Lecture 3
Recap of basic data structures, binary search, insertion/selection sort

CS 161 Design and Analysis of Algorithms
Ioannis Panageas
Outline of these notes

▷ Review of basic data structures
▷ Searching in a sorted array/binary search: the algorithm, analysis, proof of optimality
▷ Sorting, part 1: insertion sort, selection sort
Basic Data structures

Prerequisite material. Review [GT Chapters 2–4, 6] as necessary

- Arrays, dynamic arrays
- Linked lists
- Stacks, queues
- Dictionaries, hash tables
- Binary trees
Arrays, Dynamic arrays, Linked lists

Arrays:
- Numbered collection of cells or entries
- Numbering usually starts at 0
- Fixed number of entries
- Each cell has an index which uniquely identifies it.
- Accessing or modifying the contents of a cell given its index: O(1) time.
- Inserting or deleting an item in the middle of an array is slow.

Dynamic arrays:
- Similar to arrays, but size can be increased or decreased
- ArrayList in Java, list in Python

Linked lists:
- Collection of nodes that form a linear ordering.
- The list has a first node and a last node
- Each node has a next node and a previous node (possibly null)
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Stacks and Queues

- **Stacks**: Container of objects that are inserted and removed according to Last-In First-Out (LIFO) principle:
  - Only the most-recently inserted object can be removed.
  - Insert and remove are usually called **push** and **pop**.

- **Queues** (often called FIFO Queues):
  - Container of objects that are inserted and removed according to First-In First-Out (FIFO) principle:
  - Only the element that has been in the queue the longest can be removed.
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Dictionaries/Maps

A Dictionary (or Map) stores <key,value> pairs, which are often referred to as items.

- There can be at most one item with a given key.

Examples:
1. <Student ID, Student data>
2. <Object ID, Object data>
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Hashing

An efficient method for implementing a dictionary.

- **A hash table**, an array of size $N$.
- **A hash function**, which maps any key from the set of possible keys to an integer in the range $[0, N - 1]$.
- **A collision strategy**, which determines what to do when two keys are mapped to the same table location by the hash function.

Commonly used collision strategies are:
- **Chaining**
- **Open addressing**: linear probing, quadratic probing, double hashing
- **Cuckoo hashing**

Hashing is fast:
- $O(1)$ expected time for access, insertion
- Cuckoo hashing improves the access time to $O(1)$ worst-case time. Insertion time remains $O(1)$ expected time.

Disadvantages on next slide.
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We will use as a data structure and as a tool for analyzing algorithms.
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The depth of a binary tree is the maximum of the levels of all its leaves.
Traversing binary trees

Preorder: root, left subtree (in preorder), right subtree (in preorder):
ABDGHCEF

Inorder: left subtree (in inorder), root, right subtree (in inorder):
GDHBAECF

Postorder: left subtree (in postorder), right subtree (in postorder), root:
GHDBEFCA

Breadth-first order (level order): level 0 left-to-right, then level 1 left-to-right, ...:
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Facts about binary trees

1. There are at most $2^k$ nodes at level $k$.
2. A binary tree with depth $d$ has:
   - At most $2^d$ leaves.
   - At most $2^{d+1} - 1$ nodes.
3. A binary tree with $n$ leaves has depth $\geq \lceil \log_2 n \rceil$.
4. A binary tree with $n$ nodes has depth $\geq \lfloor \log_2 n \rfloor$. 
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4. A binary tree with $n$ nodes has depth $\geq \lfloor \lg n \rfloor$. 
Binary search trees

- Function as ordered dictionaries. (Can find successors, predecessors)
- Find, insert, and remove can all be done in $O(h)$ time ($h$ = tree height)
- AVL trees, Red-Black Trees, Weak AVL trees: $h = O(\log n)$, so find, insert, and remove can all be done in $O(\log n)$ time.
- Splay trees and Skip Lists: alternatives to balanced trees
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[GT] Chapters 3–4 for details
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Binary Search: Searching in a sorted array

Input is a sorted array \( A \) and an item \( x \). Problem is to locate \( x \) in the array.

Several variants of the problem, for example:

1. Determine whether \( x \) is stored in the array
2. Find the largest \( i \) such that \( A[i] \leq x \) (with a reasonable convention if \( x < A[0] \)).

We will focus on the first variant.

