Lecture 5
Av. case analysis of quick sort, divide and conquer, mergesort

CS 161 Design and Analysis of Algorithms
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def quickSort(A, first, last):
    if first < last:
        splitpoint = split(A, first, last)
        quickSort(A, first, splitpoint - 1)
        quickSort(A, splitpoint + 1, last)
The split step

```python
def split(A, first, last):
    splitpoint = first
    x = A[first]
    for k = first+1 to last do:
        if A[k] < x:
            splitpoint = splitpoint + 1
    return splitpoint
```

Loop invariants:

- \( A[\text{first}+1..\text{splitpoint}] \) contains keys \( < x \).
- \( A[\text{splitpoint}+1..k-1] \) contains keys \( \geq x \).
- \( A[k..\text{last}] \) contains unprocessed keys.
The split step

At start:

$$\begin{array}{ccc}
\text{first} & k & \text{last} \\
\downarrow & \downarrow & \downarrow \\
x & ? &
\end{array}$$

In middle:

$$\begin{array}{c}
\text{first} \\
\downarrow \\
x
\end{array} \quad \begin{array}{c}
splitpoint \\
\downarrow \\
< x \quad \geq x
\end{array} \quad \begin{array}{c}
k \\
\downarrow \\
\text{last}
\end{array}$$

At end:

$$\begin{array}{c}
\text{first} \\
\downarrow \\
x \quad < x
\end{array} \quad \begin{array}{c}
splitpoint \\
\downarrow \\
\geq x
\end{array} \quad \begin{array}{c}
\text{last}
\end{array}$$
A bad case case for Quicksort: 1, 2, 3, . . . , n − 1, n

\( \binom{n}{2} \) comparisons required. So the worst-case running time for Quicksort is \( \Theta(n^2) \).
A bad case case for Quicksort: $1, 2, 3, \ldots, n - 1, n$

$(\binom{n}{2})$ comparisons required. So the worst-case running time for Quicksort is $\Theta(n^2)$. But what about the average case . . .?
Average-case analysis of Quicksort:

Our approach:
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1. Use the binary tree of sorted lists
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1. Use the **binary tree of sorted lists**
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3. Calculate the probability that two items get compared
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1. Use the binary tree of sorted lists
2. Number the items in sorted order
3. Calculate the probability that two items get compared
4. Use this to compute the expected number of comparisons performed by Quicksort.
Average-case analysis of Quicksort:

Sorted order: 15 18 22 23 27 36 79 83
Average-case analysis of Quicksort

Number the keys in sorted order: \( S_1 < S_2 < \cdots < S_n \).

Fact about comparisons: During the run of Quicksort, two keys \( S_i \) and \( S_j \) get compared if and only if the first key from the set of keys \( \{S_i, S_i+1, \ldots, S_j\} \) to be chosen as a pivot is either \( S_i \) or \( S_j \).

If some key \( S_k \) is chosen first with \( S_i < S_k < S_j \), then \( S_i \) goes in the left half, \( S_j \) goes in the right half, and \( S_i \) and \( S_j \) never get compared.

If \( S_i \) is chosen first, it is compared against all the other keys in the split step (including \( S_j \)).

Similar if \( S_j \) is chosen first.

Examples:
- 23 and 22 (both statements true)
- 36 and 83 (both statements false)
Average-case analysis of Quicksort

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Assume:

- All \( n \) keys are distinct
- All permutations are equally likely
- The keys in sorted order are \( S_1 < S_2 < \cdots < S_n \).

Let \( P_{i,j} \), the probability that keys \( S_i \) and \( S_j \) are compared with each other during the invocation of quicksort,

Then by Fact about comparisons on previous slide:

\[ P_{i,j} = \frac{2j-i+1}{n-i} \]
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Average-case analysis of Quicksort

Define indicator random variables \( \{X_{i,j} : 1 \leq i < j \leq n\} \)

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X_{i,j} = \begin{cases} 
1 & \text{if keys } S_i \text{ and } S_j \text{ get compared} \\
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\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}
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E \left( \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j} \right) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E(X_{i,j})
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3. The expected value of \( X_{i,j} \) is:

\[
E(X_{i,j}) = P_{i,j} = \frac{2}{j - i + 1}
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Average-case analysis of Quicksort

Hence the expected number of comparisons is

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So the average time for Quicksort is \(O(n \log n)\).
Average-case analysis of Quicksort

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So the average time for Quicksort is \( O(n \lg n) \).
Divide and Conquer

Divide and conquer paradigm

1. Split problem into subproblem(s)
2. Solve each subproblem (usually via recursive call)
3. Combine solution of subproblem(s) into solution of original problem

