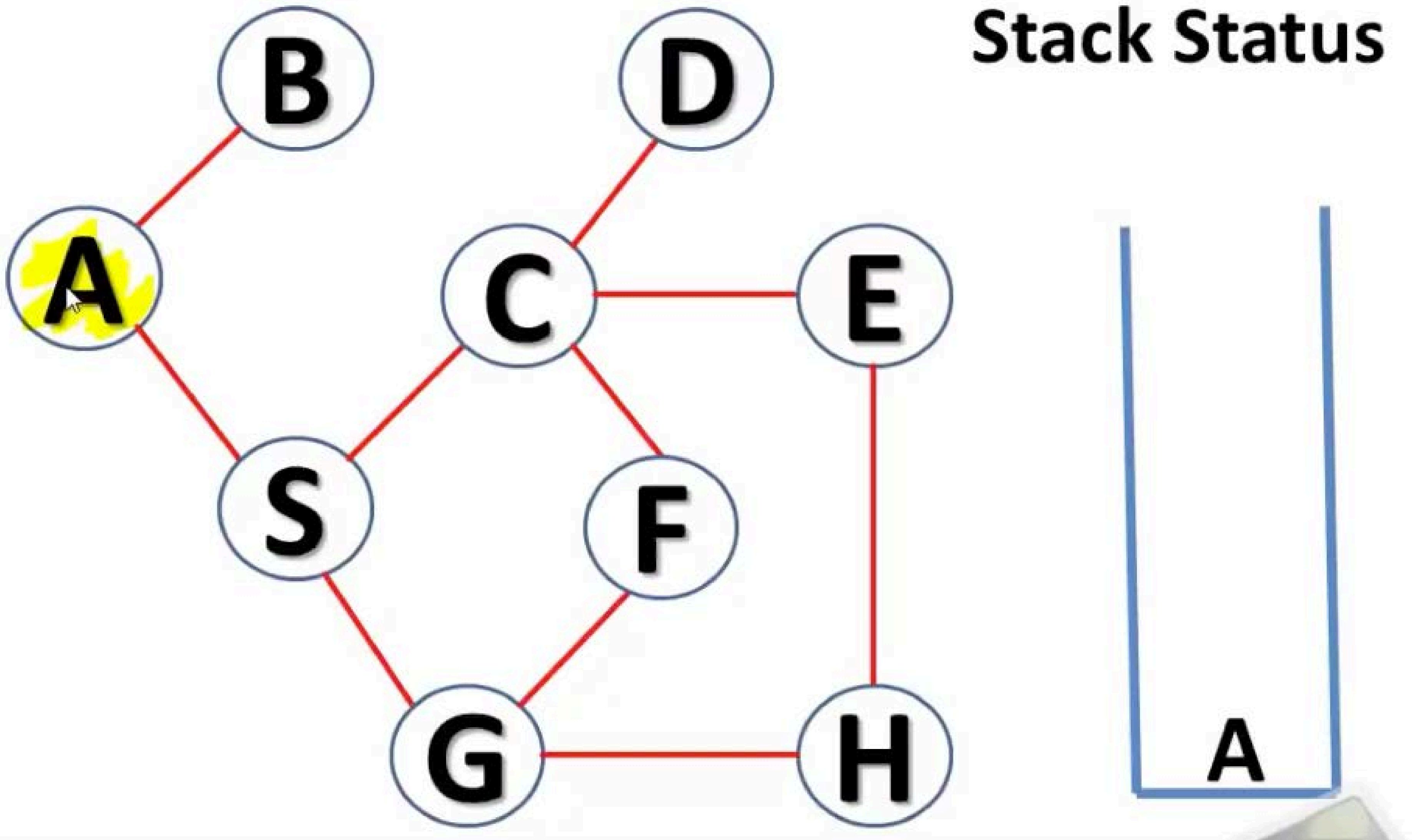
DEPTH FIRST SEARCH



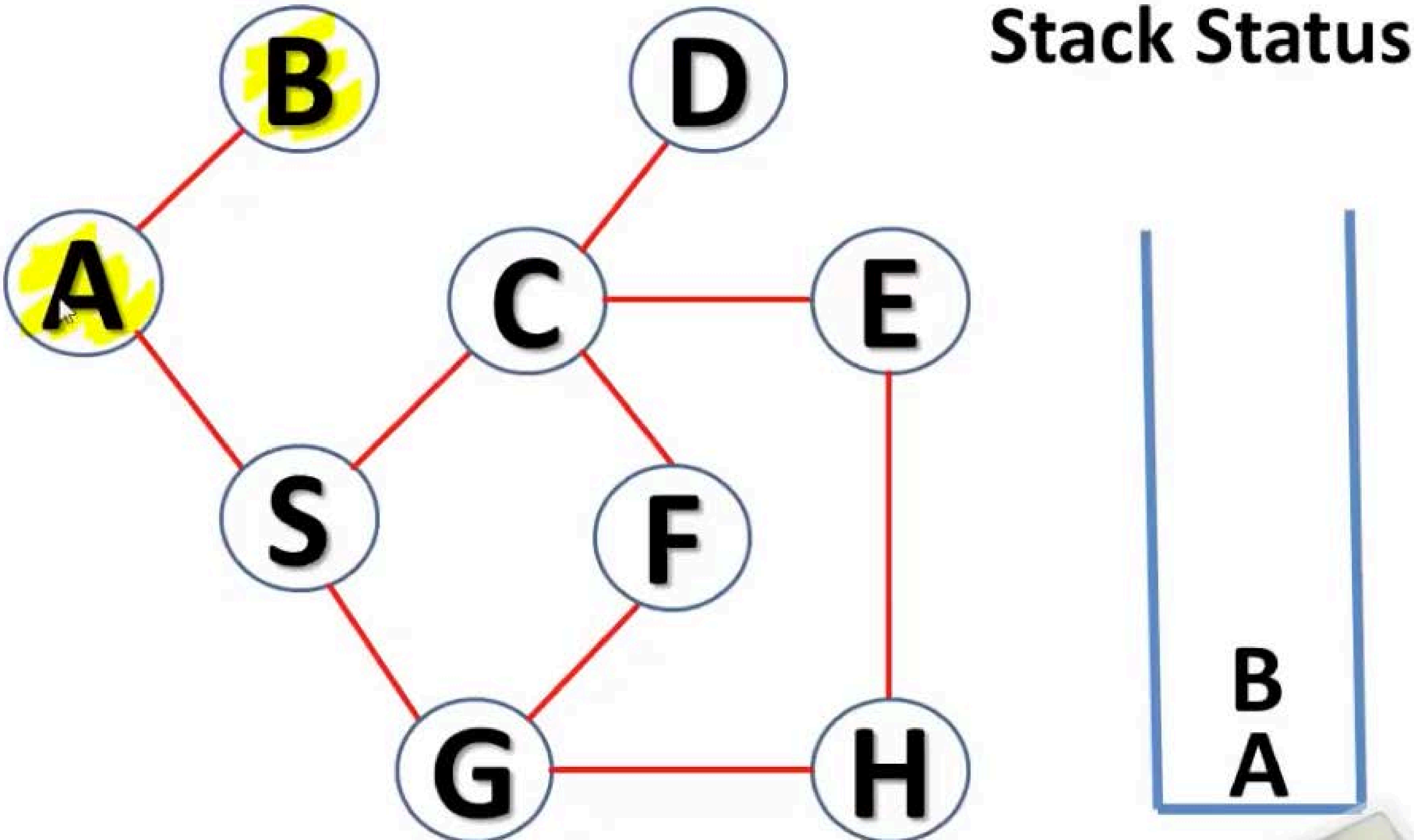
Stack Status



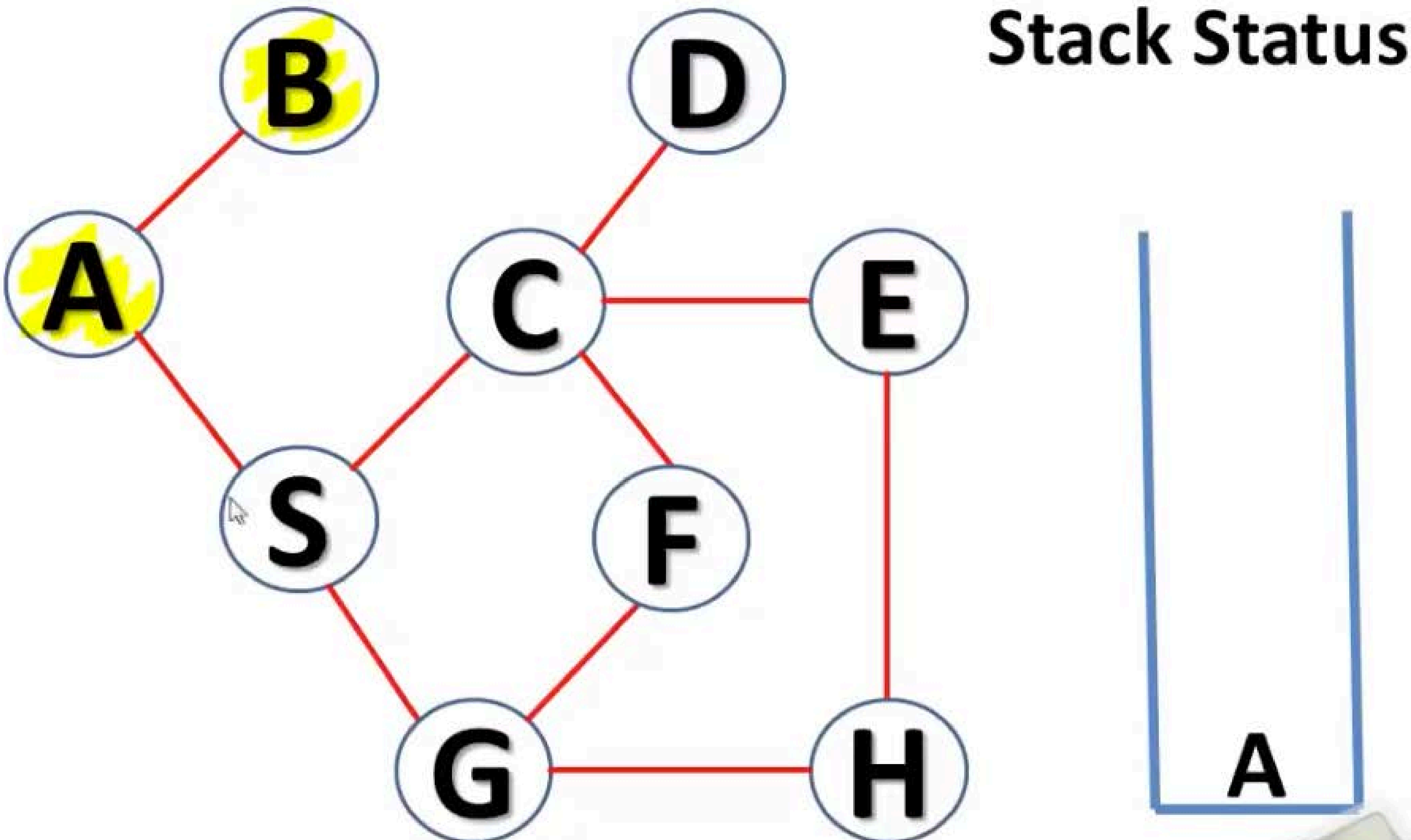
DEPTH FIRST SEARCH



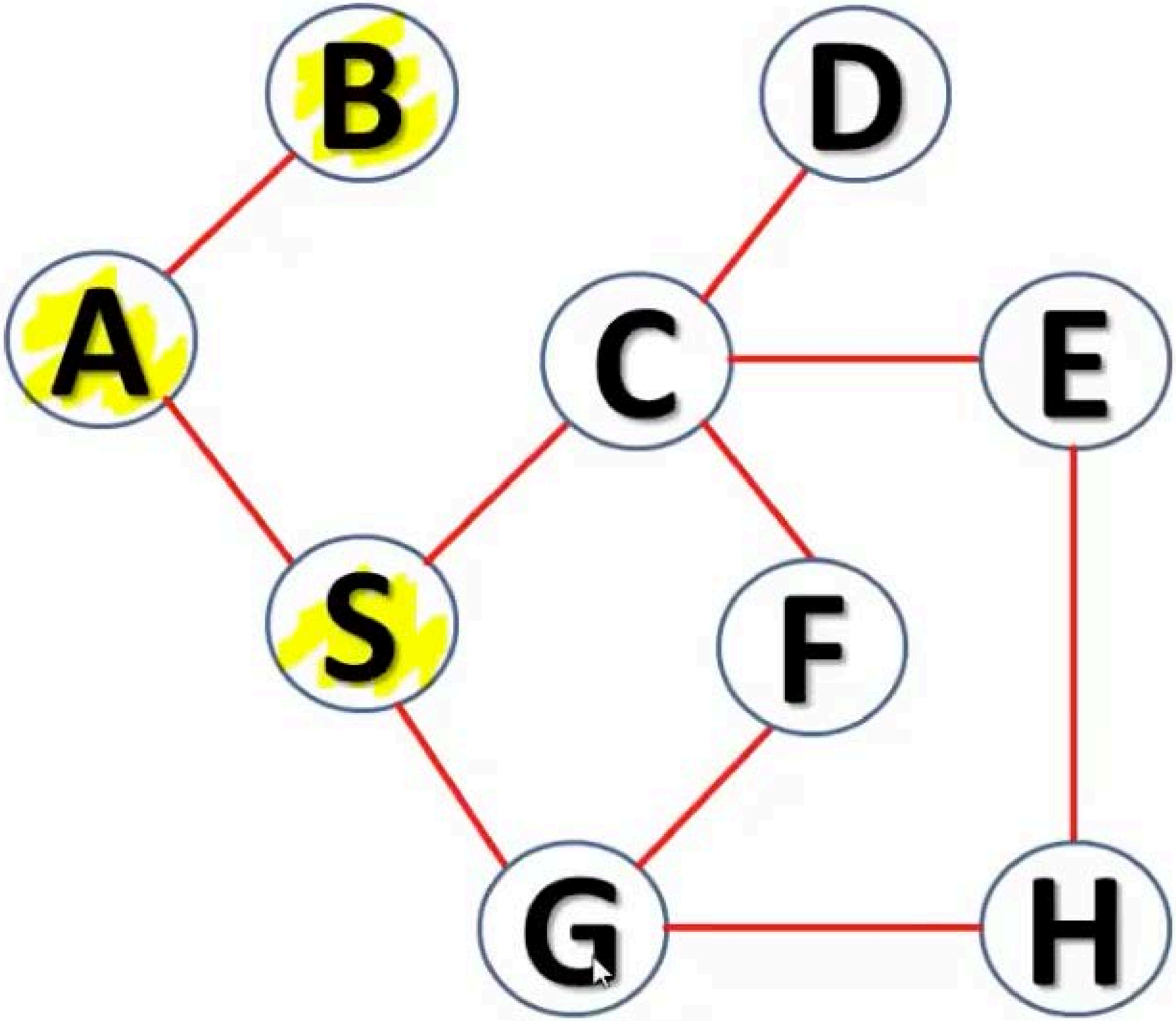
DEPTH FIRST SEARCH



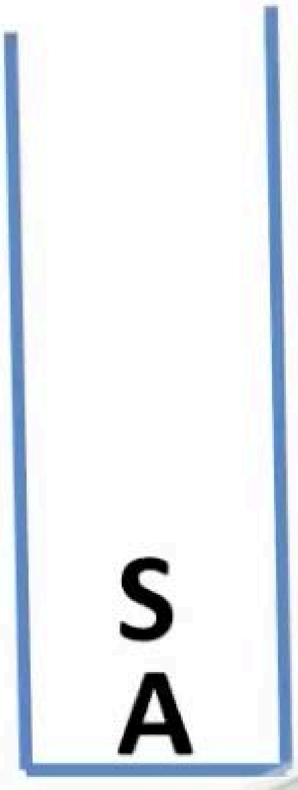
DEPTH FIRST SEARCH



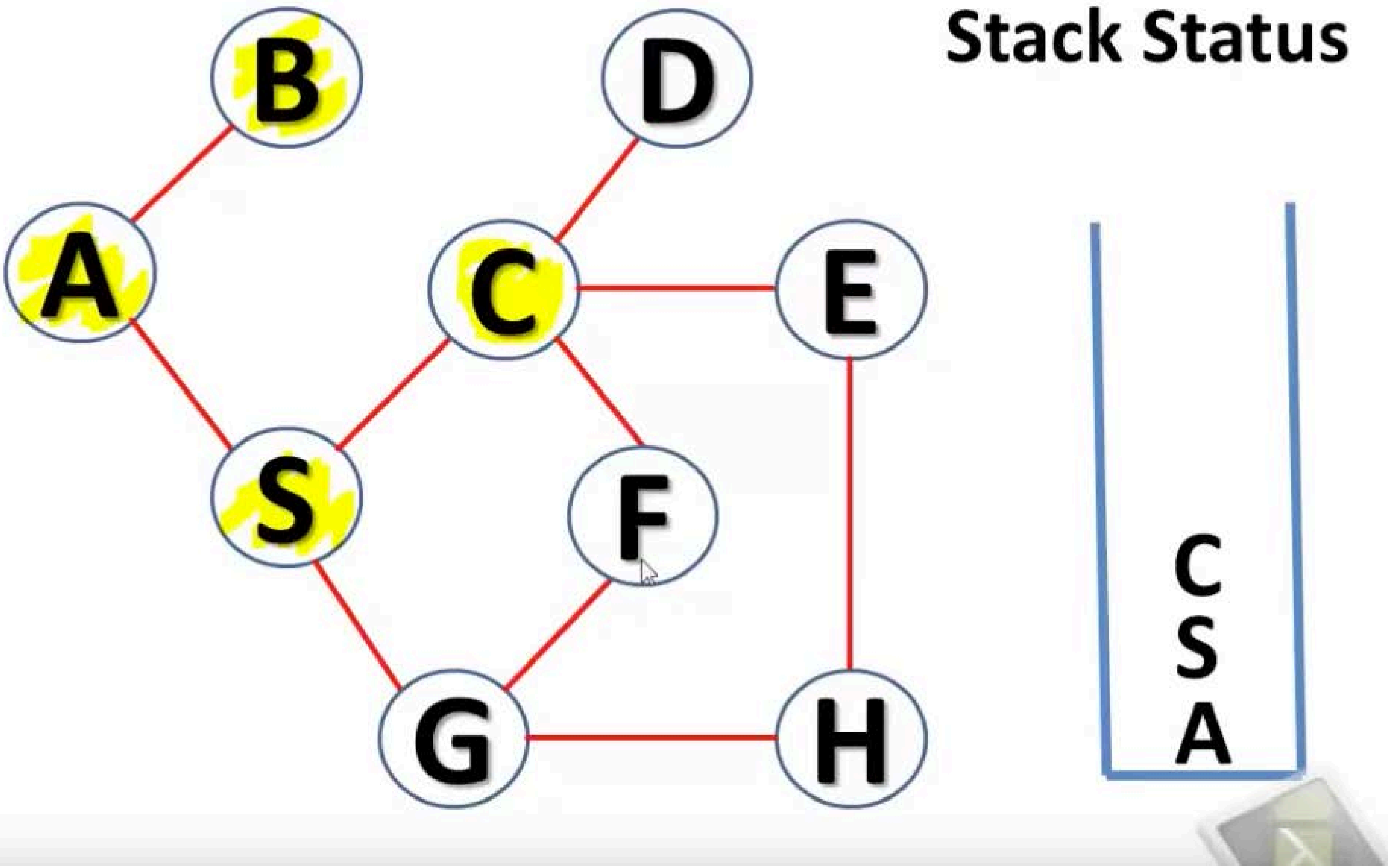
DEPTH FIRST SEARCH