We will show that binary search is an optimal algorithm for solving this problem.
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- Location of \( x \), if \( x \) found
- \(-1\), if \( x \) not found

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def binarySearch(A, x, first, last):
    if first > last:
        return (-1)
    else:
        mid = \lfloor \frac{first + last}{2} \rfloor
        if x == A[mid]:
            return mid
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Correctness of Binary Search

We need to prove two things:

1. If $x$ is in the array, its location in the array (its index) is between \textit{first} and \textit{last}, inclusive.

   Note that this is equivalent to:

   Either $x$ is not in the array, or its location is between \textit{first} and \textit{last}, inclusive.

2. On each recursive call, the difference $\textit{last} - \textit{first}$ gets strictly smaller.
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We will count the number of 3-way comparisons of $x$ against elements of $A$. (also known as decisions)

Rationale:
1. This is the essentially the same as the number of recursive calls. Every recursive call, except for possibly the very last one, results in a 3-way comparison.
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Binary Search: Analysis of Running Time (continued)

Binary search in an array of size 1: 1 decision

Binary search in an array of size $n > 1$: after 1 decision, either we are done, or the problem is reduced to binary search in a subarray with a worst-case size of $\lfloor n/2 \rfloor$

So the worst-case time to do binary search on an array of size $n$ is $T(n)$, where $T(n)$ satisfies the equation

$$T(n) = \begin{cases} 
1 & \text{if } n = 1 \\
1 + T(\lfloor n/2 \rfloor) & \text{otherwise}
\end{cases}$$

The solution to this equation is:

$$T(n) = \lfloor \log_2 n \rfloor + 1$$

This can be proved by induction.

So binary search does $\lfloor \log_2 n \rfloor + 1$ 3-way comparisons on an array of size $n$, in the worst case.
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Optimality of binary search

We will establish a lower bound on the worst-case number of decisions required to find an item in an array, using only 3-way comparisons of the item against array entries.

The lower bound we will establish is $\lceil \log_2 n \rceil + 1$ 3-way comparisons.

Since Binary Search performs within this bound, it is optimal.

Our lower bound is established using a Decision Tree model.

Note that the bound is exact (not just asymptotic).

Our lower bound is on the worst case.

It says: for every algorithm for finding an item in an array of size $n$, there is some input that forces it to perform $\lceil \log_2 n \rceil + 1$ comparisons.

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The decision tree model for searching in an array

Consider any algorithm that searches for an item \( x \) in an array \( A \) of size \( n \) by comparing entries in \( A \) against \( x \). Any such algorithm can be modeled as a decision tree:

- Each node is labeled with an integer \( \in \{0, \ldots, n-1\} \).
- A node labeled \( i \) represents a 3-way comparison between \( x \) and \( A[i] \).
- The left subtree of a node labeled \( i \) describes the decision tree for what happens if \( x < A[i] \).
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Example: Decision tree for binary search with \( n = 13 \):

```
  1 3 5 8 10 12
  0 4 7 11
  2 9
  6
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```
                          6
                         / \             / \   \
                        2   9         7   11
                       /   \           /   \   \
                      0     4       8     10
                     /     \       /     \   \
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**Example:** Decision tree for binary search with $n = 13$:
Lower bound on locating an item in an array of size $n$

Any algorithm for searching an array of size $n$ can be modeled by a decision tree with at least $n$ nodes.

Since the decision tree is a binary tree with $n$ nodes, the depth is at least $\lfloor \log_2 n \rfloor$.

The worst-case number of comparisons for the algorithm is the depth of the decision tree +1. (Remember, root has depth 0).

Hence any algorithm for locating an item in an array of size $n$ using only comparisons must perform at least $\lfloor \log_2 n \rfloor + 1$ comparisons in the worst case.

So binary search is optimal with respect to worst-case performance.
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Sorting

- Rearranging a list of items in nondescending order.
- Useful preprocessing step (e.g., for binary search).
- Important step in other algorithms.
- Illustrates more general algorithmic techniques.

We will discuss:
- Comparison-based sorting algorithms (Insertion sort, Selection Sort, Quicksort, Mergesort, Heapsort).
- Bucket-based sorting methods.
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- **Disadvantage:** under certain circumstances, specific properties of the data item can speed up the sorting process.
- Measure of time: number of comparisons

Comparison-based sorting has lower bound of $\Omega(n \log n)$ comparisons. (We will prove this.)
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- **Disadvantage:** under certain circumstances, specific properties of the data item can speed up the sorting process.
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Comparison-based sorting

- Basic operation: compare two items.
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- Comparison-based sorting has lower bound of $\Omega(n \log n)$ comparisons. (We will prove this.)
$\Theta(n \log n)$ work vs. quadratic ($\Theta(n^2)$) work

\[ y = \binom{n}{2} \]

\[ y = 10n \log n \]
Some terminology

A permutation of a sequence of items is a reordering of the sequence. A sequence of $n$ items has $n!$ distinct permutations.

Note: Sorting is the problem of finding a particular distinguished permutation of a list.

An inversion in a sequence or list is a pair of items such that the larger one precedes the smaller one.

Example: The list $18 \ 29 \ 12 \ 15 \ 32 \ 10$ has 9 inversions:

$\{(18,12), (18,15), (18,10), (29,12), (29,15), (29,10), (12,10), (15,10), (32,10)\}$
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**Example:** The list

$$18 \ 29 \ 12 \ 15 \ 32 \ 10$$

has 9 inversions:

$$\{(18,12), \ (18,15), \ (18,10), \ (29,12), \ (29,15), \ (29,10), \ (12,10), \ (15,10), \ (32,10)\}$$
Insertion sort

- Work from left to right across array
- Insert each item in correct position with respect to (sorted) elements to its left

\[
\begin{aligned}
&\text{(Sorted)} \\
&\text{(Unsorted)} \\
&\text{(Sorted)}
\end{aligned}
\]
Insertion sort

0

(Sorted)

(Unsorted)

\( k \)

(Sorted)  \( x \)  (Unsorted)

\( n - 1 \)

(Sorted)
Insertion sort

- Work from left to right across array

<table>
<thead>
<tr>
<th>(Sorted)</th>
<th>x</th>
<th>(Unsorted)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Sorted)</td>
<td></td>
<td></td>
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</table>
Insertion sort

- Work from left to right across array
- Insert each item in correct position with respect to (sorted) elements to its left

\[
\begin{array}{cccc}
0 & (\text{Unsorted}) \\
\hline
k & (\text{Sorted}) & x & (\text{Unsorted}) \\
\hline
n-1 & (\text{Sorted}) & \text{---} & \text{---} \\
\end{array}
\]
def insertionSort(n, A):
    for k = 1 to n-1:
        x = A[k]
        j = k-1
        while (j >= 0) and (A[j] > x):
            j = j-1
        A[j+1] = x
Insertion sort example

\[
\begin{array}{cccccc}
23 & 19 & 42 & 17 & 85 & 38 \\
23 & 19 & 42 & 17 & 85 & 38 \\
19 & 23 & 42 & 17 & 85 & 38 \\
19 & 23 & 42 & 17 & 85 & 38 \\
17 & 19 & 23 & 42 & 85 & 38 \\
17 & 19 & 23 & 42 & 85 & 38 \\
17 & 19 & 23 & 38 & 42 & 85
\end{array}
\]
Analysis of Insertion Sort

- **Worst-case running time:**

  - On the \( k \)th iteration of the outer loop, element \( A[k] \) is compared with at most \( k \) elements: \( A[k-1], A[k-2], \ldots, A[0] \).

  - Total number of comparisons over all iterations is at most:
    \[
    n - 1 \sum_{k=1}^{n} k = n(n-1)/2 = O(n^2)
    \]

- Insertion Sort is a bad choice when \( n \) is large. \( O(n^2) \) vs. \( O(n \log n) \).

- Insertion Sort is a good choice when \( n \) is small. (Constant hidden in the “big oh” is small).

- Insertion Sort is efficient if the input is “almost sorted”:
  \[
  \text{Time} \leq n - 1 + \text{(number of inversions)}
  \]

- **Storage:** in place: \( O(1) \) extra storage.
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Selection Sort

Two variants:
1. Repeatedly (for \(i\) from 0 to \(n - 1\)) find the minimum value, output it, delete it.
   - Values are output in sorted order
2. Repeatedly (for \(i\) from \(n - 1\) down to 1)
   - Find the maximum of \(A[0], A[1], \ldots, A[i]\).
   - Swap this value with \(A[i]\) (no-op if it is already \(A[i]\)).

Both variants run in \(O(n^2)\) time if we use the straightforward approach to finding the maximum/minimum.

They can be improved by treating the items \(A[0], A[1], \ldots, A[i]\) as items in an appropriately designed priority queue. (Next set of notes)
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