Lecture 4
Binary search (cont.), insertion/selection sort, analysis of quick sort

CS 161 Design and Analysis of Algorithms
Ioannis Panageas
Binary Search: Searching in a sorted array

- Input is a sorted array $A$ and an item $x$.
- Problem is to locate $x$ in the array.

- We will show that binary search is an optimal algorithm for solving this problem.
Binary Search: Searching in a sorted array

**Input:**

- $A$: Sorted array with $n$ entries $[0..n-1]$
- $x$: Item we are seeking

```python
def binarySearch(A, x, first, last):
    if first > last:
        return (-1)
    else:
        mid = \lfloor (first + last) / 2 \rfloor
        if x == A[mid]:
            return mid
        else if x < A[mid]:
            return binarySearch(A, x, first, mid - 1)
        else:
            return binarySearch(A, x, mid + 1, last)

binarySearch(A, x, 0, n-1)
```
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- -1, if $x$ not found

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Binary Search: Analysis of Running Time (continued)

- Binary search in an array of size 1: 1 decision
- Binary search in an array of size $n > 1$: after 1 decision, either we are done, or the problem is reduced to binary search in a subarray with a worst-case size of $\lfloor n/2 \rfloor$
- So the worst-case time to do binary search on an array of size $n$ is $T(n)$, where $T(n)$ satisfies the equation

$$T(n) = \begin{cases} 
1 & \text{if } n = 1 \\
1 + T(\lfloor n/2 \rfloor) & \text{otherwise}
\end{cases}$$

- The solution to this equation is:

$$T(n) = \lceil \lg n \rceil + 1$$

This can be proved by induction.
- So binary search does $\lceil \lg n \rceil + 1$ 3-way comparisons on an array of size $n$, in the worst case.
We will establish a lower bound on the worst-case number of decisions required to find an item in an array, using only 3-way comparisons of the item against array entries. The lower bound we will establish is $\lfloor \log_2 n \rfloor + 1$ 3-way comparisons. Since Binary Search performs within this bound, it is optimal. Our lower bound is established using a Decision Tree model. Note that the bound is exact (not just asymptotic). Our lower bound is on the worst case. It says: for every algorithm for finding an item in an array of size $n$, there is some input that forces it to perform $\lfloor \log_2 n \rfloor + 1$ comparisons. It does not say: for every algorithm for finding an item in an array of size $n$, every input forces it to perform $\lfloor \log_2 n \rfloor + 1$ comparisons.
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Consider any algorithm that searches for an item $x$ in an array $A$ of size $n$ by comparing entries in $A$ against $x$. Any such algorithm can be modeled as a decision tree:

Example: Decision tree for binary search with $n = 13$: 

```
       6
      / \  /
     2   9
    / \ / \ /
   0  4 7 11
  / \ / \ / \  
1  3  5  8 10 12
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![Decision Tree Diagram]
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Example: Decision tree for binary search with $n = 13$:
Lower bound on locating an item in an array of size $n$

Any algorithm for searching an array of size $n$ can be modeled by a decision tree with at least $n$ nodes.

Since the decision tree is a binary tree with $n$ nodes, the depth is at least $\lfloor \lg n \rfloor$.

The worst-case number of comparisons for the algorithm is the depth of the decision tree + 1. (Remember, root has depth 0).

Hence any algorithm for locating an item in an array of size $n$ using only $\text{comparisons}$ must perform at least $\lfloor \lg n \rfloor + 1$ comparisons in the worst case.

So binary search is optimal with respect to worst-case performance.
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```latex
\begin{center}
\begin{tikzpicture}
\node (root) [circle, draw] {};
\node (node1) [circle, draw, below of=root] {};
\node (node2) [circle, draw, below of=root] {};
\node (node3) [circle, draw, below of=node1] {};
\node (node4) [circle, draw, below of=node1] {};
\node (node5) [circle, draw, below of=node2] {};
\node (node6) [circle, draw, below of=node2] {};
\end{tikzpicture}
\end{center}
```
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Hence any algorithm for locating an item in an array of size $n$ using only comparisons must perform at least $\lfloor \lg n \rfloor + 1$ comparisons in the worst case.
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Hence any algorithm for locating an item in an array of size \( n \) using only comparisons must perform at least \( \lceil \log n \rceil + 1 \) comparisons in the worst case. So binary search is optimal with respect to worst-case performance.
Sorting

- Rearranging a list of items in nondescending order.
- Useful preprocessing step (e.g., for binary search).
- Important step in other algorithms.
- Illustrates more general algorithmic techniques.