We will discuss two sorting algorithms based on this paradigm:
- Quicksort (done)
- Mergesort
MergeSort

```python
def mergeSort(A, first, last):
    if first < last:
        mid = \lfloor (first + last) / 2 \rfloor
        mergeSort(A, first, mid)
        mergeSort(A, mid + 1, last)
        merge(A, first, mid, mid + 1, last)
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MergeSort

- Split array into two equal subarrays

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def merge(A, first1, last1, first2, last2):
    index1 = first1; index2 = first2; tempIndex = 0
    // Merge into temp array until one input array is exhausted
    while (index1 <= last1) and (index2 <= last2):
        if A[index1] <= A[index2]:
            temp[tempIndex++] = A[index1++]
        else:
            temp[tempIndex++] = A[index2++]
    // Copy appropriate trailer portion
    while (index1 <= last1): temp[tempIndex++] = A[index1++]
    while (index2 <= last2): temp[tempIndex++] = A[index2++]
    // Copy temp array back to A array
    tempIndex = 0; index = first1
    while (index <= last2): A[index++] = temp[tempIndex++]

Code for the merge step
Analysis of Mergesort

\( T(n) \) = number of comparisons required to sort \( n \) items in the worst case

\[
T(n) = \begin{cases} 
T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n - 1, & n > 1 \\
0, & n = 1 
\end{cases}
\]

The asymptotic solution of this recurrence equation is

\[ T(n) = \Theta(n \log n) \]

The exact solution of this recurrence equation is

\[ T(n) = n \lceil \lg n \rceil - 2 \lfloor \lg n \rfloor + 1 \]
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Geometrical Application: Counting line intersections

Input: \( n \) lines in the plane, none of which are vertical; two vertical lines \( x = a \) and \( x = b \) (with \( a < b \)).

Problem: Count/report all pairs of lines that intersect between the two vertical lines \( x = a \) and \( x = b \).

Example: \( n = 6 \) 8 intersections.

Checking every pair of lines takes \( \Theta(n^2) \) time. We can do better.
Geometrical Application: Counting line intersections

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![Diagram of line intersections between two vertical lines](image)

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**Example:** $n = 6$
Geometrical Application: Counting line intersections

► **Input:** $n$ lines in the plane, none of which are vertical; two vertical lines $x = a$ and $x = b$ (with $a < b$).

► **Problem:** Count/report all pairs of lines that intersect between the two vertical lines $x = a$ and $x = b$.

**Example:** $n = 6$ 8 intersections
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**Example:** $n = 6$ 8 intersections

Checking every pair of lines takes $\Theta(n^2)$ time. We can do better.
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1. Sort the lines according to the $y$-coordinate of their intersection with the line $x = a$. Number the lines in sorted order. $O(n \log n)$ time

2. Produce the sequence of line numbers sorted according to the $y$-coordinate of their intersection with the line $x = b$. $O(n \log n)$ time


So the problem reduces to counting/reporting inversions.
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![Diagram of line intersections with labeled numbers](image)
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Counting Inversions: An Application of Mergesort

An inversion in a sequence or list is a pair of items such that the larger one precedes the smaller one.

Example: The list \([18, 29, 12, 15, 32, 10]\) has 9 inversions:

- \((18, 12)\)
- \((18, 15)\)
- \((18, 10)\)
- \((29, 12)\)
- \((29, 15)\)
- \((29, 10)\)
- \((12, 10)\)
- \((15, 10)\)
- \((32, 10)\)

In a list of size \(n\), there can be as many as \(\frac{n^2}{2}\) inversions.

Problem: Given a list, compute the number of inversions.

Brute force solution: Check each pair \(i, j\) with \(i < j\) to see if \(L[i] > L[j]\).

This gives a \(\Theta(n^2)\) algorithm.

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Inversion Counting

Sorting is the process of removing inversions. So to count inversions:

- Run a sorting algorithm
- Every time data is rearranged, keep track of how many inversions are being removed.

In principle, we can use any sorting algorithm to count inversions. Mergesort works particularly nicely.
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Inversion Counting with MergeSort

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$$\text{last}_1 - \text{index}_1 + 1$$
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Example

2 inversions removed: (42, 31) and (71, 31)
Pseudocode for the merge step with inversion counting

```python
def merge(A, first1, last1, first2, last2):
    index1 = first1; index2 = first2; tempIndex = 0
    invCount = 0
    # Merge into temp array until one input array is exhausted
    while (index1 <= last1) and (index2 <= last2):
        if A[index1] <= A[index2]:
            temp[tempIndex++] = A[index1++]
        else:
            temp[tempIndex++] = A[index2++]
            invCount += last1 - index1 + 1;
    # Copy appropriate trailer portion
    while (index1 <= last1): temp[tempIndex++] = A[index1++]
    while (index2 <= last2): temp[tempIndex++] = A[index2++]
    # Copy temp array back to A array
    tempIndex = 0; index = first1
    while (index <= last2): A[index++] = temp[tempIndex++]
    return invCount
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Pseudocode for MergeSort with inversion counting

```python
def mergeSort(A, first, last):
    invCount = 0
    if first < last:
        mid = \lfloor (first + last)/2 \rfloor
        invCount += mergeSort(A, first, mid)
        invCount += mergeSort(A, mid+1, last)
        invCount += merge(A, first, mid, mid+1, last)
    return invCount
```

Running time is the same as standard mergeSort: $O(n \log n)$
Pseudocode for MergeSort with inversion counting

```python
def mergeSort(A, first, last):
    invCount = 0
    if first < last:
        mid = int((first + last)/2)
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        invCount += mergeSort(A, mid + 1, last)
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Running time is the same as standard mergeSort: $O(n \log n)$
Listing inversions

We have just seen that we can count inversions without increasing the asymptotic running time of Mergesort. Suppose we want to list inversions.

When we remove inversions, we list all inversions removed:

\[
\text{first1, index1, last1, first2, index2, last2, tempindex, temp, (A[index1], A[index2]), (A[index1+1], A[index2]), \ldots, (A[last1], A[index2])}.
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The extra work to do the reporting is proportional to the number of inversions reported.
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Inversion counting summary

Using a slight modification of Mergesort, we can...

- Count inversions in $O(n \log n)$ time.
- Report inversions in $O(n \log n + k)$ time, where $k$ is the number of inversions.

The same results hold for the line-intersection counting problem. The reporting algorithm is an example of an output-sensitive algorithm. The performance of the algorithm depends on the size of the output as well as the size of the input.
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