Algorithms and Data Structures

- An **algorithm** is a step-by-step procedure for performing some task in a finite amount of time.
  - Typically, an algorithm takes input data and produces an output based upon it.

- A **data structure** is a systematic way of organizing and accessing data.
Most algorithms transform input objects into output objects.

The running time of an algorithm typically grows with the input size.

Average case time is often difficult to determine.

We focus primarily on the worst case running time.
- Easier to analyze
- Crucial to applications such as games, finance and robotics.
Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation.
- Characterizes running time as a function of the input size, $n$.
- Takes into account all possible inputs.
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment.
Pseudocode

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues
Pseudocode Details

- **Control flow**
  - if ... then ... [else ...]
  - while ... do ...
  - repeat ... until ...
  - for ... do ...
  - Indentation replaces braces

- **Method declaration**
  - Algorithm *method* (arg [, arg...])
    - Input ...
    - Output ...

- **Method call**
  - *method* (arg [, arg...])

- **Return value**
  - return *expression*

- **Expressions:**
  - Assignment
    - \( \leftarrow \)
  - Equality testing
    - \( = \)
  - Superscripts and other mathematical formatting allowed
    - \( n^2 \)
Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant \( \approx 1 \)
  - Logarithmic \( \approx \log n \)
  - Linear \( \approx n \)
  - N-Log-N \( \approx n \log n \)
  - Quadratic \( \approx n^2 \)
  - Cubic \( \approx n^3 \)
  - Exponential \( \approx 2^n \)

- In a log-log chart, the slope of the line corresponds to the growth rate.
Functions Graphed Using “Normal” Scale

- $g(n) = 1$
- $g(n) = n \log n$
- $g(n) = 2^n$
- $g(n) = \log n$
- $g(n) = n^2$
- $g(n) = n^3$

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Analysis of Algorithms

Slide by Matt Stallmann included with permission.
**Primitive Operations**

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important

**Examples:**
- Evaluating an expression
- Assigning a value to a variable
- Indexing into an array
- Calling a method
Example: By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size.

```
Algorithm arrayMax(A, n):
    Input: An array A storing n ≥ 1 integers.
    Output: The maximum element in A.
    currentMax ← A[0]
    for i ← 1 to n − 1 do
        if currentMax < A[i] then
            currentMax ← A[i]
    return currentMax
```
Growth Rate of Running Time

- Changing the hardware/software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$

- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm arrayMax
### Why Growth Rate Matters

<table>
<thead>
<tr>
<th>if runtime is...</th>
<th>time for n + 1</th>
<th>time for 2n</th>
<th>time for 4n</th>
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<tbody>
<tr>
<td>( cn \lg n )</td>
<td>( c \lg (n + 1) )</td>
<td>( c (\lg n + 1) )</td>
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<td>( cn )</td>
<td>( c (n + 1) )</td>
<td>( 2cn )</td>
<td>( 4cn )</td>
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<td>( cn \lg n )</td>
<td>( \sim c n \lg n + cn )</td>
<td>( 2cn \lg n + 2cn )</td>
<td>( 4cn \lg n + 4cn )</td>
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<td>( c n^2 )</td>
<td>( \sim c n^2 + 2cn )</td>
<td><strong>4( c n^2 )</strong></td>
<td>( 16cn^2 )</td>
</tr>
<tr>
<td>( c n^3 )</td>
<td>( \sim c n^3 + 3cn^2 )</td>
<td>( 8cn^3 )</td>
<td>( 64cn^3 )</td>
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<tr>
<td>( c 2^n )</td>
<td>( c 2^{n+1} )</td>
<td>( c 2^{2n} )</td>
<td>( c 2^{4n} )</td>
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</table>

Runtime quadruples when problem size doubles.
Constant Factors

- The growth rate is minimally affected by constant factors or lower-order terms.
- Examples
  - $10^2 n + 10^5$ is a linear function.
  - $10^5 n^2 + 10^8 n$ is a quadratic function.

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The asymptotic analysis of an algorithm determines the running time in big-Oh notation.

To perform the asymptotic analysis:
- We find the worst-case number of primitive operations executed as a function of the input size.
- We express this function with big-Oh notation.

Example:
- We say that algorithm arrayMax “runs in $O(n)$ time.”

Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations.
Big-Oh Rules

- If $f(n)$ is a polynomial of degree $d$, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say “$2n$ is $O(n)$” instead of “$2n$ is $O(n^2)$”
- Use the simplest expression of the class
  - Say “$3n + 5$ is $O(n)$” instead of “$3n + 5$ is $O(3n)$”
Analyzing Recursive Algorithms

Use a function, $T(n)$, to derive a recurrence relation that characterizes the running time of the algorithm in terms of smaller values of $n$.