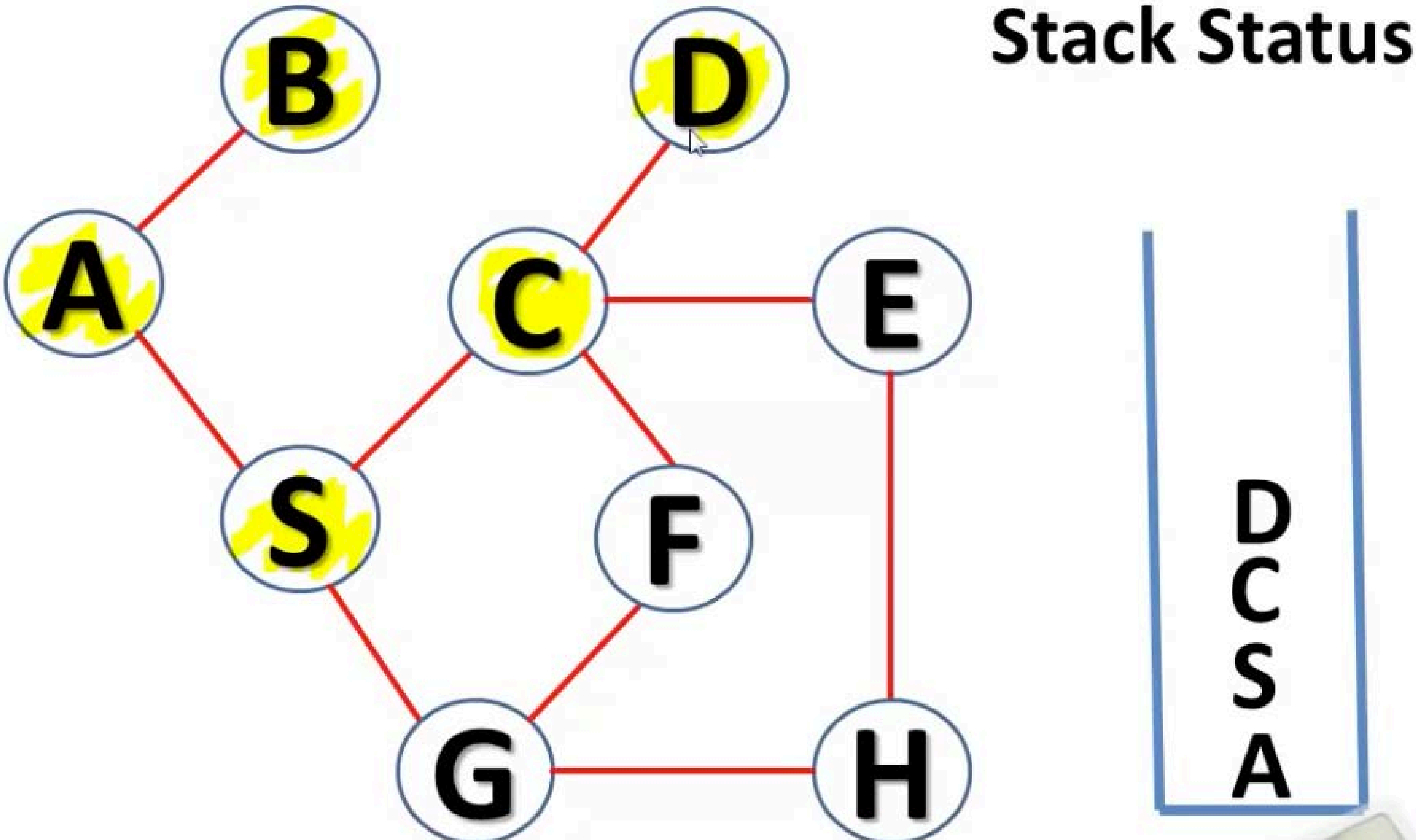
Stack Status



DEPTH FIRST SEARCH



DEPTH FIRST SEARCH



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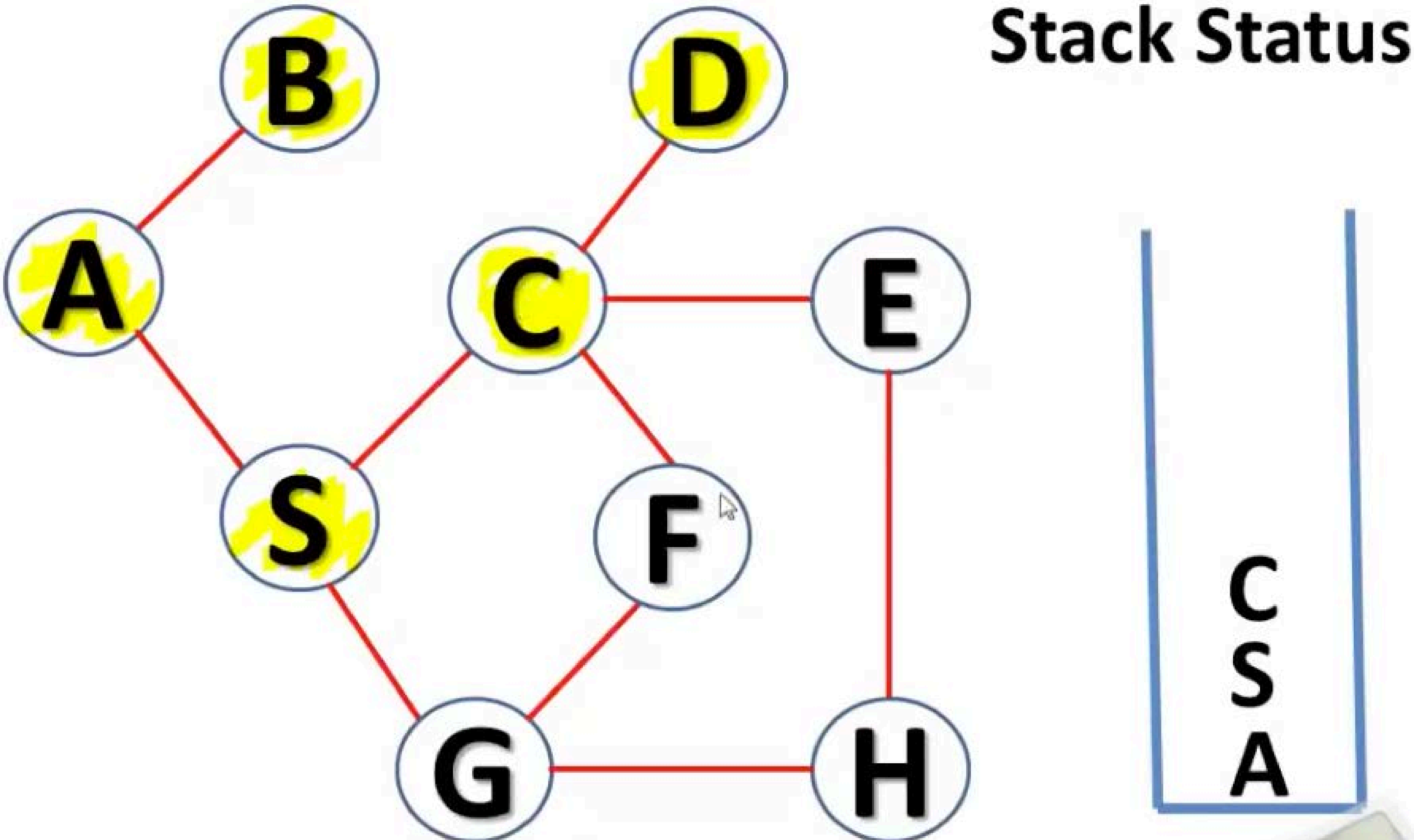

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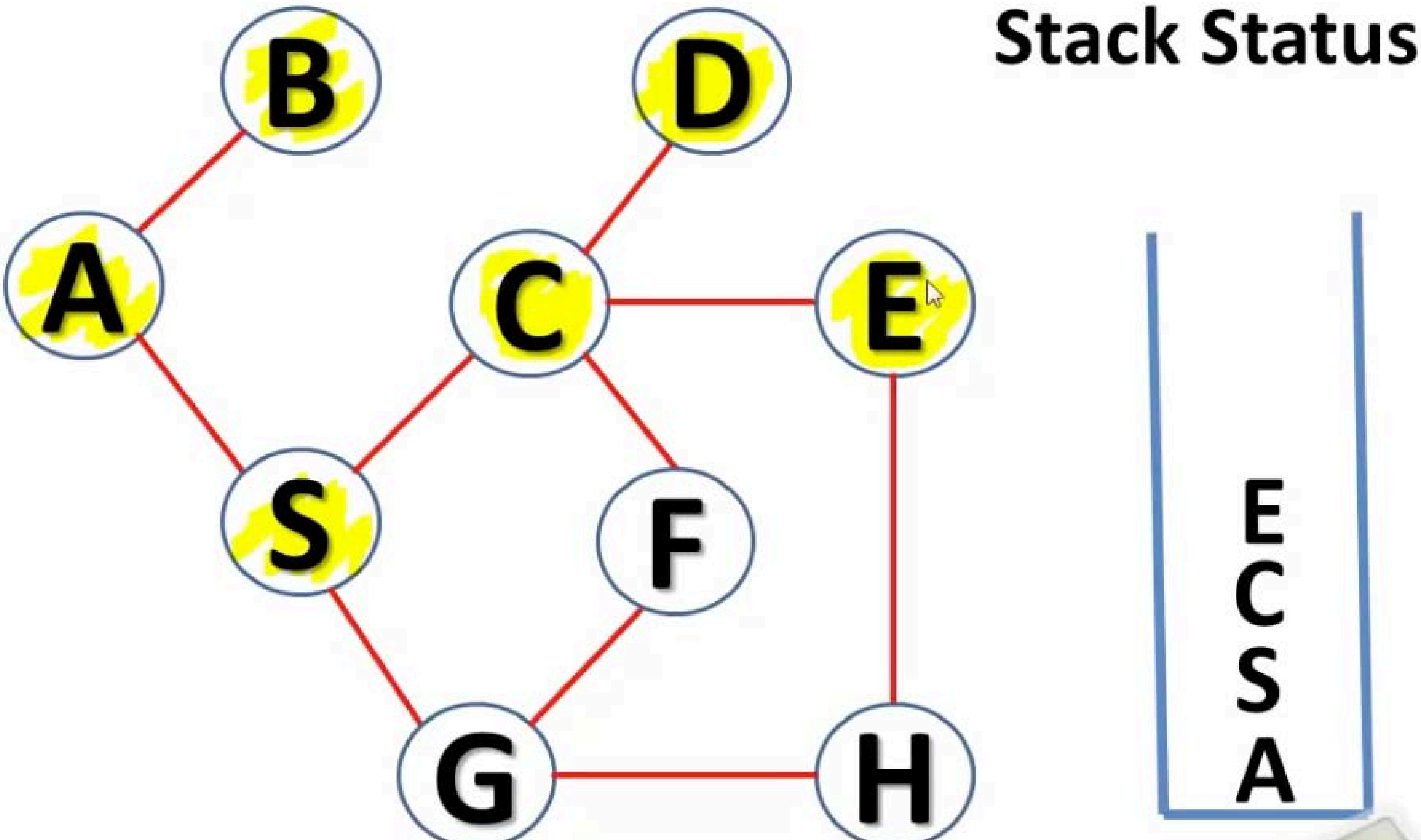
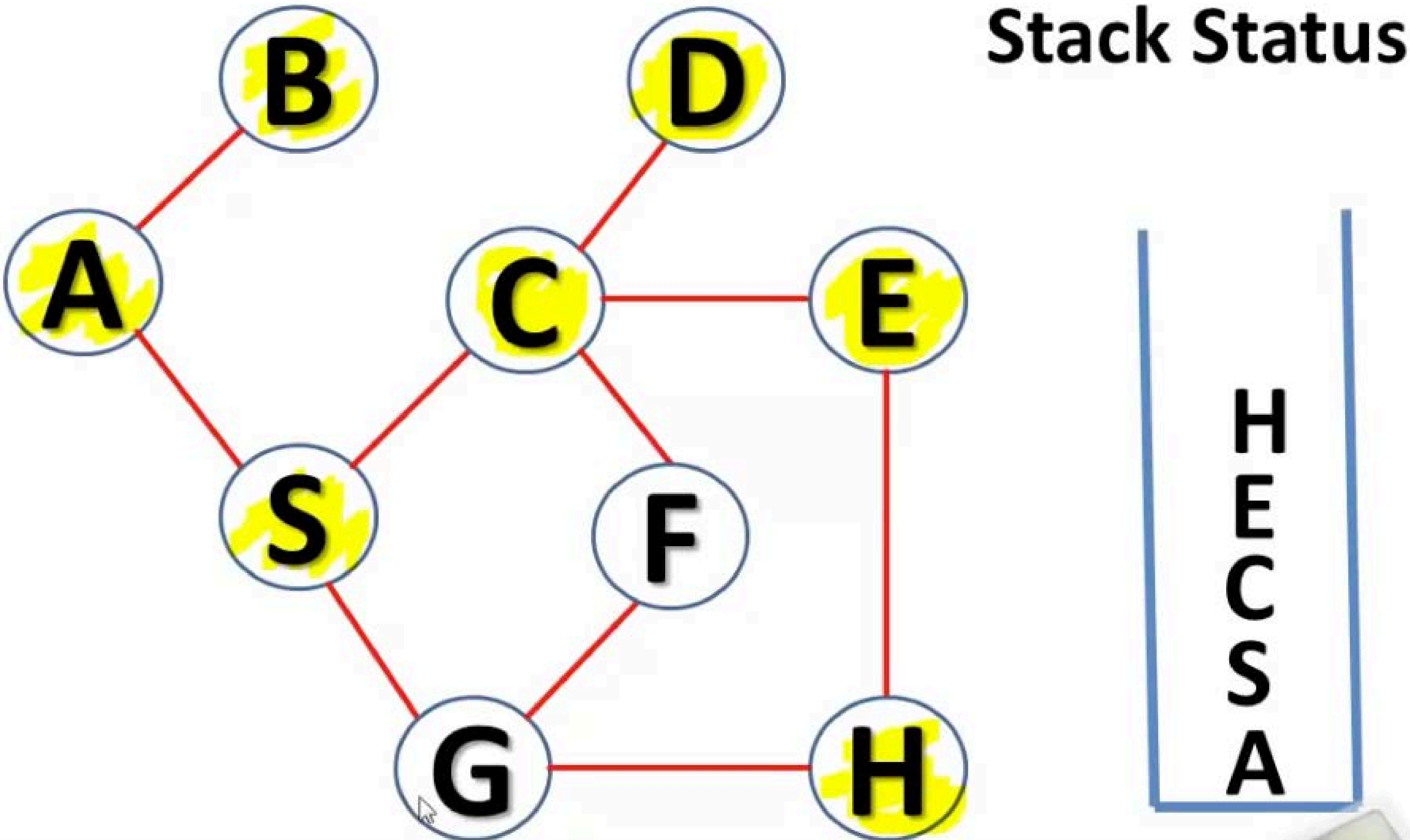
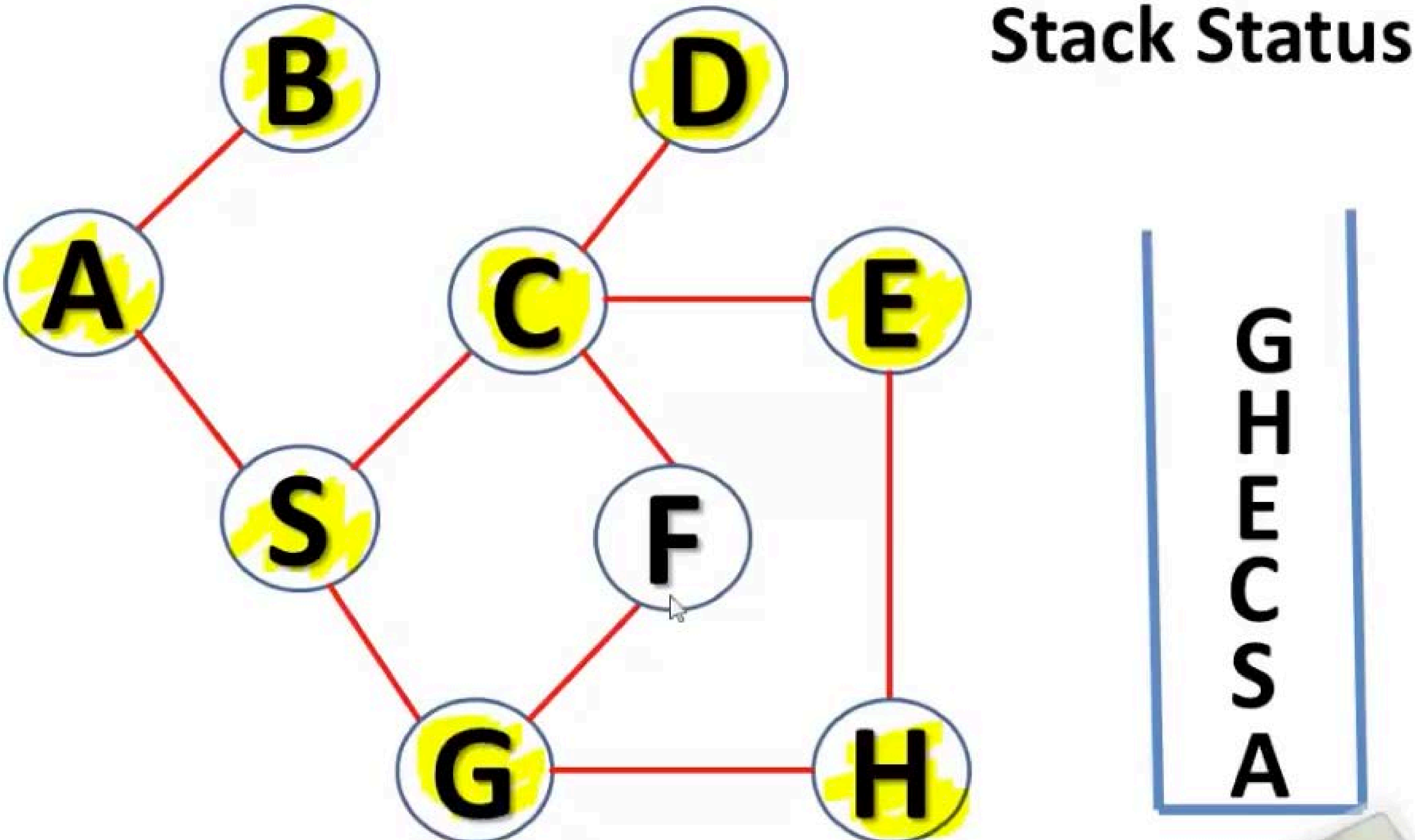