We will discuss:
- Comparison-based sorting algorithms (Insertion sort, Selection Sort, Quicksort, Mergesort, Heapsort).
- Bucket-based sorting methods.
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Bucket-based sorting methods
Comparison-based sorting

- Basic operation: compare two items.
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- Abstract model.

Advantage: doesn’t use specific properties of the data items. So same algorithm can be used for sorting integers, strings, etc.

Disadvantage: under certain circumstances, specific properties of the data item can speed up the sorting process.

Measure of time: number of comparisons

Consistent with philosophy of counting basic operations, discussed earlier.

Misleading if other operations dominate (e.g., if we sort by moving items around without comparing them)

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$\Theta(n \log n)$ work vs. quadratic ($\Theta(n^2)$) work

$y = \frac{n}{2}$

$y = 10n \log n$
Some terminology

A permutation of a sequence of items is a reordering of the sequence. A sequence of $n$ items has $n!$ distinct permutations.

Note: Sorting is the problem of finding a particular distinguished permutation of a list.

An inversion in a sequence or list is a pair of items such that the larger one precedes the smaller one.

Example: The list $18 \ 29 \ 12 \ 15 \ 32 \ 10$ has 9 inversions:

$$\{(18, 12), (18, 15), (18, 10), (29, 12), (29, 15), (29, 10), (12, 10), (15, 10), (32, 10)\}$$
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Insertion sort

Work from left to right across array

Insert each item in correct position with respect to (sorted) elements to its left

- (Sorted)
- \( x \)
- (Unsorted)
- \( k \)
- (Unsorted)
- (Sorted)
- \( n - 1 \)
Insertion sort

- Work from left to right across array
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Symbols:
- $0$: (Unsorted)
- $k$: (Sorted) $x$ (Unsorted)
- $n - 1$: (Sorted)
Insertion sort

- Work from left to right across array

0

(Sorted)  
(Unsorted)

k

(Sorted)  x  (Unsorted)

n - 1

(Sorted)
Insertion sort

- Work from left to right across array
- Insert each item in correct position with respect to (sorted) elements to its left
def insertionSort(n, A):
    for k = 1 to n-1:
        x = A[k]
        j = k-1
        while (j >= 0) and (A[j] > x):
            j = j-1
        A[j+1] = x
Insertion sort example

<table>
<thead>
<tr>
<th>23</th>
<th>19</th>
<th>42</th>
<th>17</th>
<th>85</th>
<th>38</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>19</td>
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<td>38</td>
</tr>
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Analysis of Insertion Sort

- Worst-case running time:

  On the $k$th iteration of the outer loop, element $A[k]$ is compared with at most $k$ elements: $A[k−1], A[k−2], \ldots, A[0]$.

  Total number of comparisons over all iterations is at most:

  $$n−1 \sum_{k=1}^{n} k = n(n−1)/2 = O(n^2).$$

  Insertion Sort is a bad choice when $n$ is large. ($O(n^2)$ vs. $O(n \log n)$).

  Insertion Sort is a good choice when $n$ is small. (Constant hidden in the “big oh” is small).

  Insertion Sort is efficient if the input is “almost sorted”:

  $$\text{Time} \leq n−1 + (\text{# inversions})$$

Storage: in place: $O(1)$ extra storage.
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Selection Sort

Two variants:

1. Repeatedly (for $i$ from 0 to $n - 1$) find the minimum value, output it, delete it. Values are output in sorted order.


Both variants run in $O(n^2)$ time if we use the straightforward approach to finding the maximum/minimum. They can be improved by treating the items $A[0]$, $A[1]$, ... , $A[i]$ as items in an appropriately designed priority queue. (Next set of notes)
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  1. Repeatedly (for \(i\) from 0 to \(n-1\)) find the minimum value, output it, delete it. Values are output in sorted order
  2. Repeatedly (for \(i\) from \(n-1\) down to 1) find the maximum of \(A[0], A[1], \ldots, A[i]\). Swap this value with \(A[i]\) (no-op if it is already \(A[i]\)).