**Algorithm** `recursiveMax(A, n)`:

- **Input:** An array $A$ storing $n \geq 1$ integers.
- **Output:** The maximum element in $A$.

```plaintext
if $n = 1$ then
    return $A[0]$
return \(\max\{\text{recursiveMax}(A, n - 1), A[n - 1]\}\)
```

\[
T(n) = \begin{cases} 
3 & \text{if } n = 1 \\
T(n - 1) + 7 & \text{otherwise},
\end{cases}
\]
Arithmetic Progression

- Assume the running time of $P$ is $O(1 + 2 + \ldots + n)$
- The sum of the first $n$ integers is $\frac{n(n + 1)}{2}$
  - There is a simple visual proof of this fact
- Thus, algorithm $P$ runs in $O(n^2)$ time
Math you need to Review

- Summations
- Powers
- Logarithms
- Proof techniques
- Basic probability

Properties of powers:
- $a^{(b+c)} = a^b a^c$
- $a^{bc} = (a^b)^c$
- $a^b /a^c = a^{(b-c)}$
- $b = a^{\log_a b}$
- $b^c = a^{c \log_a b}$

Properties of logarithms:
- $\log_b(xy) = \log_b x + \log_b y$
- $\log_b (x/y) = \log_b x - \log_b y$
- $\log_b xa = a \log_b x$
- $\log_b a = \log_x a / \log_x b$
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- \( g \in O(f) \) if “\( g \) grows no faster than (a constant multiple of) \( f \).”
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- \( g \in O(f) \) if \( g \) is bounded above by a constant multiple of \( f \) (for sufficiently large values of \( n \)).
- \( g \in O(f) \) if “\( g \) grows no faster than (a constant multiple of) \( f \).”
- \( g \in O(f) \) if the ratio \( g/f \) is bounded above by a constant (for sufficiently values of \( n \)).
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Formally:

$g \in O(f)$ if and only if:

$\exists C > 0 \exists n_0 > 0 \forall n > n_0 \ g(n) \leq C \cdot f(n)$.

Equivalently:

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Sometimes we write:

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- Equivalently: \( g \in O(f) \) if and only if:
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  \exists C > 0 \ \exists n_0 > 0 \ \forall n > n_0 \ \frac{g(n)}{f(n)} \leq C.
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Proof: Let $C = 1000$. Then $g(n) \leq C \cdot f(n)$ for all $n$. 
Examples of $O$-notation:

Example 2: $f(n) = n^2$, $g(n) = n^3/2$. $g \in O(f)$.

Proof: \[
\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0.
\]
Hence for any $C > 0$ the ratio is less than $C$ as long as $n$ is sufficiently large.
(Of course, how large $n$ must be to be "sufficiently large" depends on $C$).

Alternate Proof: If $n \geq 1$, $n^{1/2} \geq 1$, so $n^{3/2} \leq n^2$. Hence we can choose $C = 1$ and $n_0 = 1$. 

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Hence we can choose $C = 1$ and $n_0 = 1$. 

Examples of $O$-notation:

Example 3:
$f(n) = n^3$, $g(n) = n^4$.

Proof:
$$\lim_{n \to \infty} \frac{g(n)}{f(n)} = \infty.$$ Hence there is no $C > 0$ such that $g(n) \leq C \cdot f(n)$ for sufficiently large $n$. 
Examples of $O$-notation:

Example 3: $f(n) = n^3, g(n) = n^4$: $g \notin O(f)$. 

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Examples of $O$-notation:

Example 4:

$f(n) = n^2$, $g(n) = 5n^2 + 23n + 2$:

$g \in O(f)$.

Proof:

If $n \geq 1$, then $n \leq n^2$ and $1 \leq n^2$.

Hence:

$g(n) = 5n^2 + 23n + 2 \leq 5n^2 + 23n^2 + 2n^2 \leq 30n^2 = 30f(n)$

So we can take $C = 30$, $n_0 = 1$. 
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$$\leq 5n^2 + 23n^2 + 2n^2$$
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\( o \) ("little oh"), \( \Omega \) ("big Omega")
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  g \in o(f) \text{ if and only if } \lim_{n \to \infty} \frac{g(n)}{f(n)} = 0.
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  g \in \Omega(f) \text{ if and only if } \exists C > 0 \exists n_0 > 0 \forall n > n_0 \ g(n) \geq C \cdot f(n).
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One more definition: \( \Theta \) ("Theta")

\[ g \in \Theta(f) \text{ if and only if: } g \in O(f) \text{ and } g \in \Omega(f). \]

Equivalently, \( g \in \Theta(f) \text{ if and only if: } \exists C_1 > 0 \exists C_2 > 0 \exists n_0 > 0 \forall n > n_0 \ C_1 \leq g(n) f(n) \leq C_2. \]
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Examples of Asymptotic notation

Example 1:

\[ f(n) = n, \quad g(n) = 1000n. \]

\( g \in \Omega(f), \quad g \in \Theta(f) \)

To see that \( g \in \Omega(f) \), we can take \( C = 1 \).