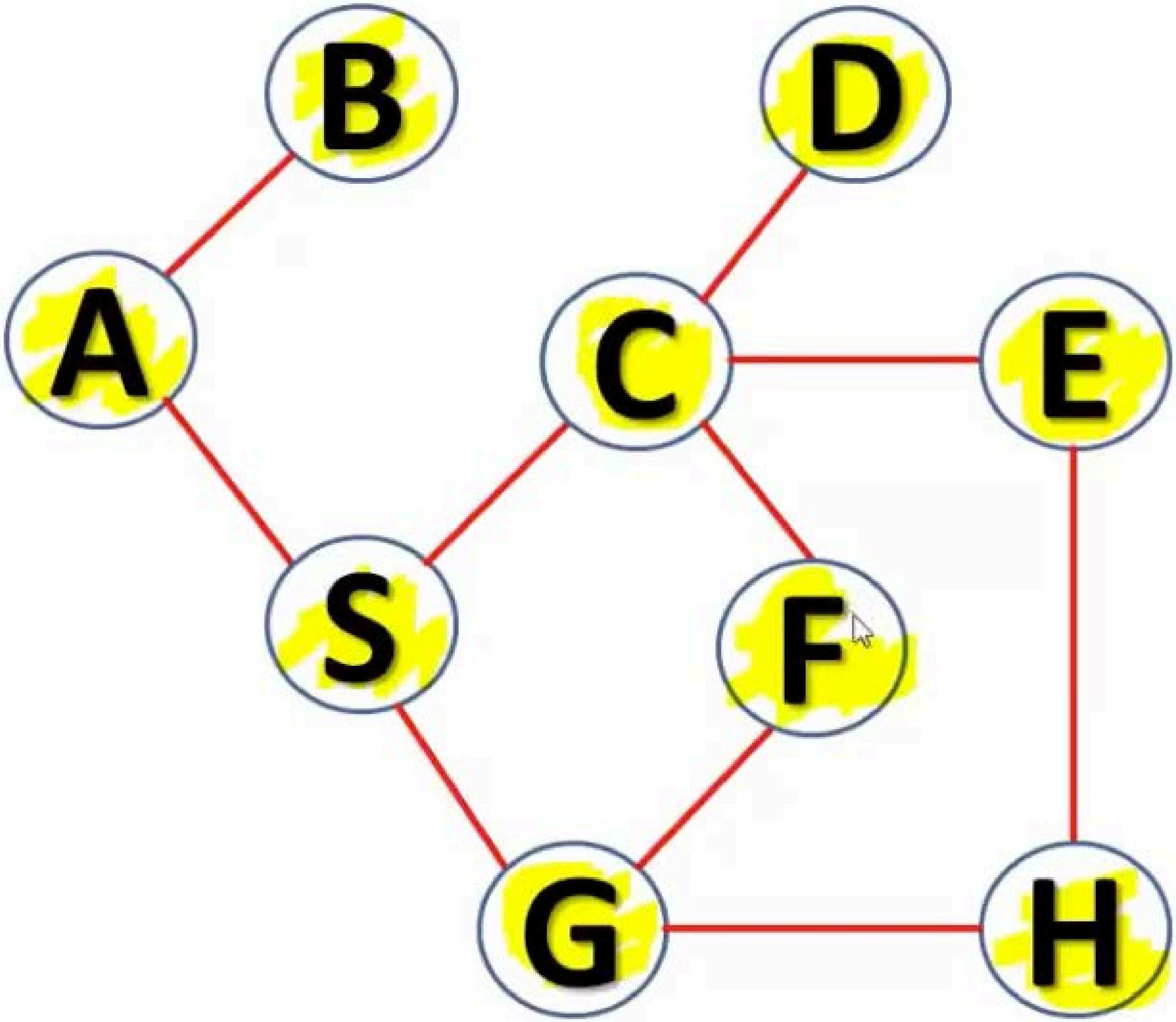
DEPTH FIRST SEARCH



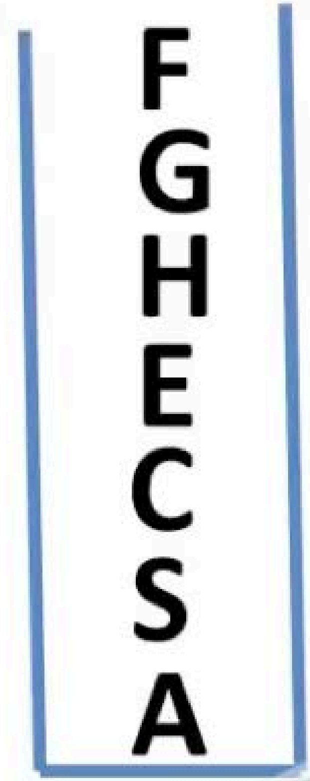
DEPTH FIRST SEARCH



DEPTH FIRST SEARCH



Stack Status



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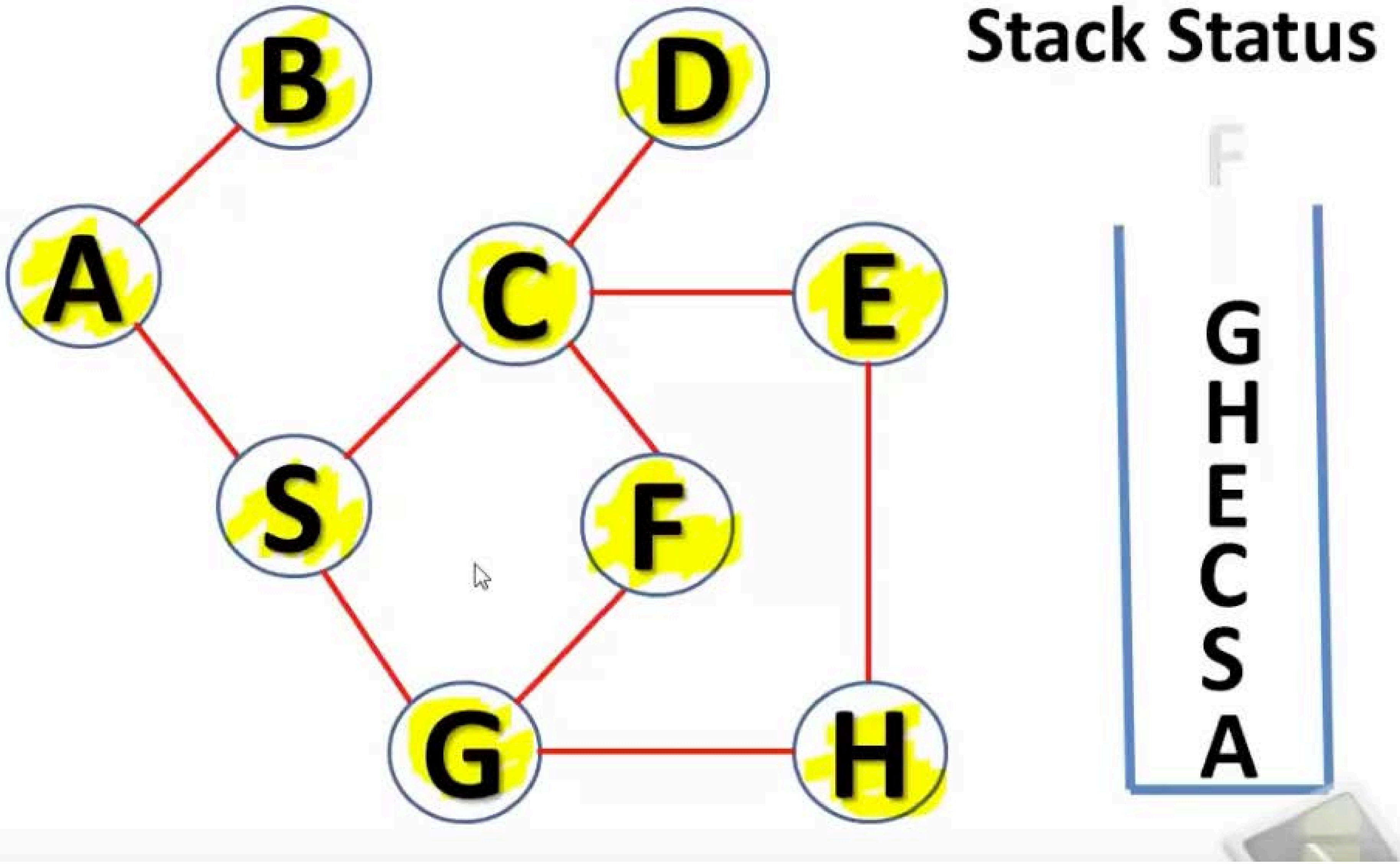
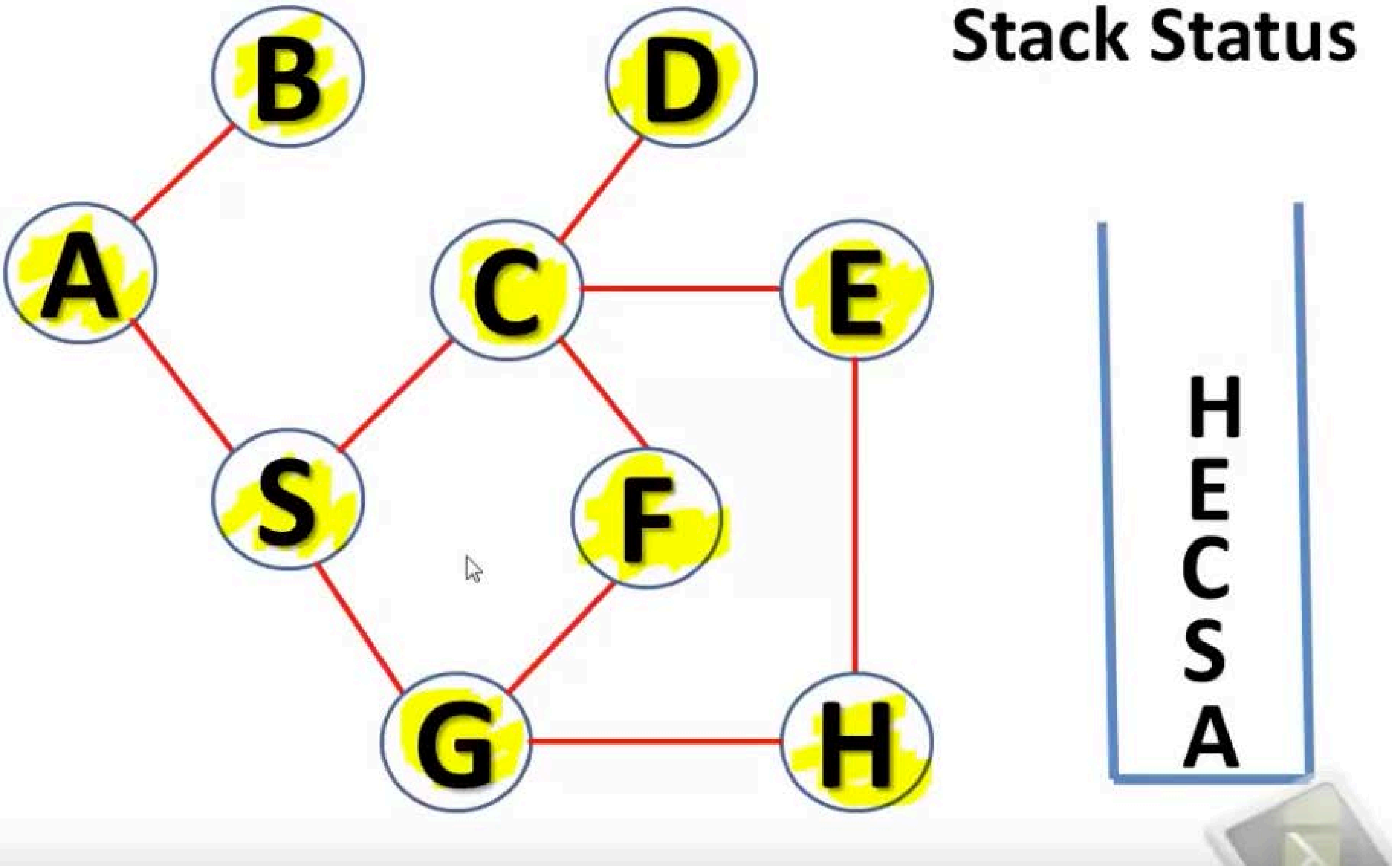
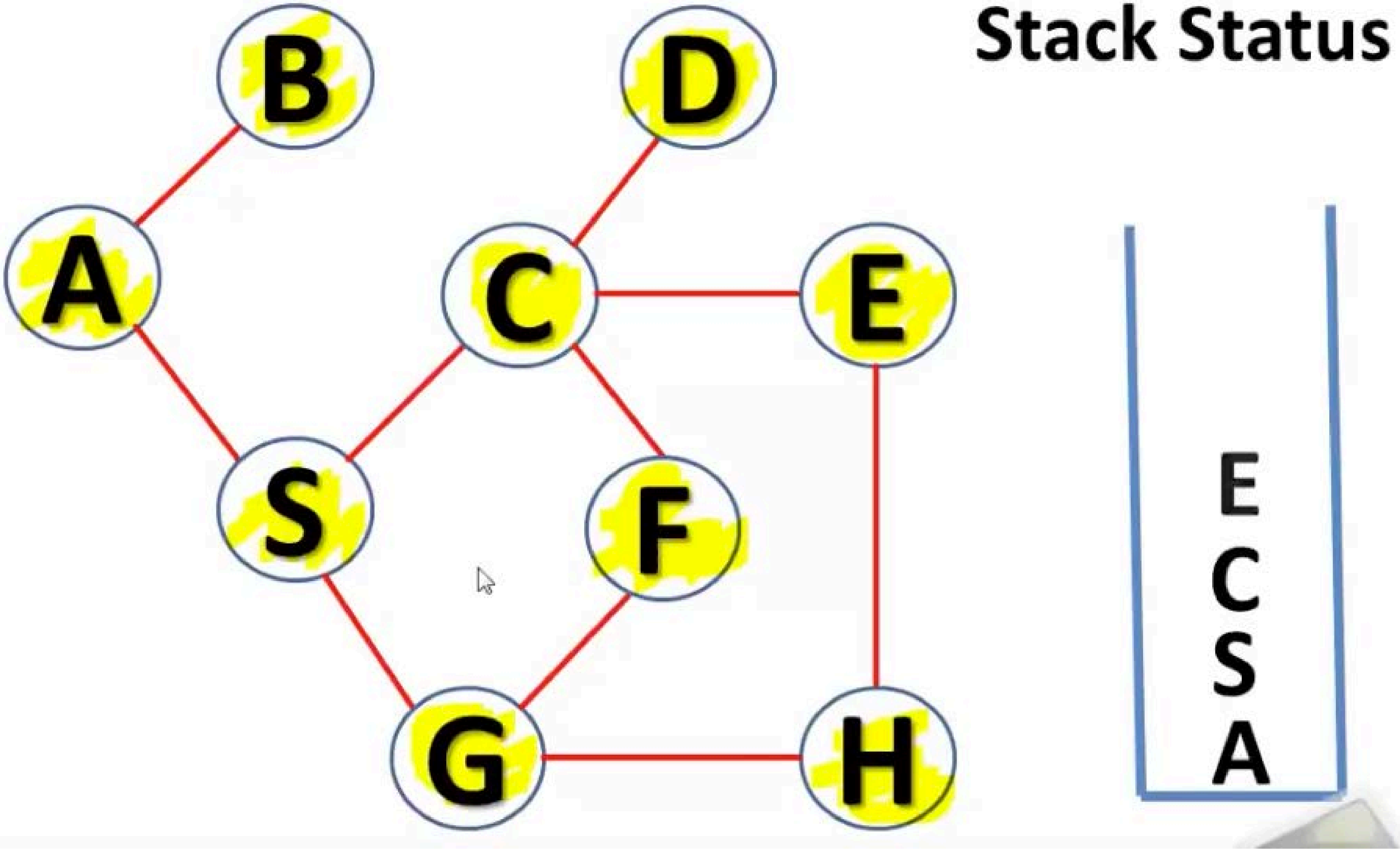