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Quicksort

Basic idea

- Classify keys as small keys or large keys. All small keys are less than all large keys.
- Rearrange keys so small keys precede all large keys.
- Recursively sort small keys, recursively sort large keys.
Quicksort

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<table>
<thead>
<tr>
<th>keys</th>
</tr>
</thead>
<tbody>
<tr>
<td>small keys</td>
</tr>
</tbody>
</table>

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Quicksort: One specific implementation

- Let the first item in the array be the pivot value \( x \) (also called the split value).
- Small keys are the keys \( < x \).
- Large keys are the keys \( \geq x \).
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```
+---+-----+-----+---+
|   |     |     |   |
| x |     |     | x |
+---+-----+-----+---+
```

```
+---+-----+-----+---+
|   |     |     |   |
| first|     |     | last |

```

```
+---+-----+-----+---+
|   |     |     |   |
|  < |     |     |  \geq |
+---+-----+-----+---+
```

```
def quickSort(A, first, last):
    if first < last:
        splitpoint = split(A, first, last)
        quickSort(A, first, splitpoint-1)
        quickSort(A, splitpoint+1, last)
The split step

def split(A, first, last):
    splitpoint = first
    x = A[first]
    for k = first+1 to last do:
        if A[k] < x:
            splitpoint = splitpoint + 1
    return splitpoint

Loop invariants:

▷ A[first+1..splitpoint] contains keys < x.
▷ A[splitpoint+1..k−1] contains keys ≥ x.
▷ A[k..last] contains unprocessed keys.
The split step

At start:

In middle:

At end:
Example of split step
Analysis of Quicksort
Analysis of Quicksort

We can visualize the lists sorted by quicksort as a binary tree.
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- The children of a node are the two sublists to be sorted.
- Identify each list with its split value.
Worst-case Analysis of Quicksort

Any pair of values $x$ and $y$ gets compared at most once during the entire run of Quicksort.

The number of possible comparisons is $\binom{n}{2} = O(n^2)$.

Hence the worst-case number of comparisons performed by Quicksort when sorting $n$ items is $O(n^2)$.

Question: Is there a better bound? Is it $o(n^2)$? Or is it $\Theta(n^2)$?

Answer: The bound is tight. It is $\Theta(n^2)$.

We will see why on the next slide.
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A bad case case for Quicksort: 1, 2, 3, \ldots, n − 1, n

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$(\frac{n}{2})$ comparisons required. So the worst-case running time for Quicksort is $\Theta(n^2)$. But what about the average case . . . ?
Average-case analysis of Quicksort:

Our approach:
Average-case analysis of Quicksort:

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1. Use the binary tree of sorted lists
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1. Use the **binary tree of sorted lists**
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4. Use this to compute the expected number of comparisons performed by Quicksort.
Average-case analysis of Quicksort:

Sorted order: 15 18 22 23 27 36 79 83
Average-case analysis of Quicksort

Number the keys in sorted order: $S_1 < S_2 < \cdots < S_n$.

Fact about comparisons: During the run of Quicksort, two keys $S_i$ and $S_j$ get compared if and only if the first key from the set of keys $\{S_i, S_{i+1}, \ldots, S_j\}$ to be chosen as a pivot is either $S_i$ or $S_j$.

If some key $S_k$ is chosen first with $S_i < S_k < S_j$, then $S_i$ goes in the left half, $S_j$ goes in the right half, and $S_i$ and $S_j$ never get compared.

If $S_i$ is chosen first, it is compared against all the other keys in the split step (including $S_j$).

Similar if $S_j$ is chosen first.

Examples:

- 23 and 22 (both statements true)
- 36 and 83 (both statements false)
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Assume:
▶ All $n$ keys are distinct
▶ All permutations are equally likely
▶ The keys in sorted order are $S_1 < S_2 < \cdots < S_n$.

Let $P_{i,j}$, the probability that keys $S_i$ and $S_j$ are compared with each other during the invocation of quicksort.

Then by Fact about comparisons on previous slide:
$P_{i,j}$, the probability that the first key from $\{S_i, S_{i+1}, \ldots, S_j\}$ to be chosen as a pivot value is either $S_i$ or $S_j$:
$$P_{i,j} = 2^{j-i+1}.$$
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\[
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Average-case analysis of Quicksort

Define indicator random variables \( \{ X_{i,j} : 1 \leq i < j \leq n \} \)

\[
X_{i,j} = \begin{cases} 
1 & \text{if keys } S_i \text{ and } S_j \text{ get compared} \\
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3. The expected value of \( X_{i,j} \) is:

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Average-case analysis of Quicksort

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\]
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Average-case analysis of Quicksort

Hence the expected number of comparisons is

\[ \sum_{i=1}^{n} \sum_{j=i+1}^{n} E(X_{i,j}) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j - i + 1} \]

\[ = \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k} \quad (k = j - i + 1) \]

\[ < \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{2}{k} \]

\[ = 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k} \]

\[ = 2 \sum_{i=1}^{n} H_n = 2nH_n \in O(n \lg n). \]

So the average time for Quicksort is \( O(n \lg n) \).