Then \( g(n) = 1000 \cdot n > 1 \cdot n = C \cdot f(n) \).

To see that \( g \in \Theta(f) \), we could argue that \( g \in O(f) \) (shown earlier) and \( g \in \Omega(f) \) (shown above).

Or we can take \( C_1 = 1, \quad C_2 = 1000 \). Then \( C_1 \leq g(n) \leq C_2 \).
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To see that \( g \in \Omega(f) \), we can take \( C = 1. \)

Then \( g(n) = 1000 \cdot n > 1 \cdot n = C \cdot f(n). \)

To see that \( g \in \Theta(f) \), we could argue that \( g \in O(f) \) (shown earlier) and \( g \in \Omega(f) \) (shown above).

Or we can take \( C_1 = 1, \ C_2 = 1000. \) Then

\[
C_1 \leq \frac{g(n)}{f(n)} \leq C_2.
\]
Examples of Asymptotic notation

Example 2:

\[ f(n) = n^2, \quad g(n) = \frac{n^3}{2} : \]

\[ g \in \Theta(f) \]

Because \( \lim_{n \to \infty} \frac{g(n)}{f(n)} = 0. \)
Examples of Asymptotic notation

Example 2: $f(n) = n^2$, $g(n) = n^{3/2}$.
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Example 2: $f(n) = n^2$, $g(n) = n^{3/2}$:

$g \in o(f)$

Because $\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$. 
Examples of Asymptotic notation

Example 3:

\( f(n) = n^3, \quad g(n) = n^4 \):

\( g \in \Omega(f) \)

Because \( \lim_{n \to \infty} \frac{g(n)}{f(n)} = \infty \), so we can choose any \( C \) we want.
Examples of Asymptotic notation

Example 3: $f(n) = n^3$, $g(n) = n^4$: $g(n) \in \Omega(f(n))$.
Example 3: \( f(n) = n^3, \ g(n) = n^4: \)

\[ g \in \Omega(f) \]
Examples of Asymptotic notation

Example 3: \( f(n) = n^3, \ g(n) = n^4: \)

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Because \( \lim_{n \to \infty} \frac{g(n)}{f(n)} = \infty, \) so we can choose any \( C \) we want.
Examples of Asymptotic notation

Example 4: \( f(n) = n^2 \), \( g(n) = 5n^2 - 23n + 2 \):

\[ g \in \Omega(f). \]

Proof: If \( n \geq 23 \), then \( 23 \leq n^2 \).

Hence if \( n \geq 23 \):

\[ g(n) = 5n^2 - 23n + 2 \geq 5n^2 - n^2 = 4n^2 = 4f(n) \]

So we can take \( C = 4 \), \( n_0 = 23 \).
Examples of Asymptotic notation

Example 4: \( f(n) = n^2, \ g(n) = 5n^2 - 23n + 2: \)

\( g \in \Omega(f). \)
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Examples of Asymptotic notation

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Example 4: \( f(n) = n^2, \ g(n) = 5n^2 - 23n + 2: \)

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Proof: If \( n \geq 23 \), then \( 23n \leq n^2 \). Hence if \( n \geq 23: \)

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\begin{align*}
g(n) &= 5n^2 - 23n + 2 \\
&\geq 5n^2 - n^2
\end{align*}
\]
Examples of Asymptotic notation

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\begin{align*}
g(n) &= 5n^2 - 23n + 2 \\
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&\geq 4n^2
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Examples of Asymptotic notation

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Examples of Asymptotic notation

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So we can take \( C = 4, \ n_0 = 23 \).
Another Example

Example 5: \( \ln n = o(n) \) \((n)\)

Proof:
Examine the ratio \( \frac{\ln n}{n} \) as \( n \to \infty \).

If we try to evaluate the limit directly, we obtain the "indeterminate form" \( \frac{\infty}{\infty} \).

We need to apply L'Hôpital's rule (from calculus). (Continued on next slide)
Example 5: $\ln n = o(n)$
Another Example

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We need to apply L’Hôpital’s rule (from calculus).

(Continued on next slide)
Example 5, continued:
\[ \ln n = o(n) \]

**L’Hôpital’s rule:** If the ratio of limits

\[
\lim_{n \to \infty} \frac{g(n)}{f(n)}
\]

is an indeterminate form (i.e., \(\infty/\infty\) or \(0/0\)), then

\[
\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{g'(n)}{f'(n)}
\]

where \(f'\) and \(g'\) are, respectively, the derivatives of \(f\) and \(g\).
Example 5, continued:

\( \ln n = o(n) \)

Let \( f(n) = n \), \( g(n) = \ln n \).
Example 5, continued:

\[ \ln n = o(n) \]

Let \( f(n) = n, \ g(n) = \ln n. \)

Then \( f'(n) = 1, \ g'(n) = 1/n. \)
Example 5, continued:

\( \ln n = \mathcal{o}(n) \)

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By L’Hôpital’s rule:
Example 5, continued:

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By L’Hôpital’s rule:

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\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{g'(n)}{f'(n)}
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Example 5, continued:

$\ln n = o(n)$

Let $f(n) = n$, $g(n) = \ln n$.