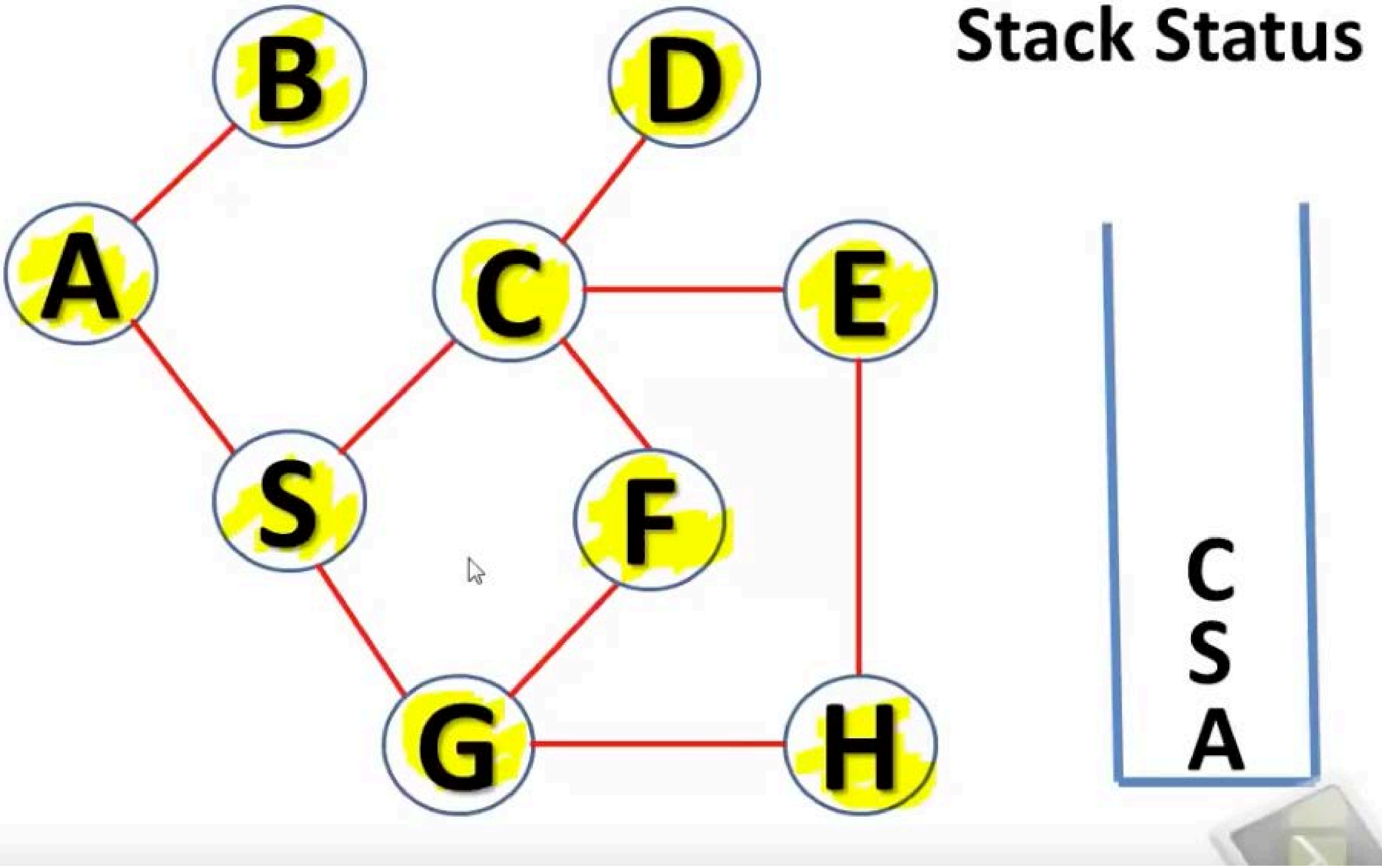
DEPTH FIRST SEARCH



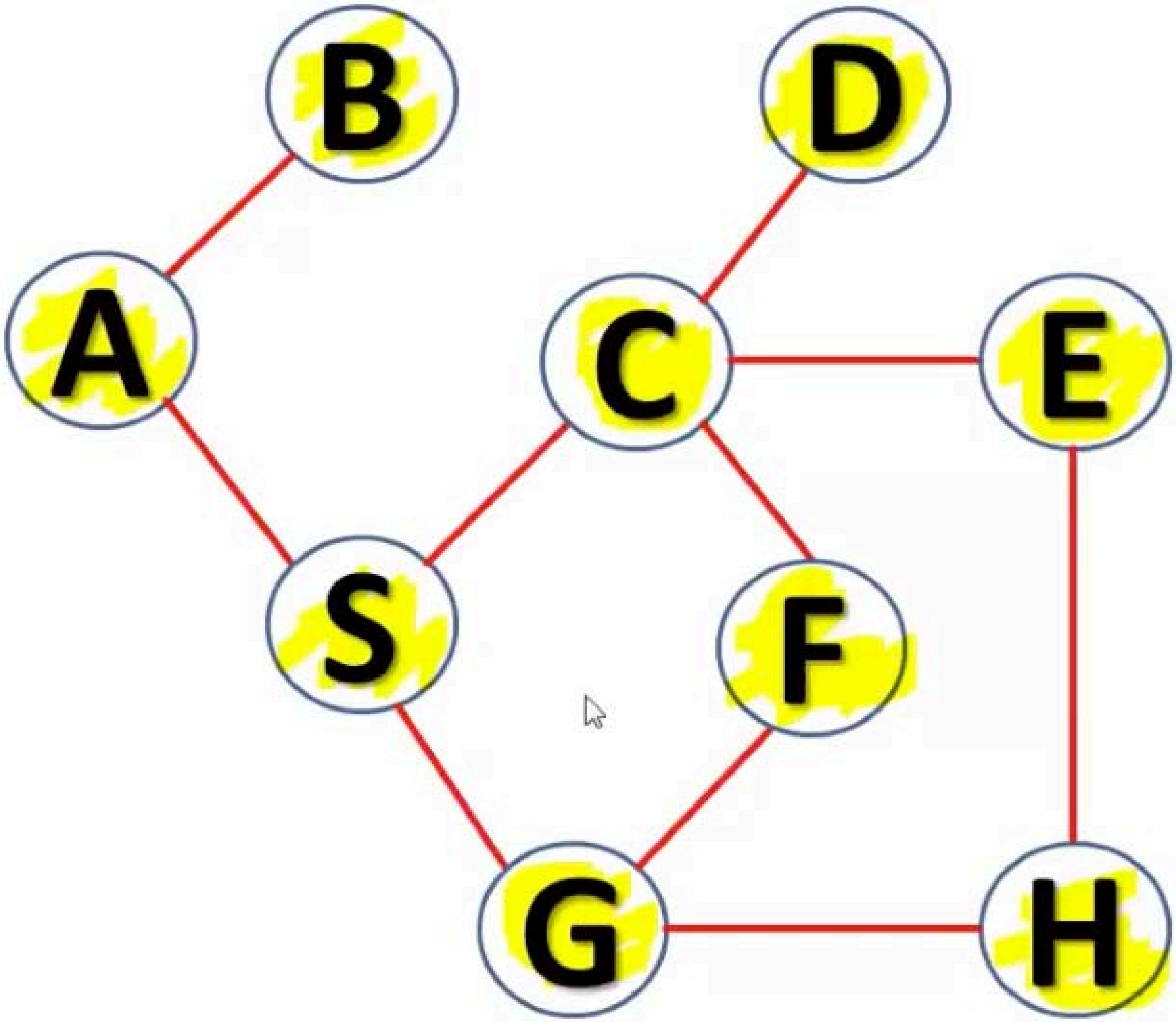
DEPTH FIRST SEARCH



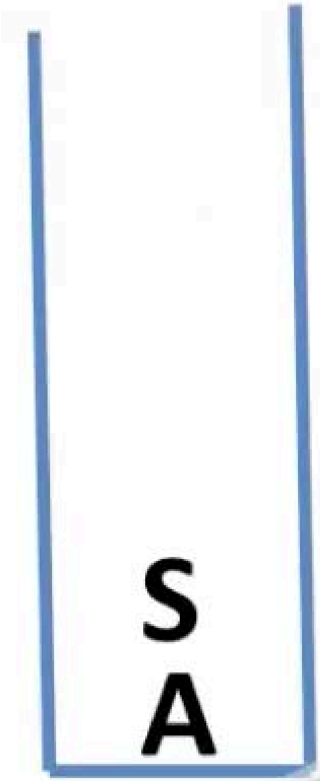
DEPTH FIRST SEARCH



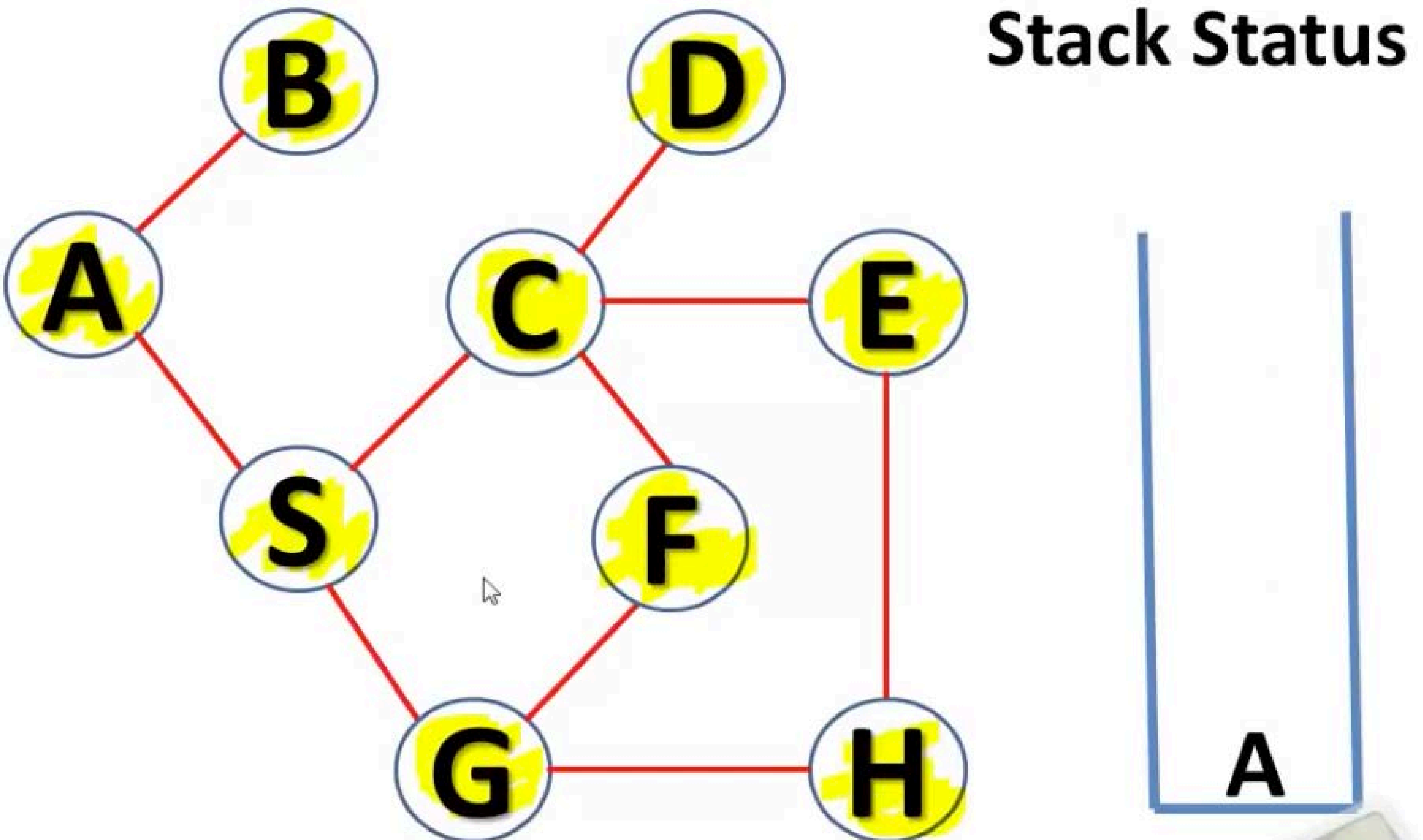
DEPTH FIRST SEARCH



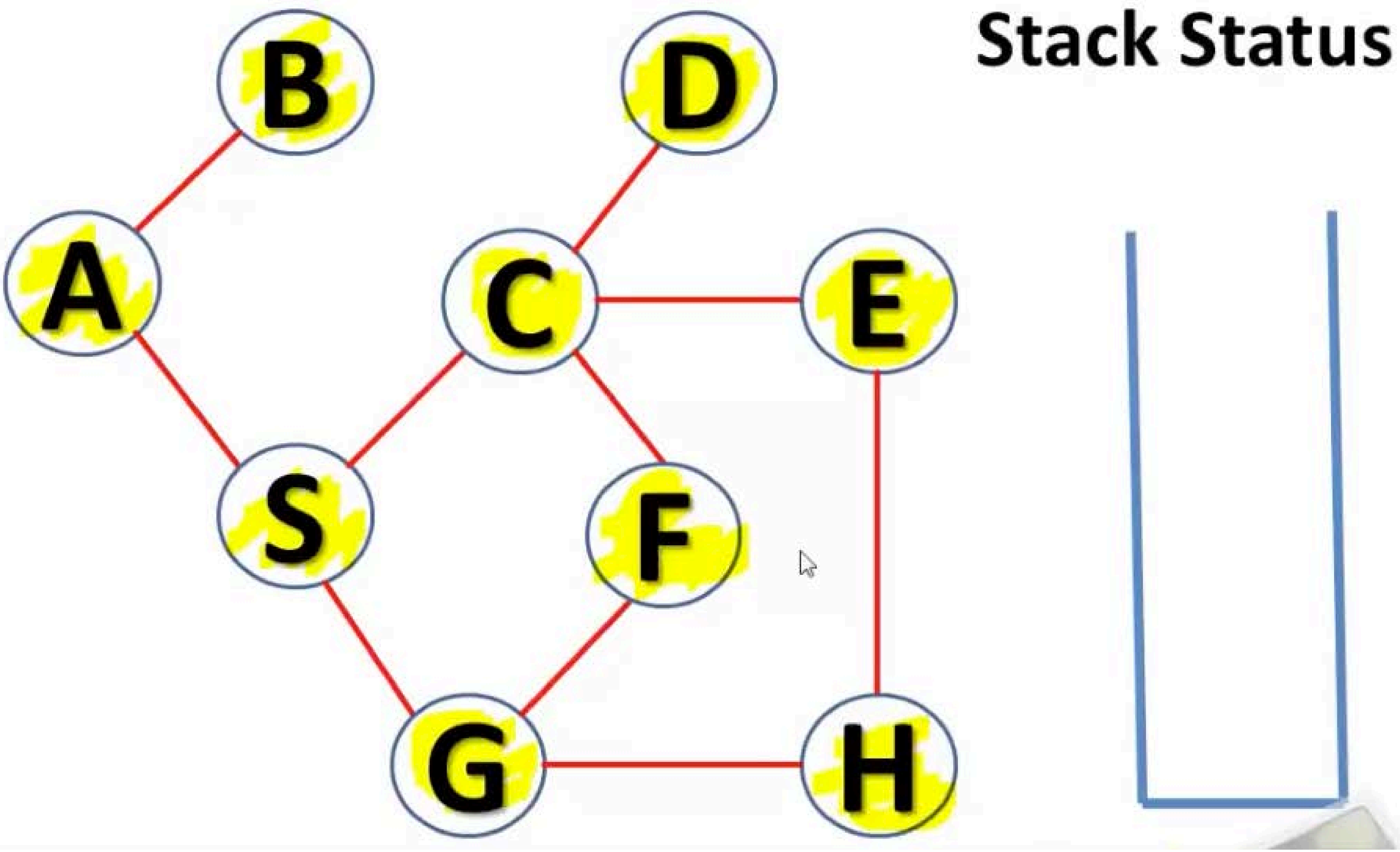
Stack Status



DEPTH FIRST SEARCH

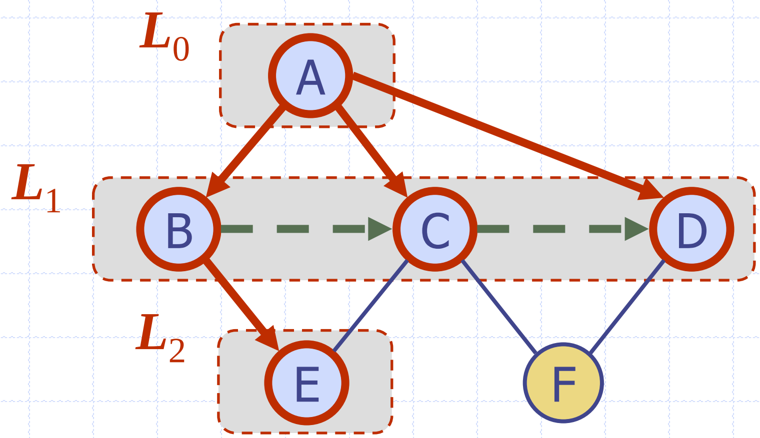


DEPTH FIRST SEARCH



Presentation for use with the textbook, Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Breadth-First Search



Breadth-First Search

- Breadth-first search (BFS) is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- BFS on a graph with n vertices and m edges takes $O(n + m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

BFS Algorithm

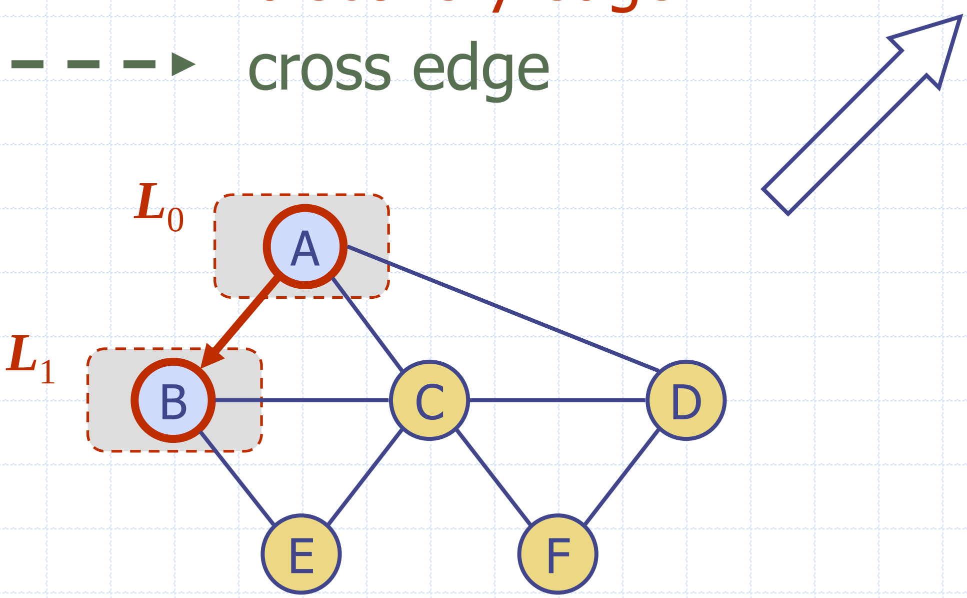
- The algorithm uses "levels" L_i and a mechanism for setting and getting "labels" of vertices and edges.

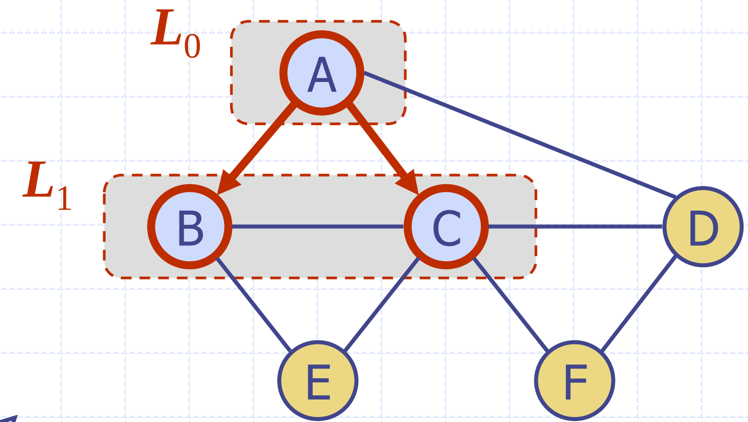
Algorithm BFS(G, s):
Input: A graph G and a vertex s of G
Output: A labeling of the edges in the connected component of s as discovery edges and cross edges

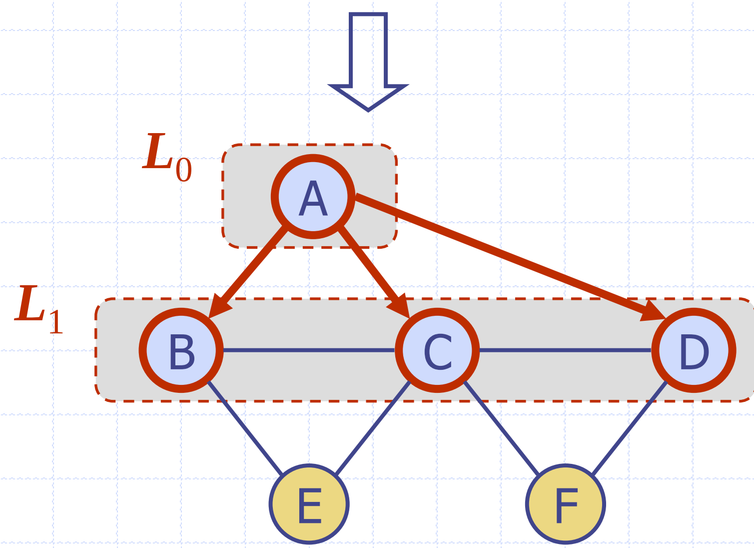
```
Create an empty list, L_0
Mark s as explored and insert s into L_0
i ← 0
while L_i is not empty do
    create an empty list, L_{i+1}
    for each vertex, v, in L_i do
        for each edge, e = (v, w), incident on v in G do
            if edge e is unexplored then
                if vertex w is unexplored then
                    Label e as a discovery edge
                    Mark w as explored and insert w into L_{i+1}
                else
                    Label e as a cross edge
    i ← i + 1
```

Example

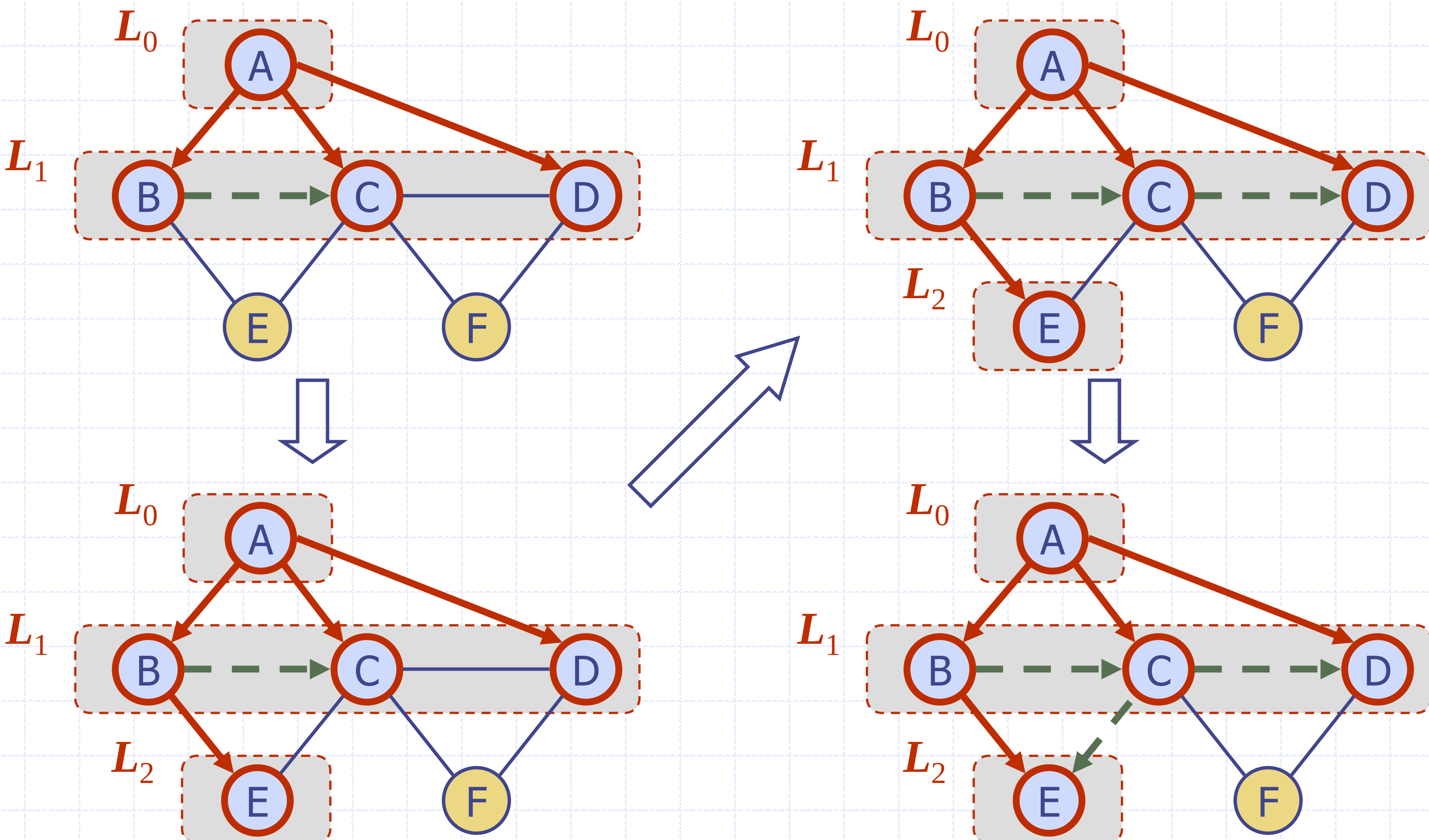
- unexplored vertex
- visited vertex
- unexplored edge
- discovery edge
- cross edge