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By L'Hôpital's rule:

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{g'(n)}{f'(n)}$$

$$= \lim_{n \to \infty} \frac{1/n}{1}$$

Hence $g(n) = o(f(n))$. 
Example 5, continued:

$\ln n = o(n)$

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Hence \( g(n) = o(f(n)) \).
Math background
Math background

- Sums, Summations
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- Logarithms, Exponents Floors, Ceilings, Harmonic Numbers
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- Proof Techniques
Math background

- Sums, Summations
- Logarithms, Exponents Floors, Ceilings, Harmonic Numbers
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- Basic Probability
Sums, Summations

Summation notation:

\[ \sum_{i=a}^{b} f(i) = f(a) + f(a+1) + \cdots + f(b) \]

Special cases:

What if \( a = b \)?

What if \( a > b \)?

If \( S = \{s_1, \ldots, s_n\} \) is a finite set:

\[ \sum_{x \in S} f(x) = f(s_1) + f(s_2) + \cdots + f(s_n) \]
Sums, Summations

- Summation notation:

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Geometric sum

\[ \sum_{i=0}^{n} a_i = 1 + a_1 + a_2 + \cdots + a_n = \frac{1 - a^{n+1}}{1 - a}, \]
provided that \( a \neq 1 \).

Previous formula holds for \( a = 0 \) because \( a^0 = 1 \) even when \( a = 0 \).

Special case of geometric sum:

\[ \sum_{i=0}^{n} 2^i = 1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1. \]
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\]
Infinite Geometric sum

From the previous slide:

\[ \sum_{i=0}^{n} a_i = 1 + a_1 + a_2 + \cdots + a_n = 1 - a_{n+1}, \]

provided that \( a \neq 1. \)

If \( |a| < 1, \) we can take the limit as \( n \to \infty: \)

\[ \sum_{i=0}^{\infty} a_i = 1 + a_1 + a_2 + \cdots = \frac{1}{1-a}, \]

Special case of infinite geometric sum:

\[ \sum_{i=0}^{\infty} \frac{1}{2^i} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2. \]
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Other Summations

- Sum of first $n$ integers:
  $$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \Theta(n^2)$$

- Sum of first $n$ squares:
  $$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3)$$

- In general, for any fixed positive integer $k$:
  $$\sum_{i=1}^{n} i^k = 1^k + 2^k + 3^k + \cdots + n^k = \Theta(n^{k+1})$$
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Logarithms

Definition: $\log_b x = y$ if and only if $b^y = x$.

Some useful properties:

1. $\log_b 1 = 0$.
2. $\log_b b^a = a$.
3. $\log_b (xy) = \log_b x + \log_b y$.
4. $\log_b (x^a) = a \log_b x$.
5. $x \log_b y = y \log_b x$.
6. $\log_x b = \frac{1}{\log_b x}$.
7. $\log_a x = \frac{1}{\log_b a} \log_b x$.
8. $\log_a x = (\log_b x)(\log_a b)$.

Exercise: Prove the above properties.
Definition: $\log_b x = y$ if and only if $b^y = x$. 

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Exercise: Prove the above properties.
Logarithms

Example (#2): Prove \( \log_b a = x \).

Solution: Let \( y = \log_b a \) [by definition of log]

\[ y = b^x \]

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Logarithms

Example (#2): Prove $\log_b b^a = a$. 
Logarithms

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Logarithms

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$by = b^a$ [by definition of log]
Example (#2): Prove $\log_b b^a = a$.

Solution: Let $y = \log_b b^a$

\[ b^y = b^a \quad \text{[by definition of log]} \]

\[ y = a \]
Logarithms

Special Notations:

▶ \( \ln x = \log_e x \) 
\( e = 2.71828... \)

▶ \( \lg x = \log_2 x \)

Some conversions (from Rules #7 and #8 on previous slides):

▶ \( \ln x = (\log_2 x)(\log_e 2) = 0.693 \lg x \)

▶ \( \lg x = \log_e x \log_e 2 = \ln x / 0.693 = 1.44 \ln x \)
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Floors and ceilings
Floors and ceilings

$\lfloor x \rfloor = \text{largest integer } \leq x$. (Read as Floor of $x$)
Floors and ceilings

- $\lfloor x \rfloor =$ largest integer $\leq x$. (Read as Floor of $x$)
- $\lceil x \rceil =$ smallest integer $\geq x$ (Read as Ceiling of $x$)
Factorials

\[ n! = 1 \cdot 2 \cdots n \]
Factorials

- \( n! = 1 \cdot 2 \cdots n \)
- \( n! \) represents the number of distinct permutations of \( n \) objects.
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- \( n! \) represents the number of distinct permutations of \( n \) objects.

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 3 & 2 \\
2 & 1 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
3 & 2 & 1 \\
\end{array}
\]
Combinations

$\binom{n}{k} =$ The number of different ways of choosing $k$ objects from a collection of $n$ objects. (Pronounced "$n$ choose $k".")

Example: $\binom{5}{2} = 10$

$\{1, 2\}$  $\{1, 3\}$  $\{1, 4\}$  $\{1, 5\}$  $\{2, 3\}$  $\{2, 4\}$  $\{2, 5\}$  $\{3, 4\}$  $\{3, 5\}$  $\{4, 5\}$

Formula: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Special cases:
- $\binom{n}{0} = 1$
- $\binom{n}{1} = n$
- $\binom{n}{2} = \frac{n(n-1)}{2}$
Combinations

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\[
\begin{align*}
\{1, 2\} & \quad \{1, 3\} & \quad \{1, 4\} & \quad \{1, 5\} & \quad \{2, 3\} \\
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\end{align*}
\]
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Harmonic Numbers

The $n$th Harmonic number is the sum:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

These numbers go to infinity:

$$\lim_{n \to \infty} H_n = \infty$$
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The harmonic numbers are closely related to logs. Recall:

\[ \ln x = \int_1^x \frac{1}{t} \, dt \]

We will show that \( H_n = \Theta(\log n) \).
Harmonic Numbers

\[ y = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n} < \ln n < \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{n} - 1 \]

\( H_n - \frac{1}{n} < \ln n < H_n - 1 \), so \( H_n = \Theta(\log n) \).
Harmonic Numbers

\[ y = \frac{1}{x} \]
Harmonic Numbers

\[ \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \]
Harmonic Numbers

\begin{align*}
\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} &< \ln n < 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \\
H_n - 1 &< \ln n < H_n - \frac{1}{n}
\end{align*}
Harmonic Numbers

\[\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \quad < \quad \ln n \quad < \quad 1 + \frac{1}{2} + \ldots + \frac{1}{n-1}\]

\[H_n - 1 \quad < \quad \ln n \quad < \quad H_n - \frac{1}{n}\]

Hence \(\ln n + \frac{1}{n} < H_n < \ln n + 1\), so \(H_n = \Theta(\log n)\).
Proof/Justification Techniques

Proof by Example

A statement of the form "There exists..." is true.
A statement of the form "For all..." is false.
A statement of the form "If P then Q" is false.

Illustration:
Consider the statement:

All numbers of the form $2^k - 1$ are prime.

This statement is False: $2^4 - 1 = 15 = 3 \cdot 5$.

Note: The statement can be rewritten as:
If $n$ is an integer of the form $2^k - 1$, then $n$ is prime.
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1. Direct proof: Assume P is true. Show that Q must be true.

2. Indirect proof: Assume Q is false. Show that P must be false. This is also known as a proof by contraposition.

3. Proof by contradiction: Assume P is true and Q is false. Show that there is a contradiction.

See [GT] Section 1.3.3 for examples.
Proof/Justification Techniques

Suppose we want to prove a statement of the form “If $P$ then $Q$” is true. There are three approaches:

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See [GT] Section 1.3.3 for examples.
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Proof/Justification Techniques: Induction
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- Let $P(n)$ be a statement with an integer parameter, $n$. 
  **Mathematical induction** is a technique for proving that $P(n)$ is true for all integers $\geq$ some base value $b$. 
Proof/Justification Techniques: Induction

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- Let $P(n)$ be a statement with an integer parameter, $n$. **Mathematical induction** is a technique for proving that $P(n)$ is true for all integers $\geq$ some base value $b$.
- Usually, the base value is 0 or 1.
Proof/Justification Techniques: Induction

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Proof/Justification Techniques:
Induction

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- Usually, the base value is 0 or 1.
- To show $P(n)$ holds for all $n \geq b$, we must show two things:
  1. Base Case: $P(b)$ is true (where $b$ is the base value).
  2. Inductive step: If $P(k)$ is true, then $P(k + 1)$ is true.
Example: Show that for all $n \geq 1$

$$\sum_{i=1}^{n} i \cdot 2^i = (n - 1) \cdot 2^{(n+1)} + 2$$
Induction Example

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Base Case: \( (n = 1) \)
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LHS
Induction Example

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$$\text{LHS} = \sum_{i=1}^{1} i \cdot 2^i = 1 \cdot 2^1 = 2.$$
Example: Show that for all \( n \geq 1 \)