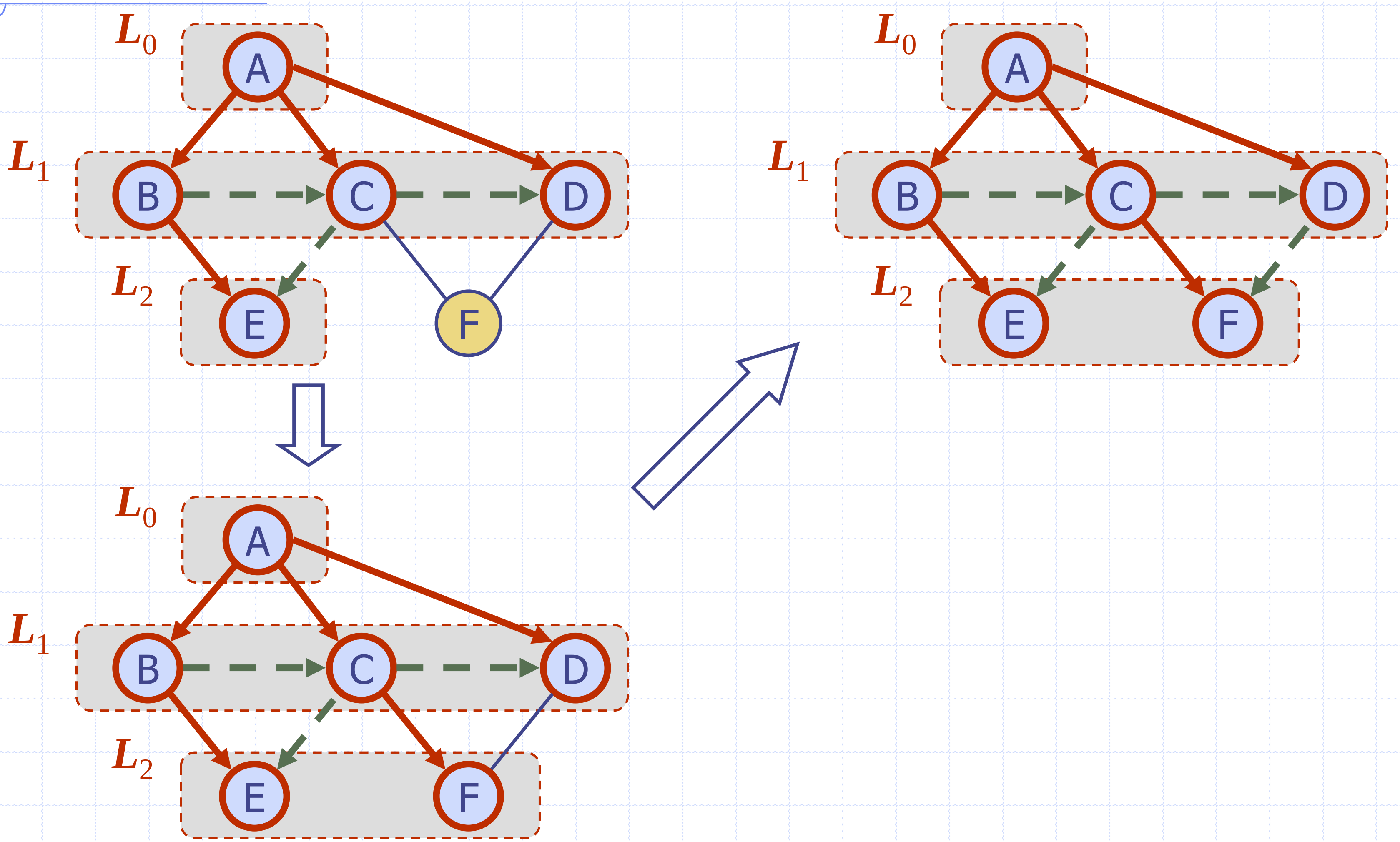




Example (cont.)



Example (cont.)



Properties

Notation

G_s: connected component of s

Property 1

$BFS(G, s)$ visits all the vertices and edges of G_s

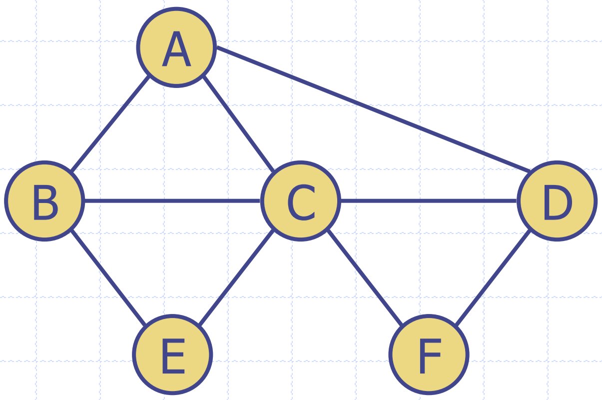
Property 2

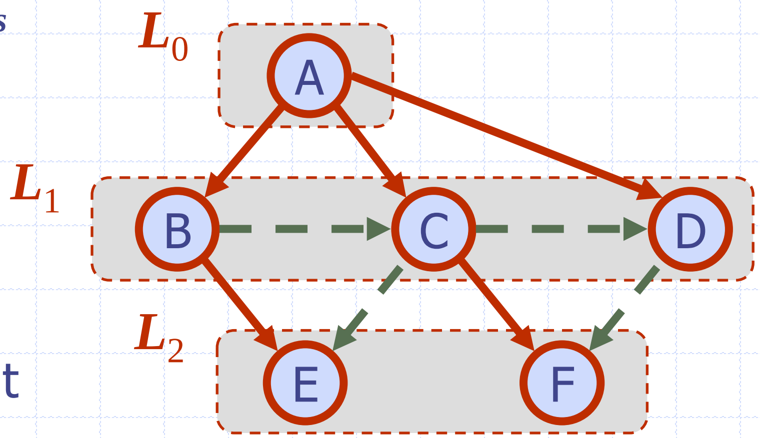
The discovery edges labeled by $BFS(G, s)$ form a spanning tree T_s of G_s

Property 3

For each vertex v in L_i

- The path of T_s from s to v has i edges
- Every path from s to v in G_s has at least i edges





Analysis

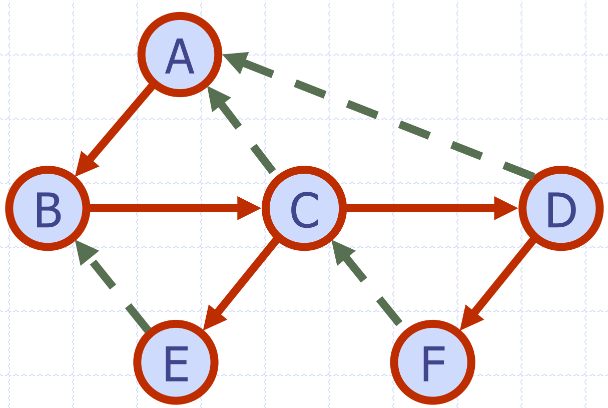
- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method incidentEdges is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

Applications

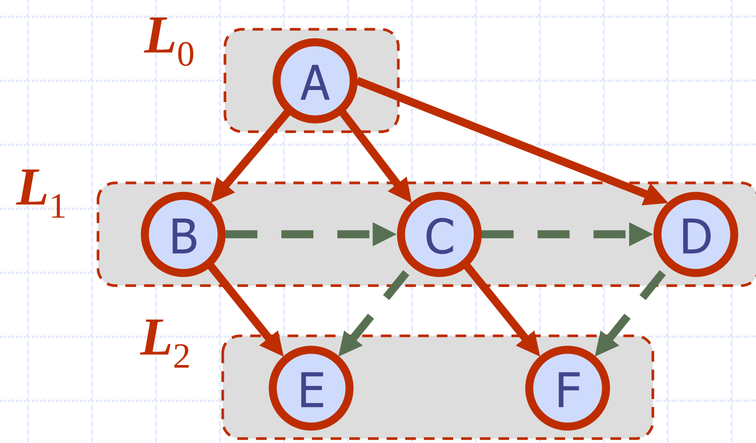
- We can use the BFS traversal algorithm, for a graph G, to solve the following problems in $O(n + m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

DFS vs. BFS

Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
Biconnected components	√	



DFS



BFS

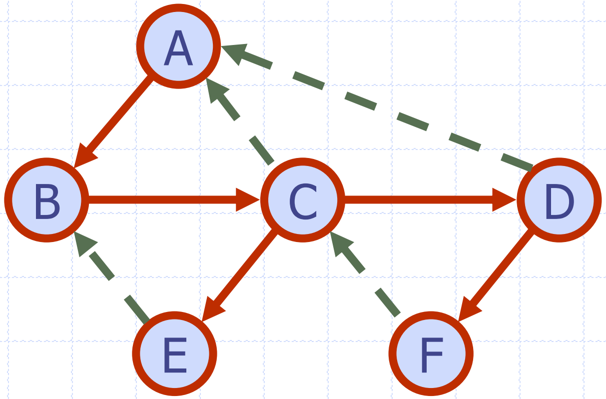
DFS vs. BFS (cont.)

Back edge (v,w)

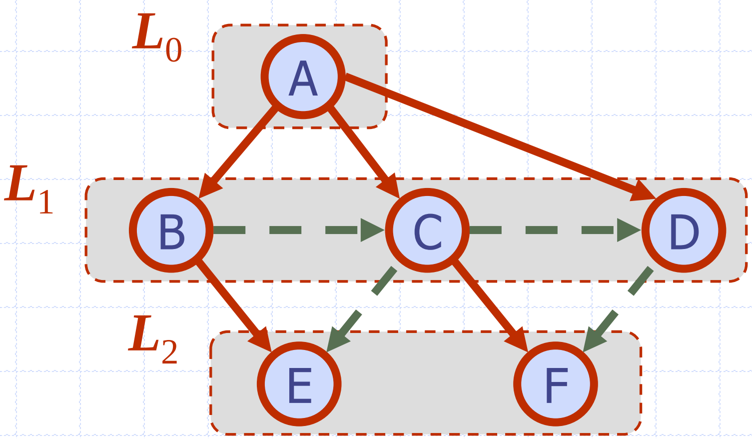
- w is an ancestor of v in the tree of discovery edges

Cross edge (v,w)

- w is in the same level as v or in the next level



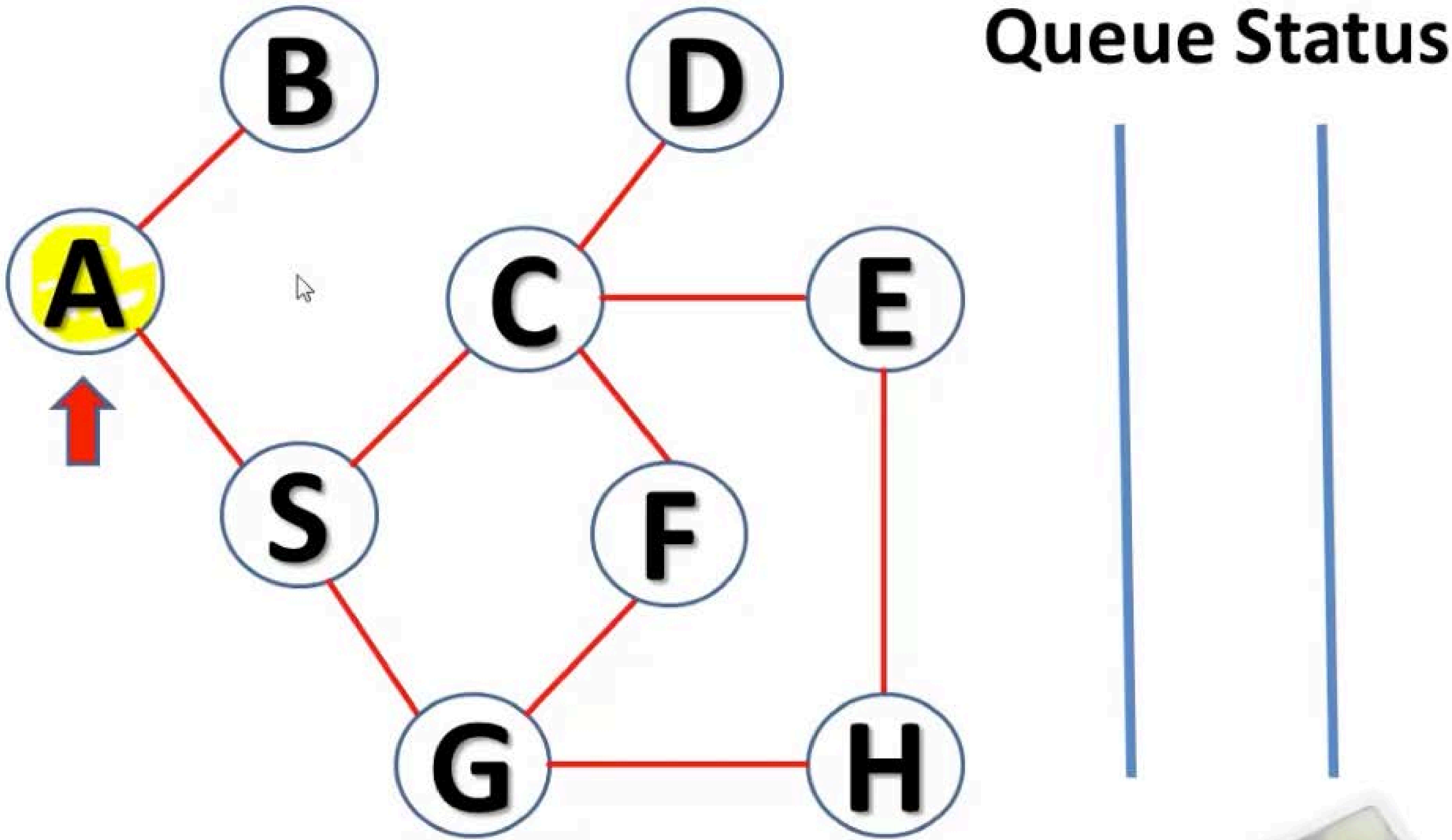
DFS



BFS

2. BFS WITH QUEUE

BREADTH FIRST SEARCH



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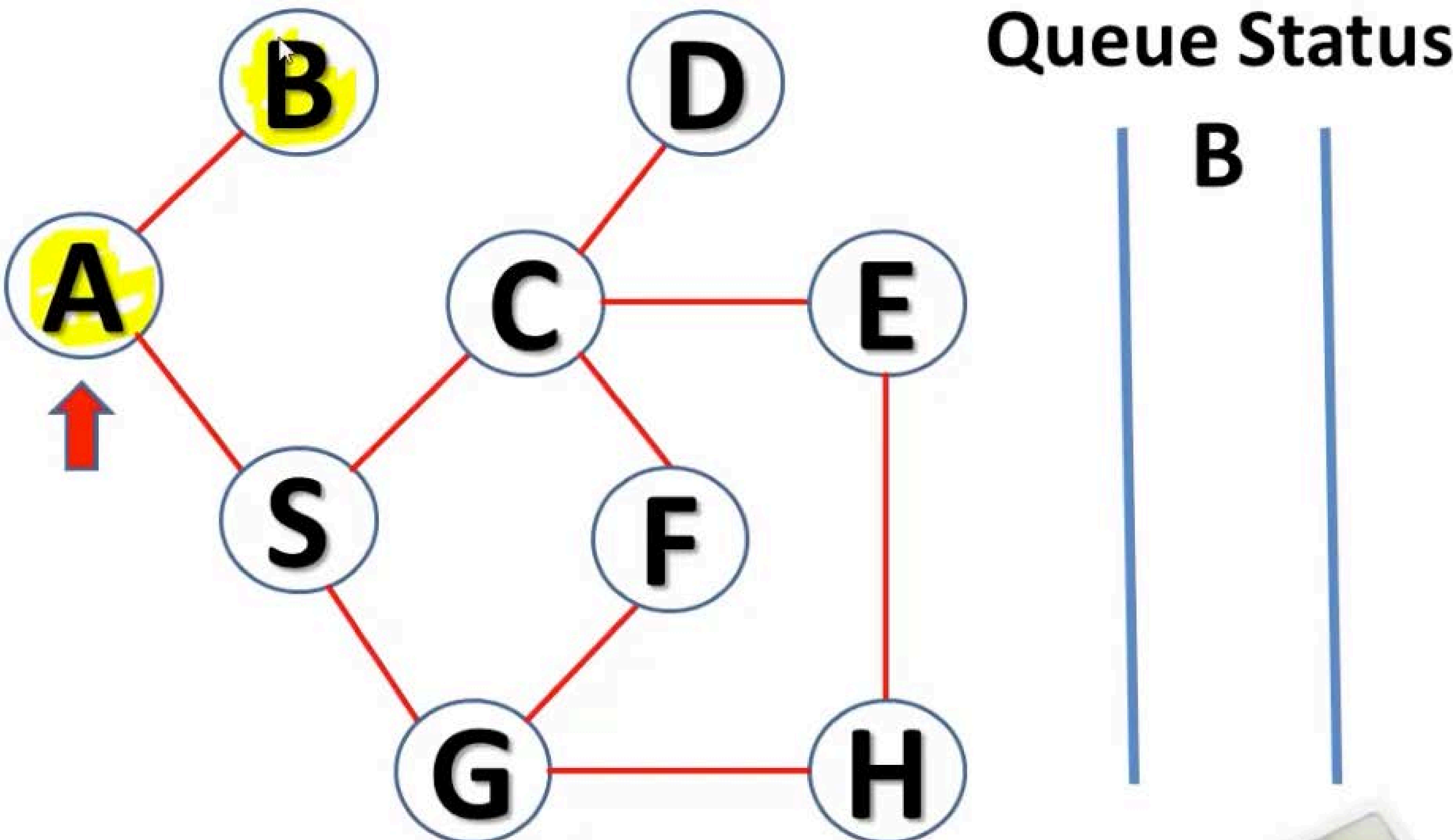

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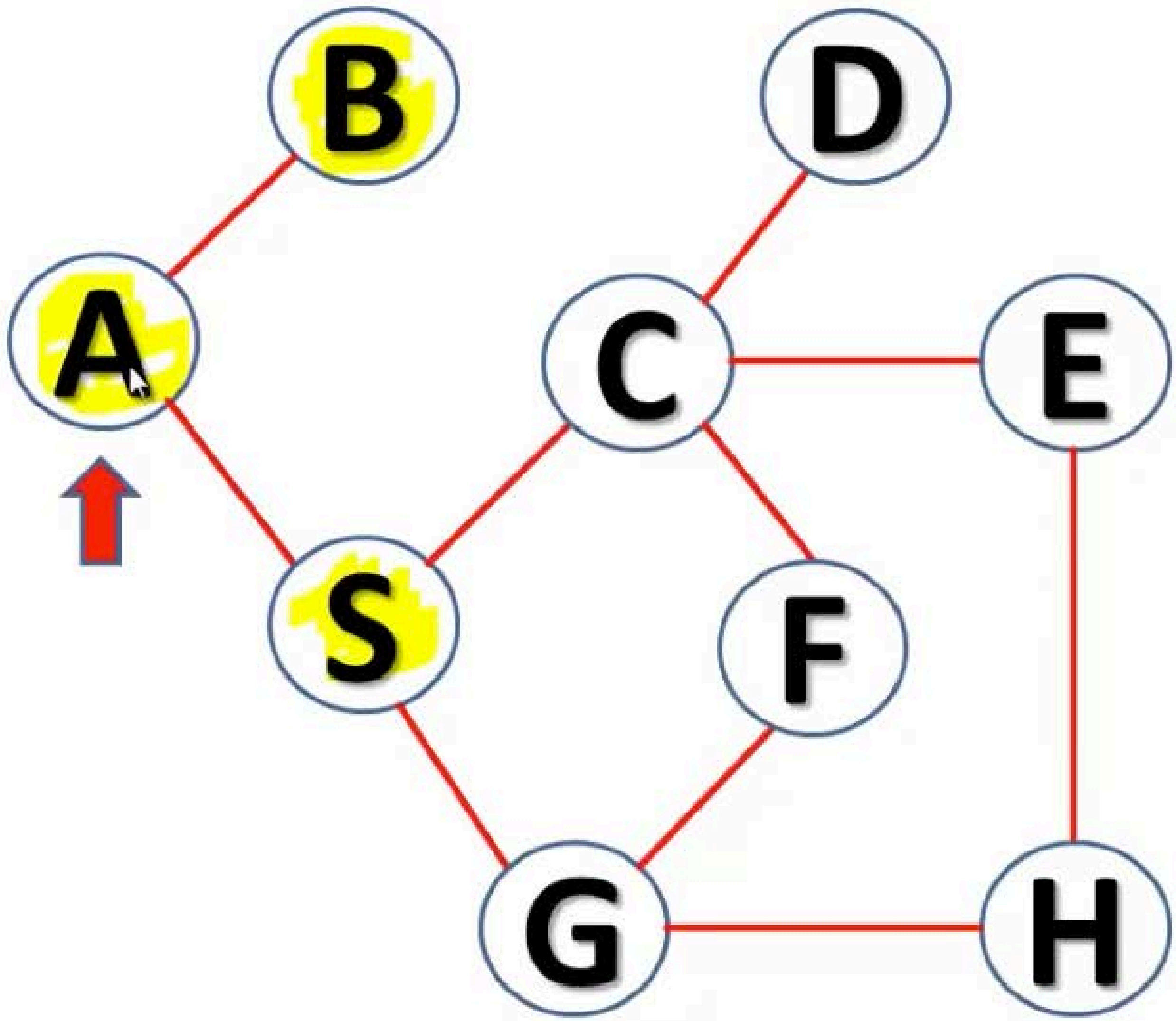