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\sum_{i=1}^{n} i \cdot 2^i = (n - 1) \cdot 2^{(n+1)} + 2
\]

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\[
\text{LHS} = \sum_{i=1}^{1} i \cdot 2^i = 1 \cdot 2^1 = 2.
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RHS
Induction Example

Example: Show that for all $n \geq 1$

$$\sum_{i=1}^{n} i \cdot 2^i = (n - 1) \cdot 2^{(n+1)} + 2$$

Base Case: $(n = 1)$

LHS \hspace{1cm} RHS

$$\begin{align*}
\text{LHS} &= \sum_{i=1}^{1} i \cdot 2^i = 1 \cdot 2^1 = 2. \\
\text{RHS} &= (1 - 1) \cdot 2^{1+1} + 2 = 0 + 2 = 2.
\end{align*}$$
**Induction Example**

**Example:** Show that for all $n \geq 1$

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LHS = RHS

✓
Induction Example, continued

Inductive Step:

Assume $P(k)$ is true:

$$\sum_{i=1}^{k} i \cdot 2^i = (k-1) \cdot 2^{(k+1)} + 2.$$ 

Show $P(k+1)$ is true:

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2.$$
Induction Example, continued

Inductive Step:
Assume $P(k)$ is true:

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Assume: \[ \sum_{i=1}^{k} i \cdot 2^i = (k - 1) \cdot 2^{(k+1)} + 2. \]

Show: \[ \sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2. \]
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Show: \[ \sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2. \]

\[ \sum_{i=1}^{k+1} i \cdot 2^i = \sum_{i=1}^{k} i \cdot 2^i + (k + 1) \cdot 2^{(k+1)} \]
Induction Example, continued

Assume: \( \sum_{i=1}^{k} i \cdot 2^i = (k - 1) \cdot 2^{(k+1)} + 2. \)

Show: \( \sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2. \)

\[
\begin{align*}
\sum_{i=1}^{k+1} i \cdot 2^i &= \sum_{i=1}^{k} i \cdot 2^i + (k + 1) \cdot 2^{(k+1)} \\
&= (k - 1) \cdot 2^{(k+1)} + 2 + (k + 1) \cdot 2^{(k+1)} \\
&= k \cdot 2^{(k+2)} + 2
\end{align*}
\]
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&= 2k \cdot 2^{(k+1)} + 2
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\]
Probability

Defined in terms of a sample space, $S$.

Sample space consists of a finite set of outcomes, also called elementary events.

An event is a subset of the sample space. (So an event is a set of outcomes).

Sample space can be infinite, even uncountable. In this course, it will generally be finite.

Example: (2-coin example.) Flip two coins.

Sample space $S = \{HH, HT, TH, TT\}$.

The event "first coin is heads" is the subset $\{HH, HT\}$.
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**Example:** (2-coin example.) Flip two coins.

- Sample space \( S = \{HH, HT, TH, TT\} \).
- The event “first coin is heads” is the subset \( \{HH, HT\} \).
Probability function

A probability function $P(\cdot)$ that maps events (subsets of the sample space $S$) to real numbers such that:

1. $P(\emptyset) = 0$.
2. $P(S) = 1$.
3. For every event $A$, $0 \leq P(A) \leq 1$.
4. If $A, B \subseteq S$ and $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

Note: Property 4 implies that if $A \subseteq B$ then $P(A) \leq P(B)$.
A probability function is a function $P(\cdot)$ that maps events (subsets of the sample space $S$) to real numbers such that:

1. $P(\emptyset) = 0$.
2. $P(S) = 1$.
3. For every event $A$, $0 \leq P(A) \leq 1$.
4. If $A, B \subseteq S$ and $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

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Probability function (continued)

For finite sample spaces, this can be simplified:

- Sample space $S = \{s_1, \ldots, s_k\}$
- Each outcome $S_i$ is assigned a probability $P(s_i)$, with $\sum_{i=1}^{k} P(s_i) = 1$.

The probability of an event $E \subseteq S$ is:

$$P(E) = \sum_{s_i \in E} P(s_i).$$

Example: (2-coin example, continued). Define $P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$.

Then $P$ (first coin is heads) = $P(HH) + P(HT) = \frac{1}{2}$. 
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Random variables

Intuitive definition: a random variable is a variable whose value depends on the outcome of some experiment.

Formal definition: a random variable is a function that maps outcomes in a sample space $S$ to real numbers.

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- **Special case:** An Indicator variable is a random variable that is always either 0 or 1.
Expectation

The expected value, or expectation, of a random variable $X$ represents its "average value". Formally: Let $X$ be a random variable with a finite set of possible values $V = \{x_1, \ldots, x_k\}$. Then $E(X) = \sum_{x \in V} x \cdot P(X = x)$.

Example: (2-coin example, continued). Let $X$ be the number of heads when two coins are thrown. Then $E(X) = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) + 2 \cdot P(X = 2) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$. 
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$$= 0 \cdot \left(\frac{1}{4}\right) + 1 \cdot \left(\frac{1}{2}\right) + 2 \cdot \left(\frac{1}{4}\right)$$

$$= 1$$
Expectation

Example: Throw a single six-sided die. Assume the die is fair, so each possible throw has a probability of $\frac{1}{6}$.

The expected value of the throw is:

\[
1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5.
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Linearity of Expectation

For any two random variables $X$ and $Y$, 

$$E(X + Y) = E(X) + E(Y).$$

Proof: see [GT], 1.3.4

Very useful, because usually it is easier to compute $E(X)$ and $E(Y)$ and apply the formula than to compute $E(X + Y)$ directly.

Example 1: Throw two six-sided dice. Let $X$ be the sum of the values. Then 

$$E(X) = E(X_1 + X_2) = E(X_1) + E(X_2) = 3.5 + 3.5 = 7,$$

where $X_i$ is the value on die $i$ ($i = 1, 2$).

Example 2: Throw 100 six-sided dice. Let $Y$ be the sum of the values. Then 

$$E(Y) = 100 \cdot 3.5 = 350.$$
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Independent events

Two events $A_1$ and $A_2$ are independent iff

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2).$$

Example: (2-coin example, continued).

Let $A_1 = \text{coin 1 is heads} = \{HH, HT\}$

$A_2 = \text{coin 2 is tails} = \{HT, TT\}$

Then

$$P(A_1) = \frac{1}{2}, \quad P(A_2) = \frac{1}{2}, \quad \text{and} \quad P(A_1 \cap A_2) = P(HT) = \frac{1}{4} = P(A_1) \cdot P(A_2).$$

So $A_1$ and $A_2$ are independent.
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A collection of \( n \) events \( C = \{ A_1, A_2, \ldots, A_n \} \) is mutually independent (or simply independent) if:

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Example: Suppose we flip 10 coins. Suppose the flips are fair \( (P(H) = P(T) = 1/2) \) and independent. Then the probability of any particular sequence of flips (e.g., HHTTTHTHTH) is \( 1/(2^{10}) \).
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Example: Probability and counting

Example: Suppose we flip a coin 10 times. Suppose the flips are fair and independent. What is the probability of getting exactly 7 heads out of the 10 flips?

Solution:

▶ The outcomes consist of the set of possible sequences of 10 flips (e.g., HHTTTHTHTH).
▶ The probability of each outcome is $1/2^{10}$.
▶ The number of successful outcomes is $\binom{10}{7}$.
▶ Hence the probability of getting exactly 7 heads is:

$$\binom{10}{7} \cdot 2^{10} = 120 \cdot 1024 = 0.117.$$
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$$\frac{\binom{10}{7}}{2^{10}} = \frac{120}{1024} = 0.117.$$
An average-case result about finding the maximum

An algorithm to find the maximum:

\[ v = -\infty \]

\[
\begin{align*}
&\text{for } i = 0 \text{ to } n-1: \\
&\quad \text{if } A[i] > v: \\
&\qquad v = A[i]
\end{align*}
\]

return \( v \)

▶ Worst-case number of comparisons is \( n \).
▶ This can be reduced to \( n - 1 \).

How many times is the running maximum updated?

In the worst case \( n \).

What about the average case? . . .
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\[
\text{for } i = 0 \text{ to } n-1:\n\hspace{1cm} \text{if } A[i] > v:\n\hspace{2cm} v = A[i]\n\]

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  - What about the average case? ...
Average number of updates to the running maximum

Assume all possible orderings (permutations) of $A$ are equally likely.

All $n$ elements of $A$ are distinct.

The running maximum gets updated on iteration $i$ of the loop iff $\max\{A[0], \ldots, A[i]\} = A[i]$.

The probability of this happening is $1 / (i + 1)$.

Define indicator variables $X_i$: $X_i = \begin{cases} 1 & \text{if } v \text{ gets updated on iteration } i \\ 0 & \text{if } v \text{ does not get updated on iteration } i \end{cases}$

Then $E(X_i) = 1 / (i + 1)$.