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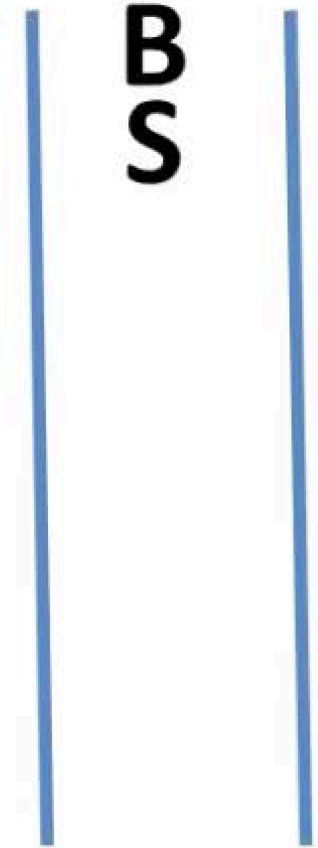

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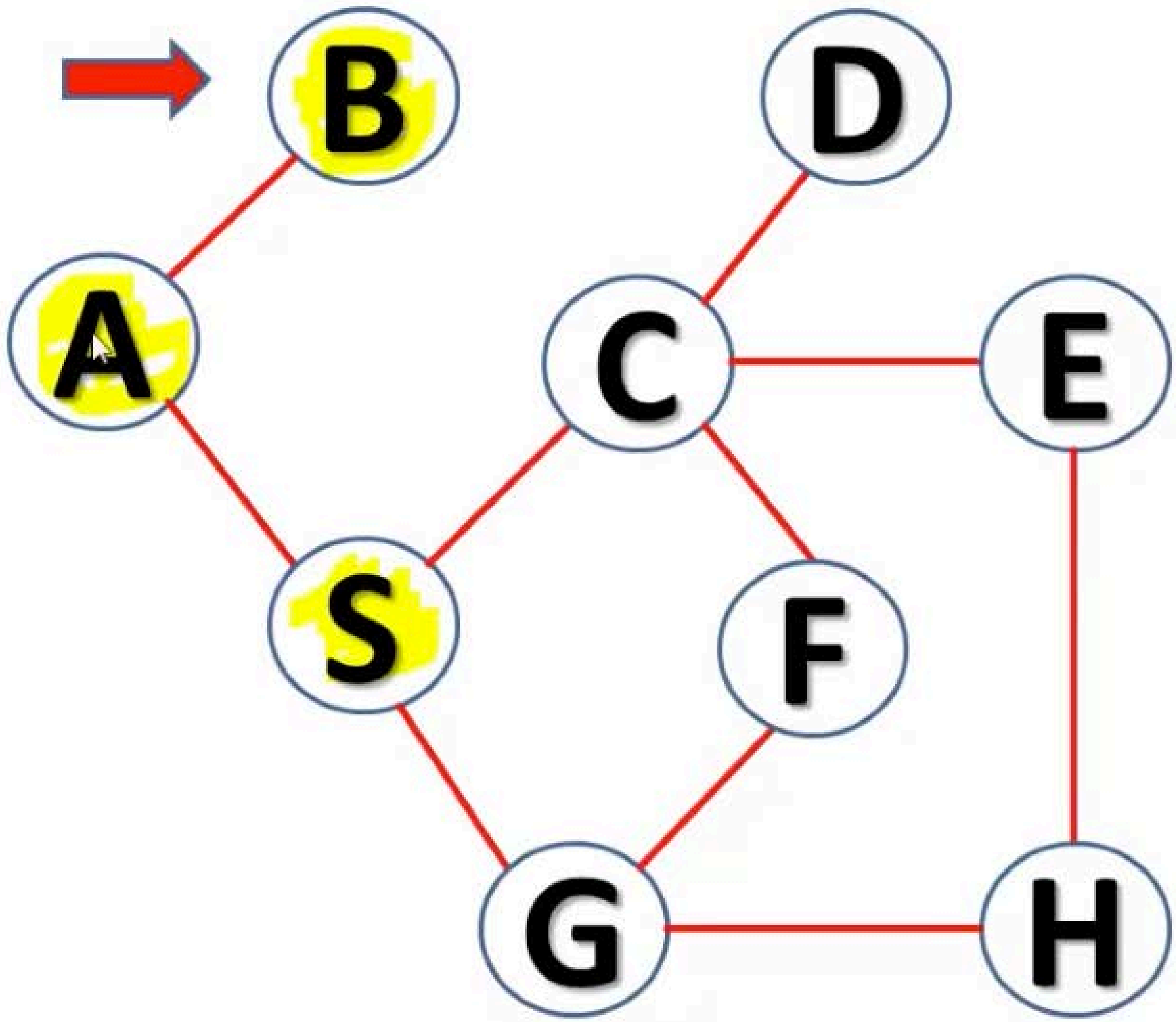

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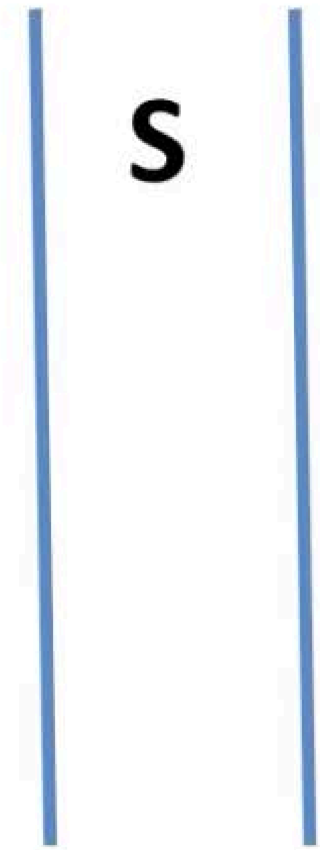

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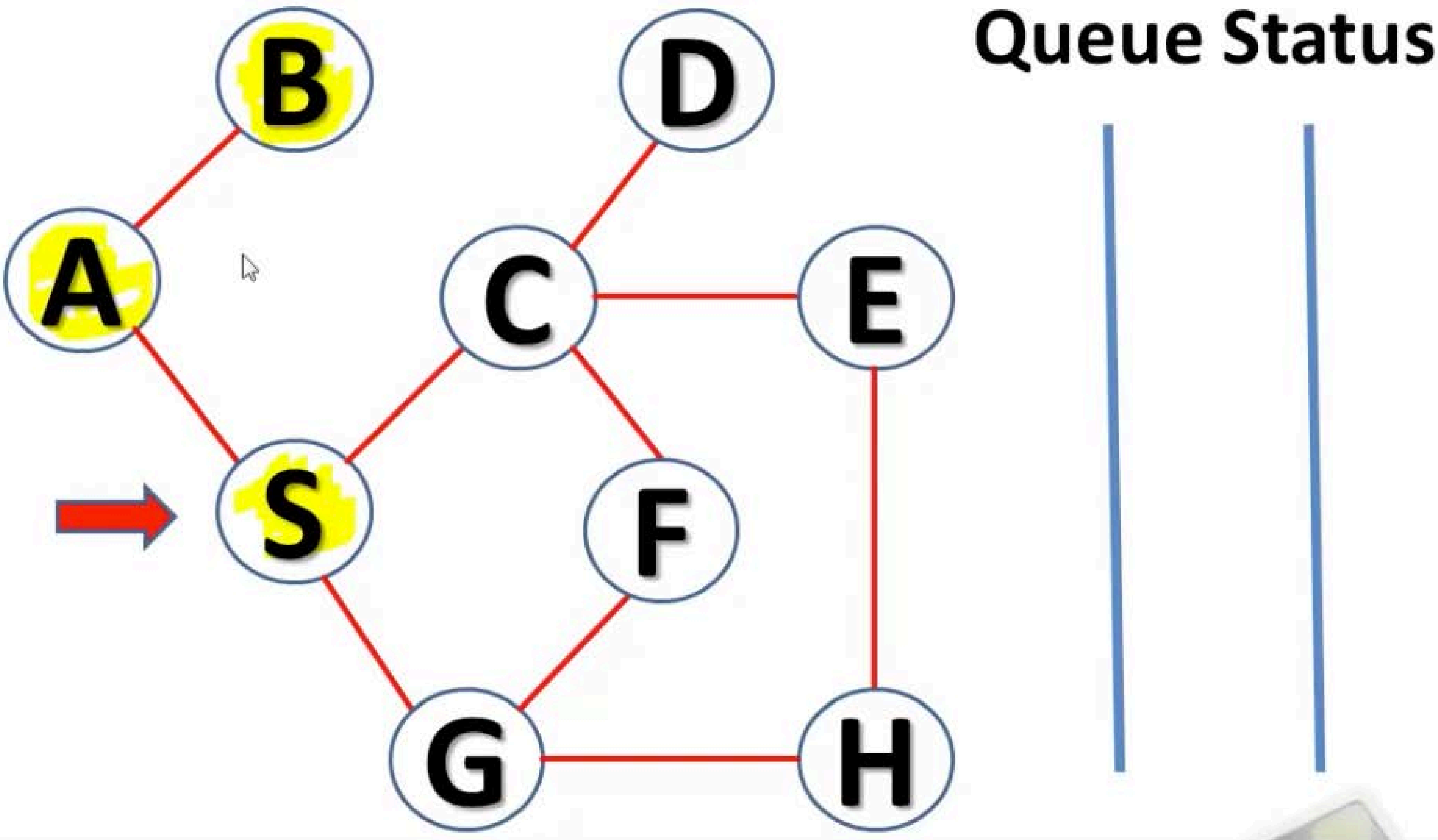

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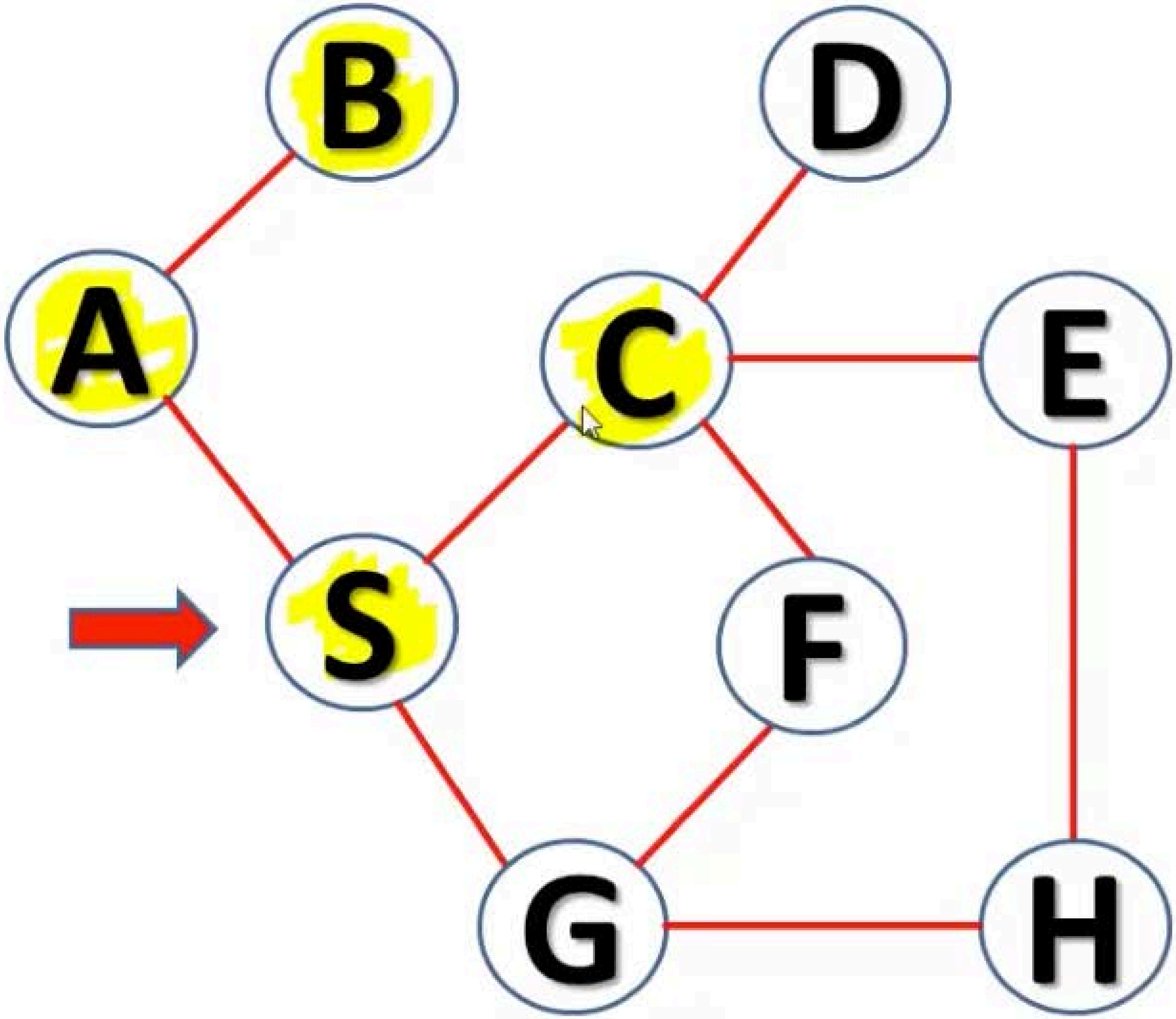

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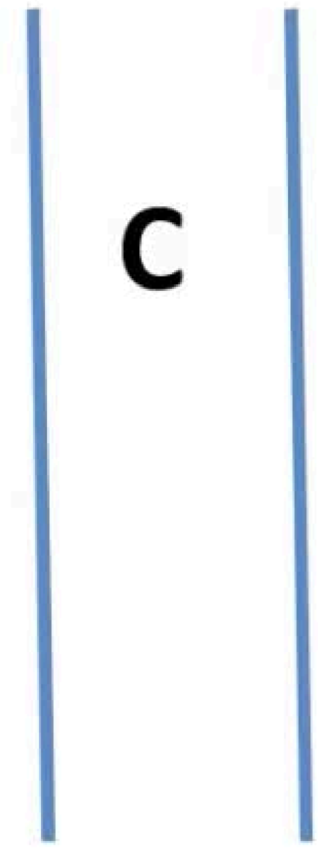

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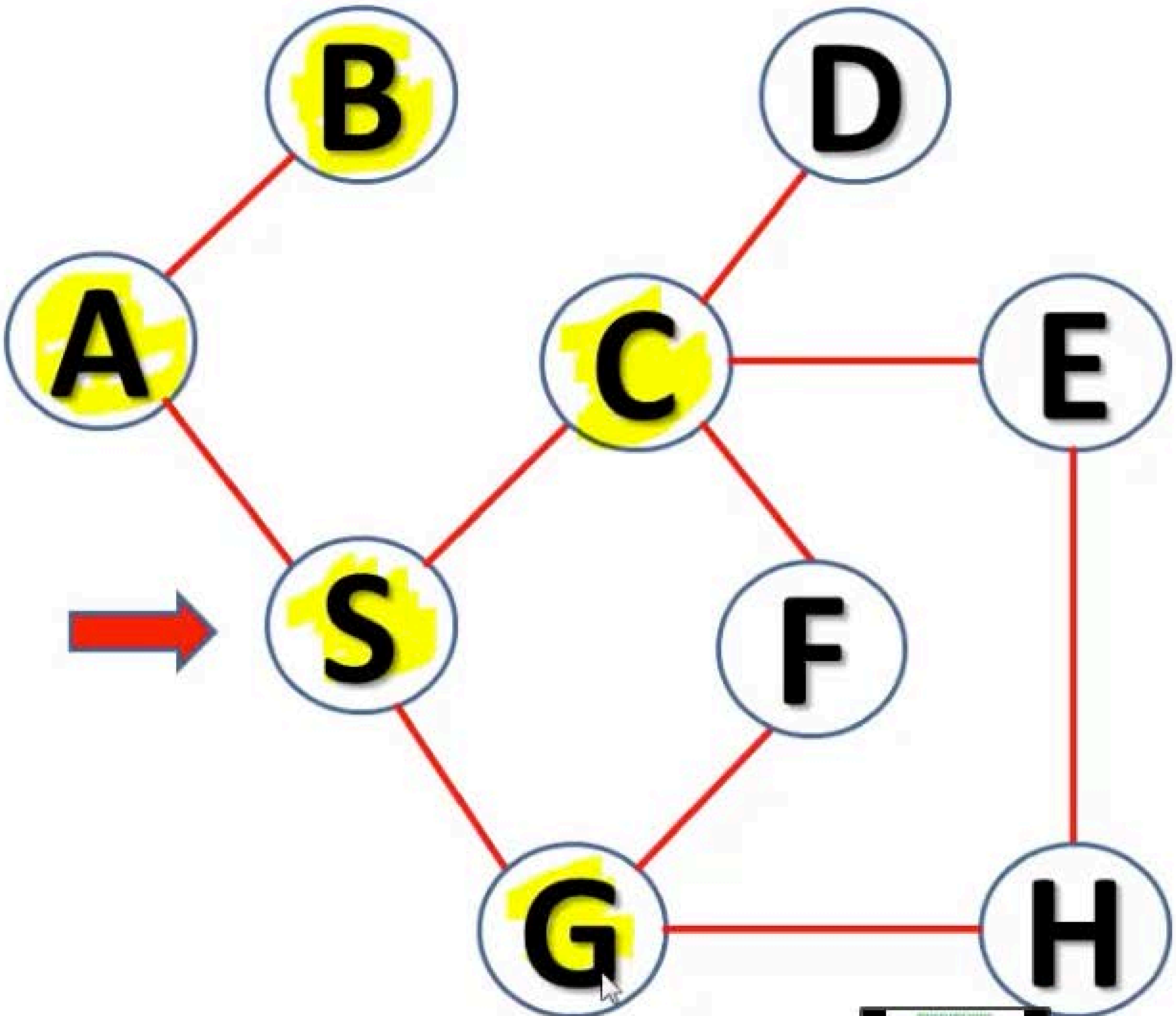

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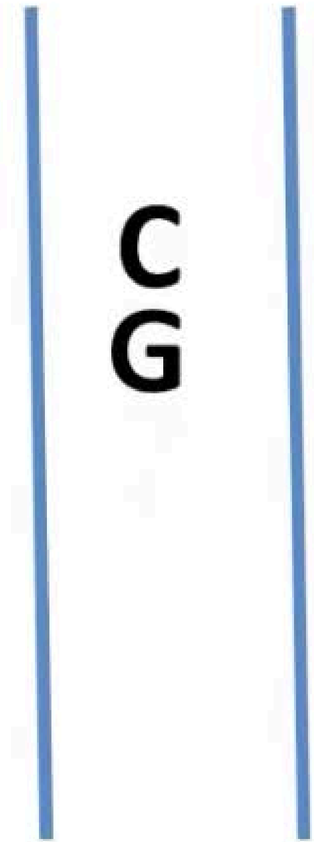