The total number of times that $v$ gets updated is:

$$X = n - 1 \sum_{i=0}^{n-1} X_i$$
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- The total number of times that $v$ gets updated is:
  \[
  X = \sum_{i=0}^{n-1} X_i
  \]
Average number of updates to the running maximum (continued)

The expected total number of times that $v$ gets updated is:

$$E(X) = E(n - 1 \sum_{i=0}^{n-1} X_i) = n - 1 \sum_{i=0}^{n-1} E(X_i) = n - 1 \sum_{i=1}^{n} \frac{1}{i} = H_n = O(\log n)$$

It can be shown that $H_n = \ln n + \gamma + o(1)$, where $\gamma = 0.5772157\ldots$

If there are 30,000 elements in the list, the expected update count is about 10.9

If there are 3,000,000,000 elements in the list, the expected update count is about 22.4
Average number of updates to the running maximum (continued)

The expected total number of times that $v$ gets updated is:

$$E(X)$$
Average number of updates to the running maximum (continued)

The expected total number of times that \( v \) gets updated is:

\[
E(X) = E \left( \sum_{i=0}^{n-1} X_i \right)
\]

It can be shown that

\[
H_n = \ln n + \gamma + o(1),
\]

where \( \gamma = 0.5772157 \ldots \)

If there are 30,000 elements in the list, the expected update count is about 10.9
If there are 3,000,000,000 elements in the list, the expected update count is about 22.4
The expected total number of times that $v$ gets updated is:

$$E(X) = E\left(\sum_{i=0}^{n-1} X_i\right) = \sum_{i=0}^{n-1} E(X_i)$$
Average number of updates to the running maximum (continued)

The expected total number of times that \( v \) gets updated is:

\[
E(X) = E \left( \sum_{i=0}^{n-1} X_i \right) = \sum_{i=0}^{n-1} E(X_i) = \sum_{i=0}^{n-1} \frac{1}{i + 1}
\]

It can be shown that \( H_n = \ln n + \gamma + o(1) \), where \( \gamma = 0.5772157 \ldots \).

If there are 30,000 elements in the list, the expected update count is about 10.9.
If there are 3,000,000,000 elements in the list, the expected update count is about 22.4.
Average number of updates to the running maximum (continued)

The expected total number of times that $v$ gets updated is:

$$E(X) = E\left(\sum_{i=0}^{n-1} X_i\right) = \sum_{i=0}^{n-1} E(X_i) = \sum_{i=0}^{n-1} \frac{1}{i+1} = \sum_{i=1}^{n} \frac{1}{i}$$

If there are 30,000 elements in the list, the expected update count is about 10.9

If there are 3,000,000,000 elements in the list, the expected update count is about 22.4
Average number of updates to the running maximum (continued)

The expected total number of times that \( v \) gets updated is:

\[
E(X) = E \left( \sum_{i=0}^{n-1} X_i \right) = \sum_{i=0}^{n-1} E(X_i) = \sum_{i=0}^{n-1} \frac{1}{i+1} = \sum_{i=1}^{n} \frac{1}{i} = H_n
\]

It can be shown that \( H_n = \ln n + \gamma + o(1) \), where \( \gamma = 0.5772157 \ldots \)

If there are 30,000 elements in the list, the expected update count is about 10.9

If there are 3,000,000,000 elements in the list, the expected update count is about 22.4
The expected total number of times that $\nu$ gets updated is:

$$E(X) = E \left( \sum_{i=0}^{n-1} X_i \right) = \sum_{i=0}^{n-1} E(X_i) = \sum_{i=0}^{n-1} \frac{1}{i+1} = \sum_{i=1}^{n} \frac{1}{i} = H_n = O(\log n)$$

It can be shown that $H_n = \ln n + \gamma + o(1)$, where $\gamma = 0.5772157...$
Average number of updates to the running maximum (continued)

The expected total number of times that \( v \) gets updated is:

\[
E(X) = E\left(\sum_{i=0}^{n-1} X_i \right) = \sum_{i=0}^{n-1} E(X_i) = \sum_{i=0}^{n-1} \frac{1}{i+1} = \sum_{i=1}^{n} \frac{1}{i} = H_n = O(\log n)
\]

It can be shown that

\[
H_n = \ln n + \gamma + o(1), \quad \text{where } \gamma = 0.5772157\ldots
\]
The expected total number of times that $v$ gets updated is:

$$E(X) = E\left(\sum_{i=0}^{n-1} X_i\right) = \sum_{i=0}^{n-1} E(X_i) = \sum_{i=0}^{n-1} \frac{1}{i+1} = \sum_{i=1}^{n} \frac{1}{i} = H_n = O(\log n)$$

It can be shown that

$$H_n = \ln n + \gamma + o(1), \quad \text{where } \gamma = 0.5772157\ldots$$

If there are 30,000 elements in the list, the expected update count is about 10.9
Average number of updates to the running maximum (continued)

The expected total number of times that \( v \) gets updated is:

\[
E(X) = E\left(\sum_{i=0}^{n-1} X_i\right) = \sum_{i=0}^{n-1} E(X_i) = \sum_{i=0}^{n-1} \frac{1}{i+1} = \sum_{i=1}^{n} \frac{1}{i} = H_n = O(\log n)
\]

It can be shown that

\[
H_n = \ln n + \gamma + o(1), \quad \text{where } \gamma = 0.5772157 \ldots
\]

If there are 30,000 elements in the list, the expected update count is about 10.9

If there are 3,000,000,000 elements in the list, the expected update count is about 22.4