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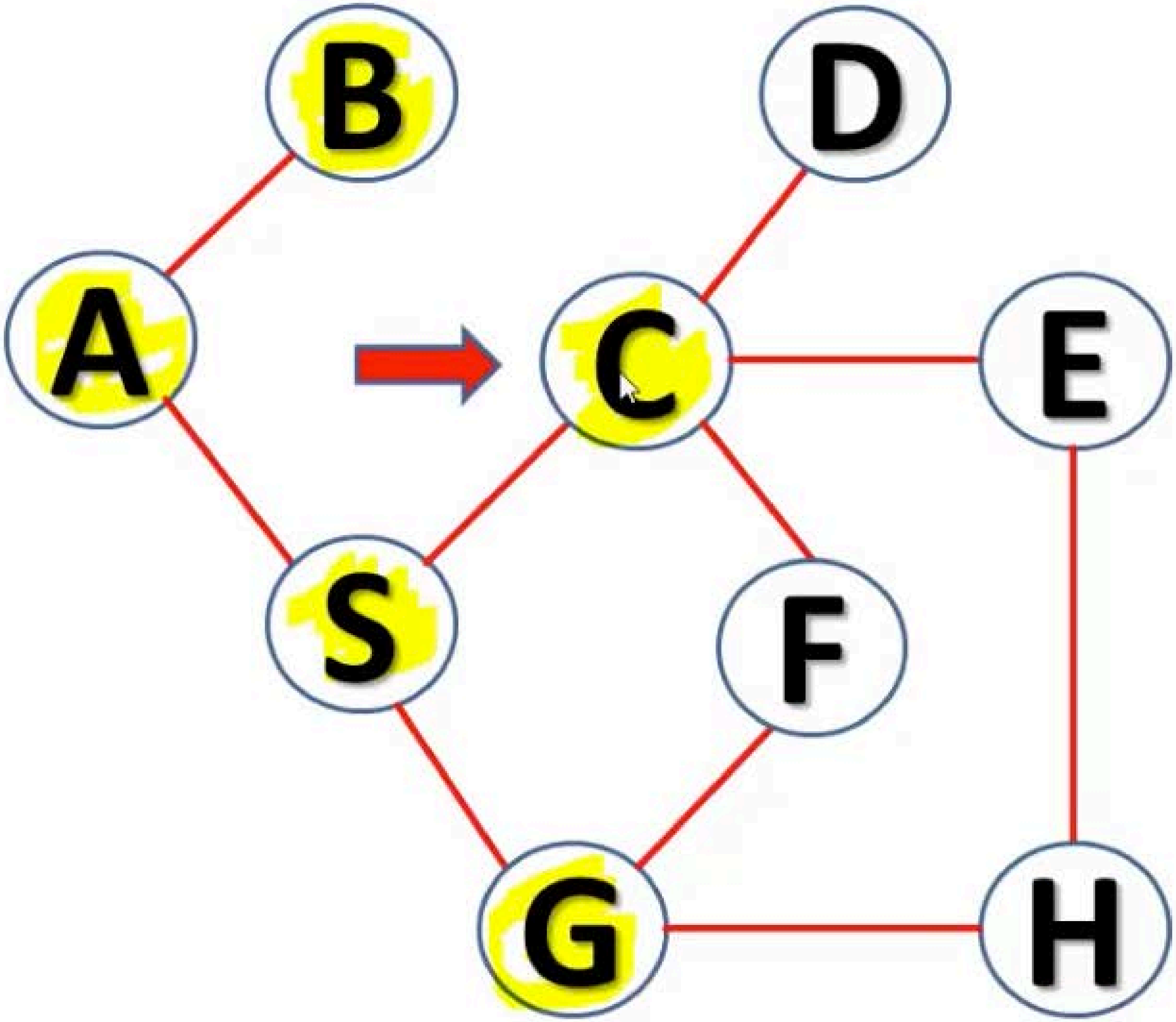

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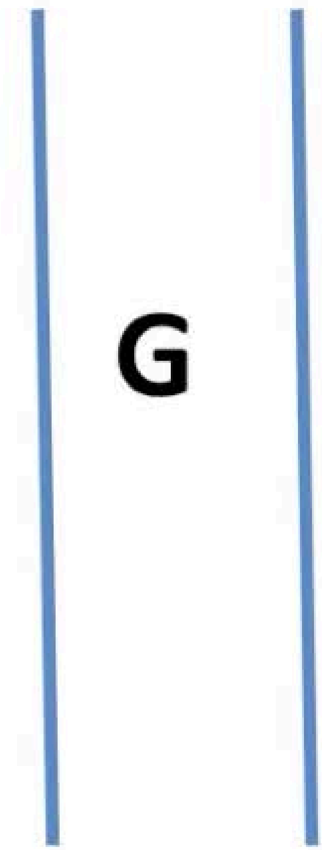

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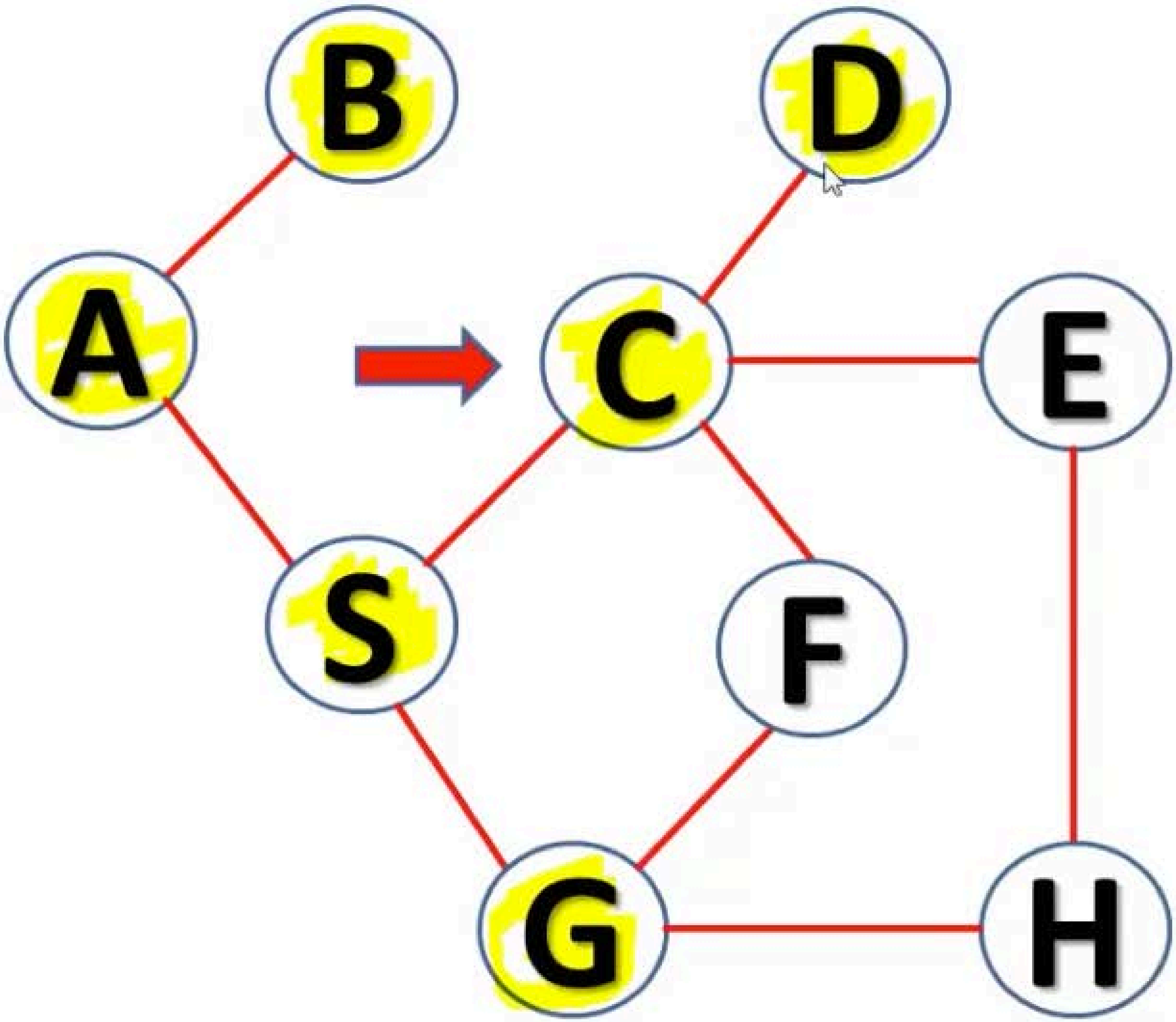

Queue Status

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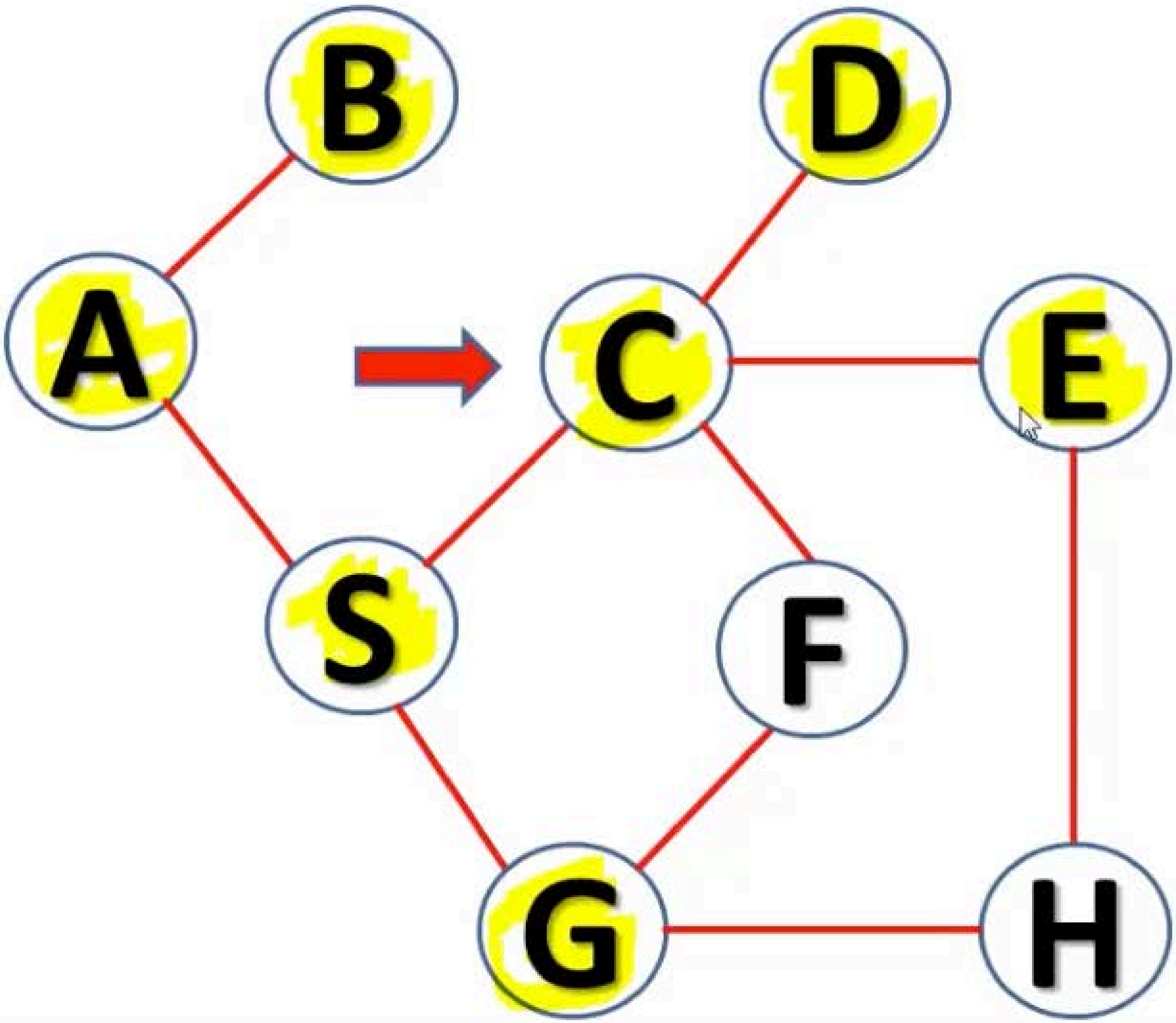
BREADTH FIRST SEARCH



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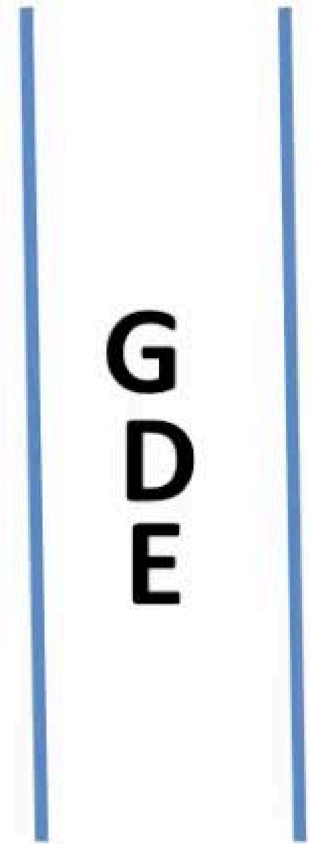

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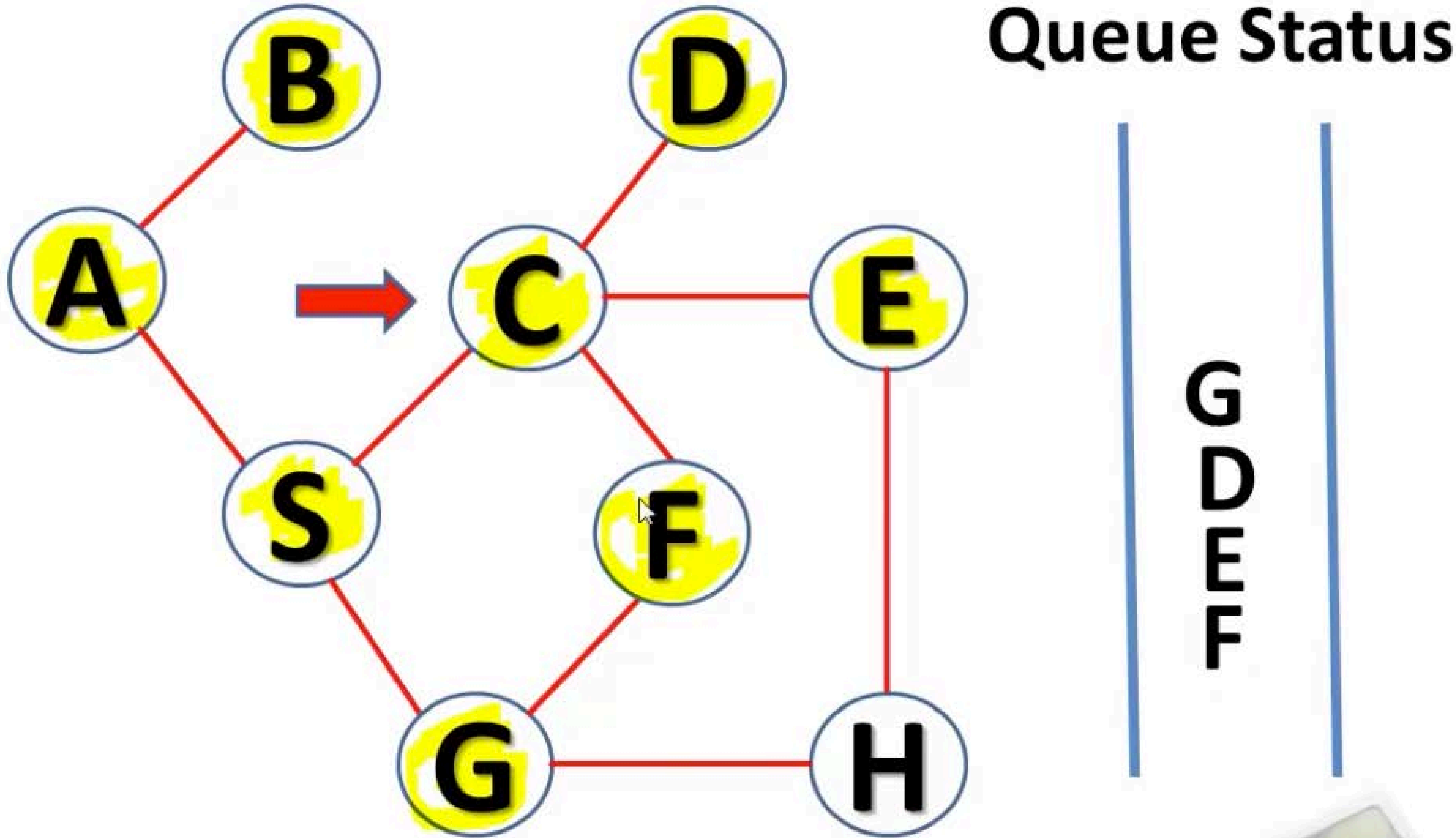

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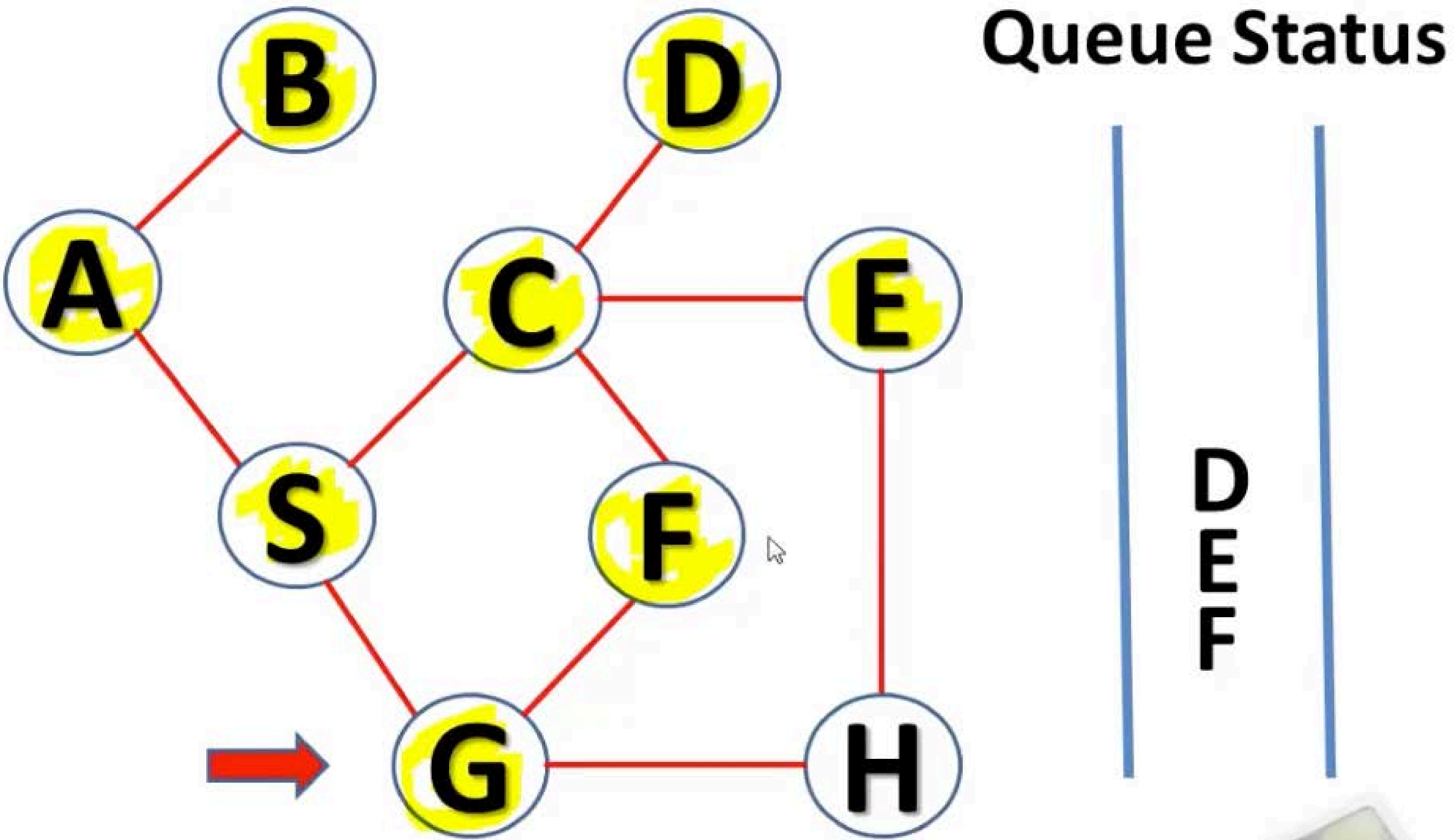

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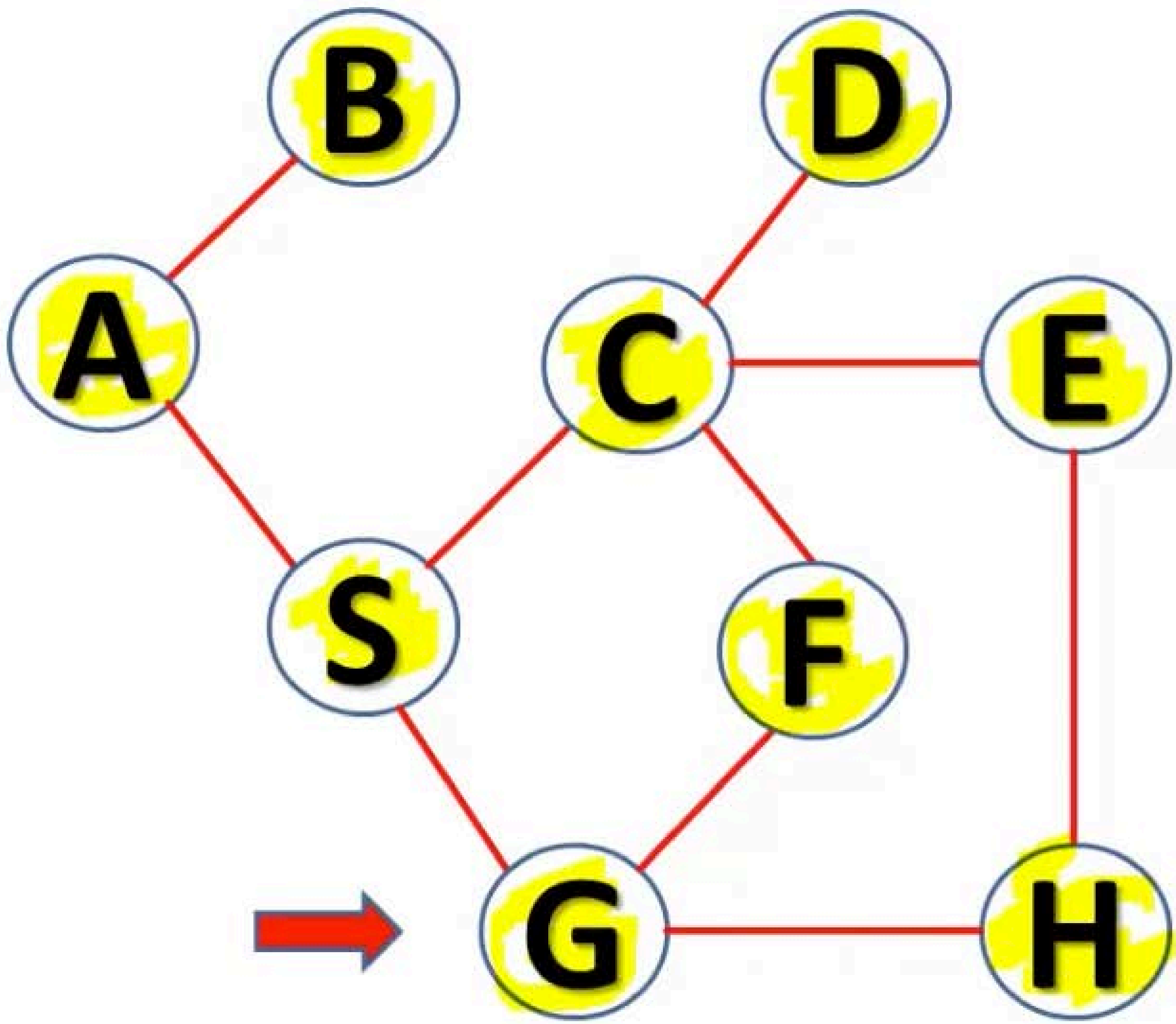

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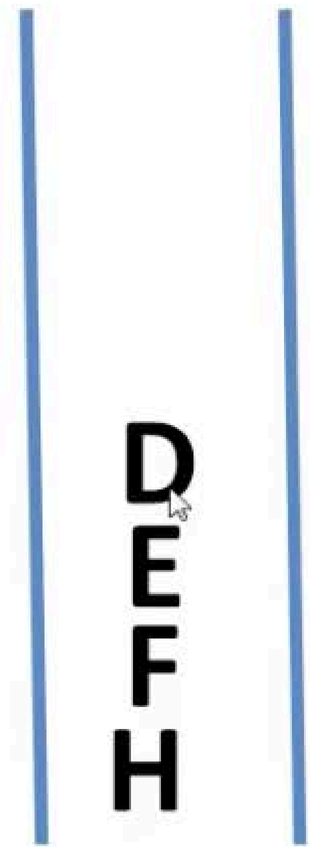

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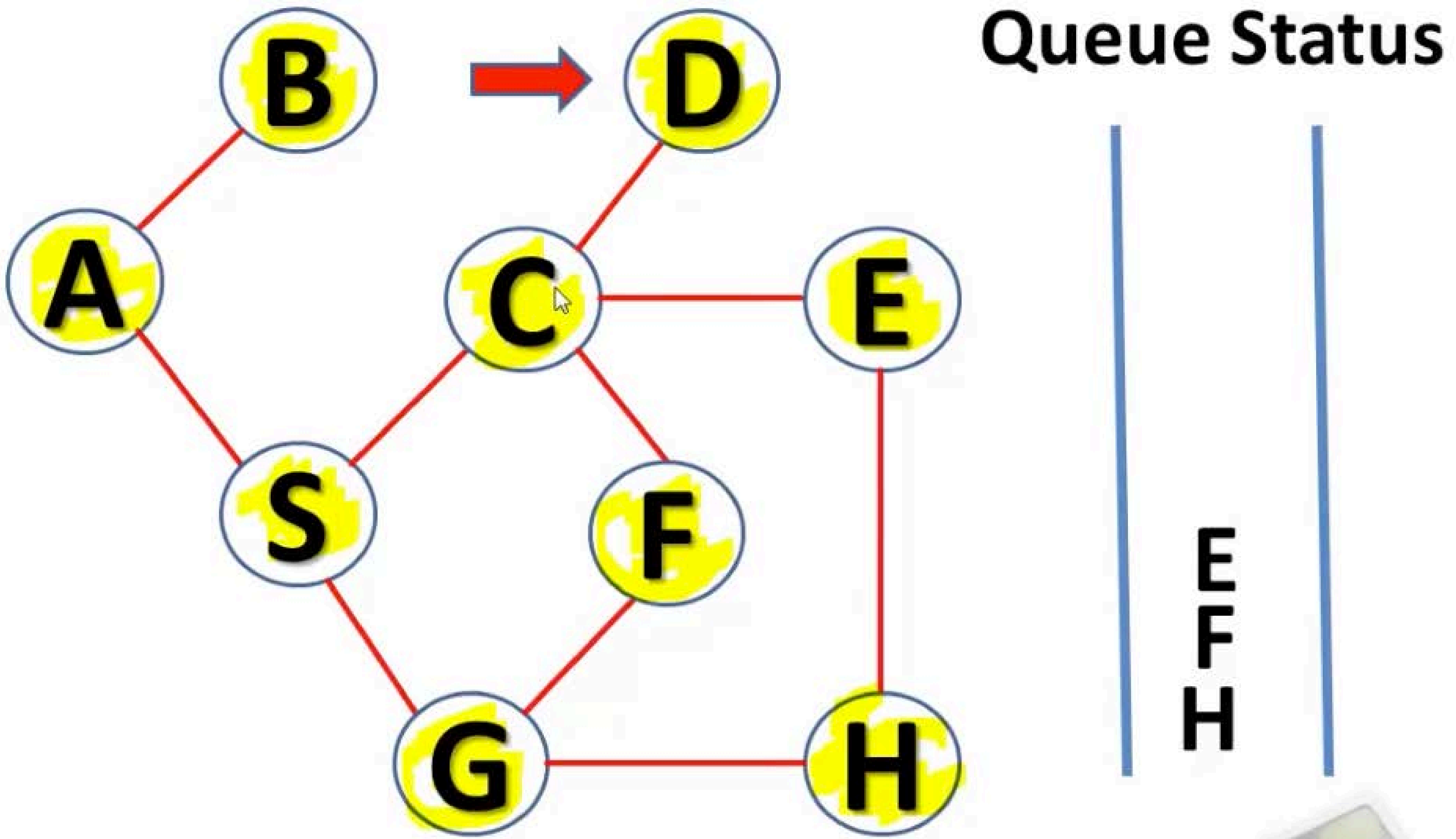

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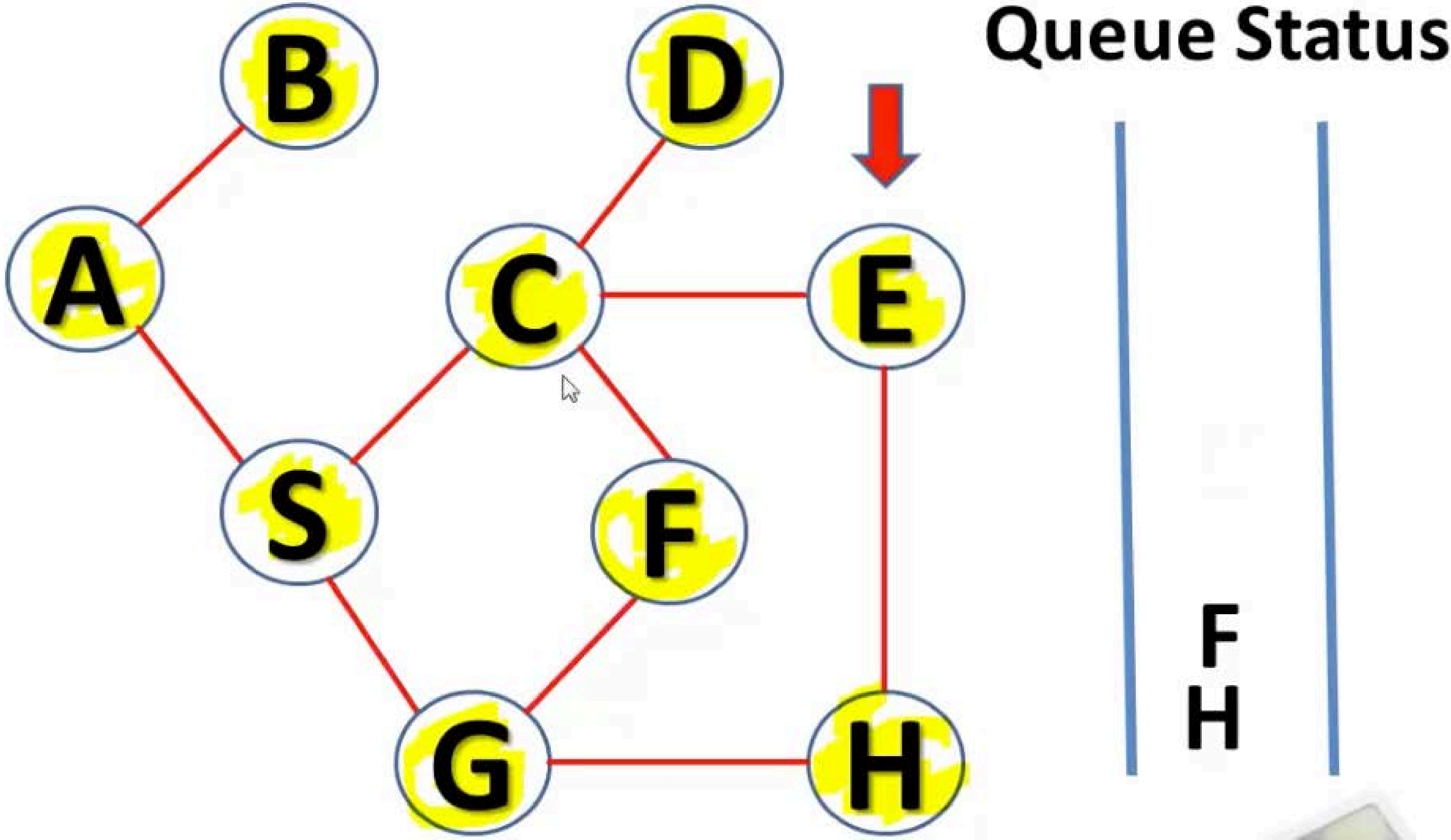

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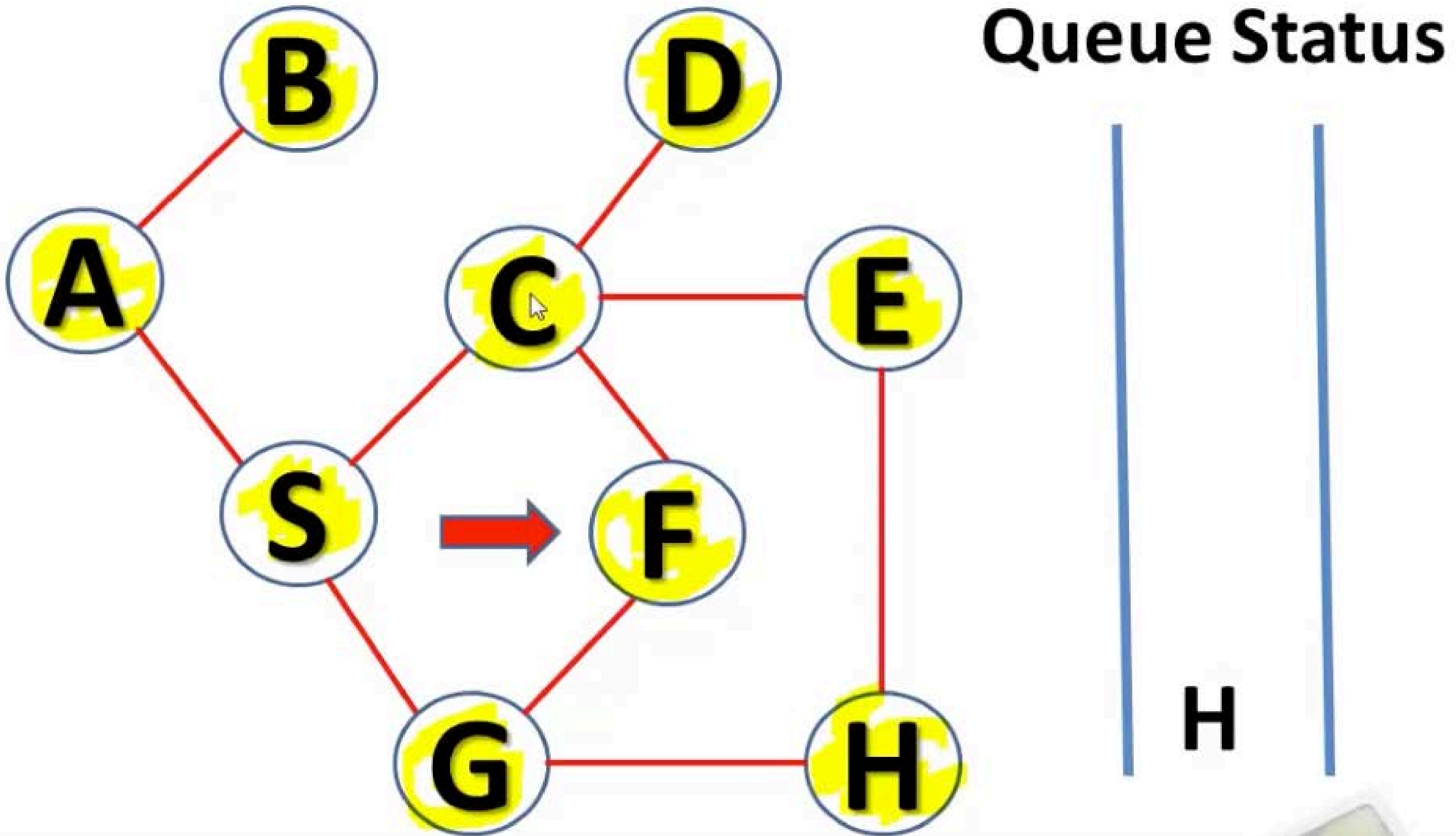

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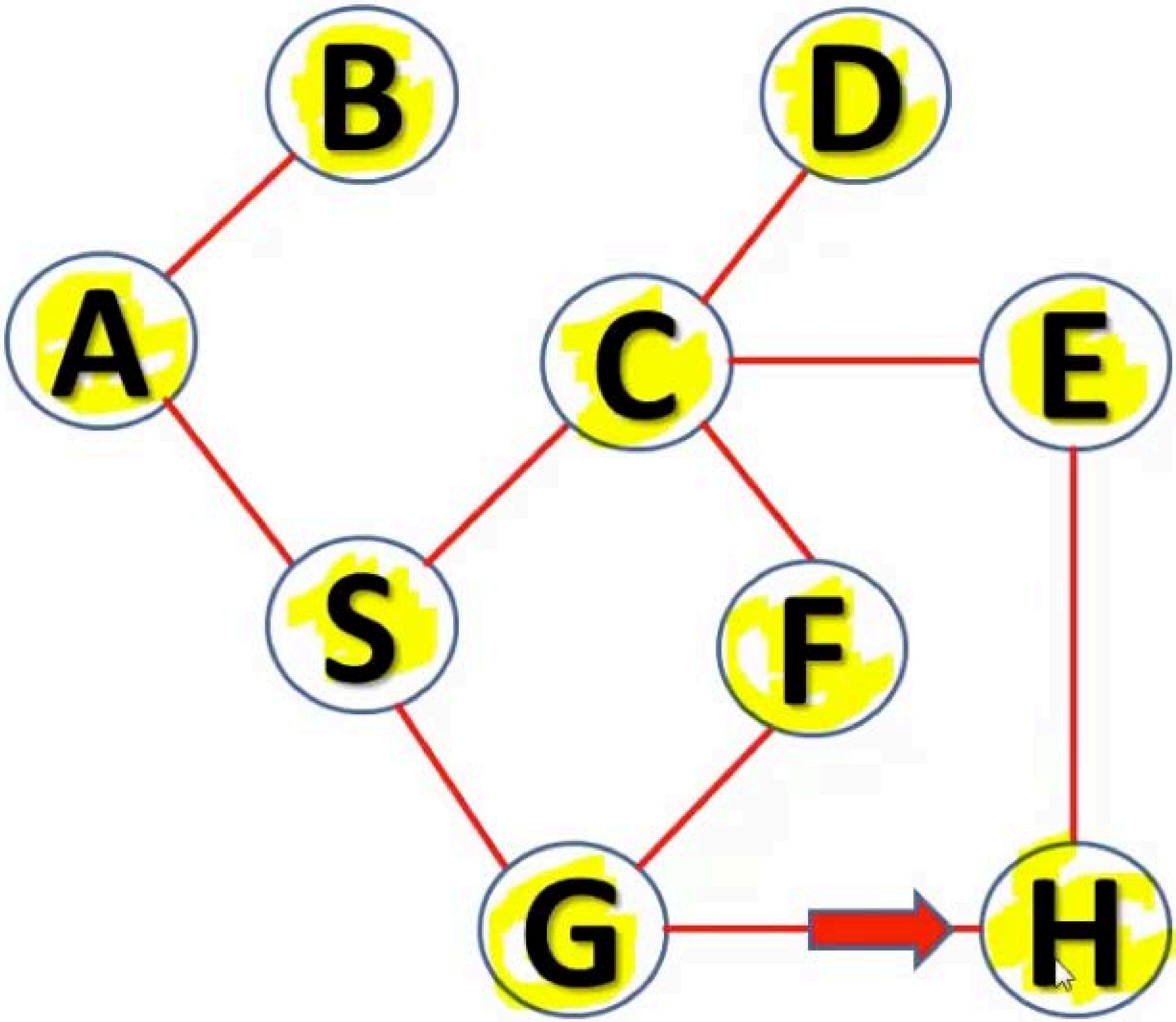

Queue Status