Lecture 2
Math overview

CS 161 Design and Analysis of Algorithms

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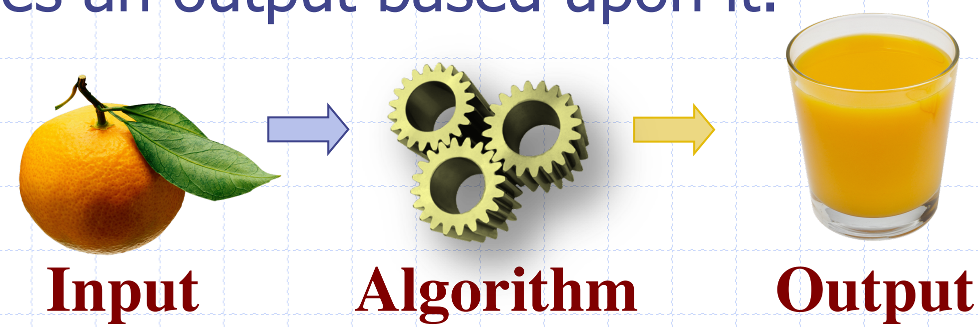
Algorithms and Data Structures

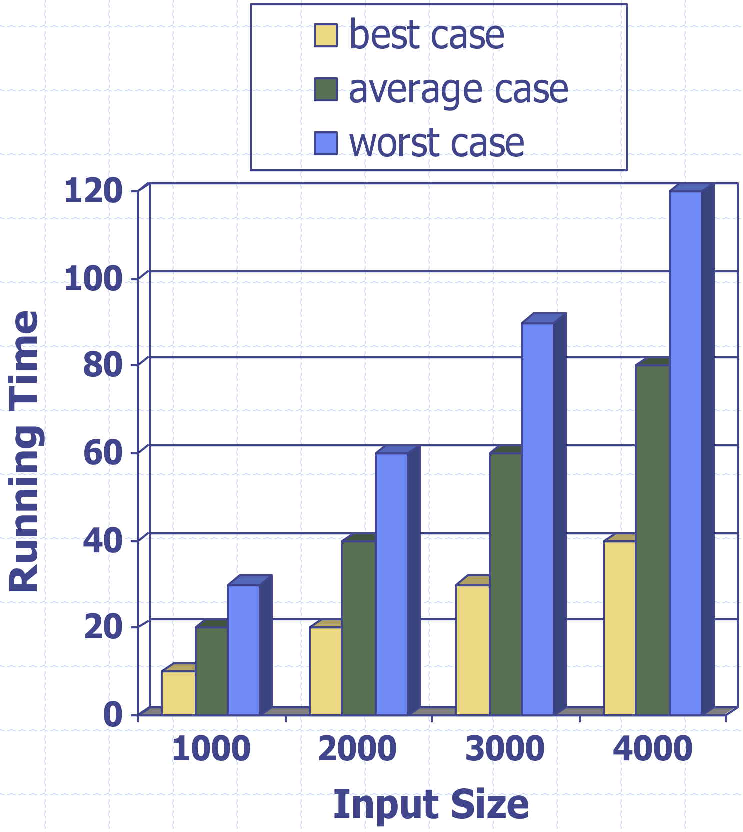
- An **algorithm** is a step-by-step procedure for performing some task in a finite amount of time.
  - Typically, an algorithm takes input data and produces an output based upon it.



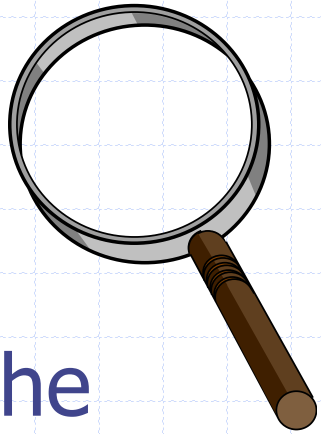
- A **data structure** is a systematic way of organizing and accessing data.

Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus primarily on the **worst case running time**.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



Theoretical Analysis



- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, n
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Pseudocode

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Pseudocode Details

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  Algorithm *method* (*arg* [, *arg*…])

  Input …

  Output …
- Method call

  method (*arg* [, *arg*…])
- Return value

  return *expression*
- Expressions:

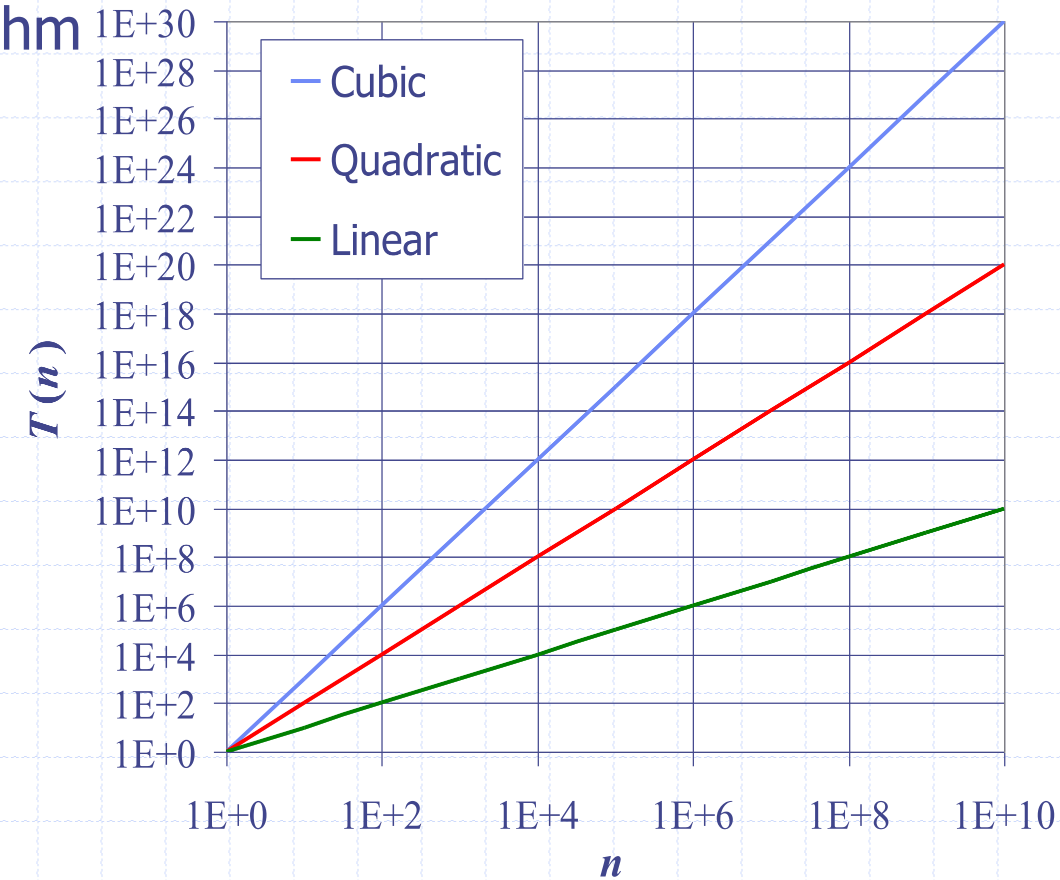
  $\leftarrow$ Assignment

  $=$ Equality testing

  n^2 Superscripts and other mathematical formatting allowed

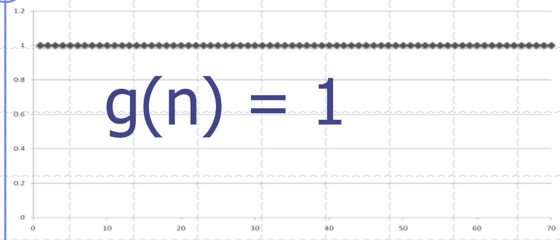
Seven Important Functions

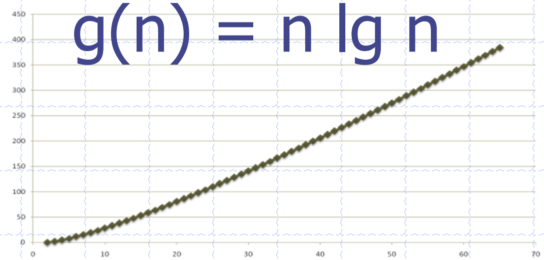
- Seven functions that often appear in algorithm analysis:
  - Constant ≈ 1
  - Logarithmic ≈ $\log n$
  - Linear ≈ n
  - N-Log-N ≈ $n \log n$
  - Quadratic ≈ n^2
  - Cubic ≈ n^3
  - Exponential ≈ 2^n
- In a log-log chart, the slope of the line corresponds to the growth rate

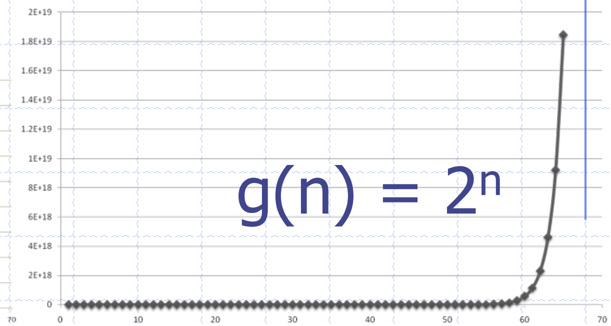


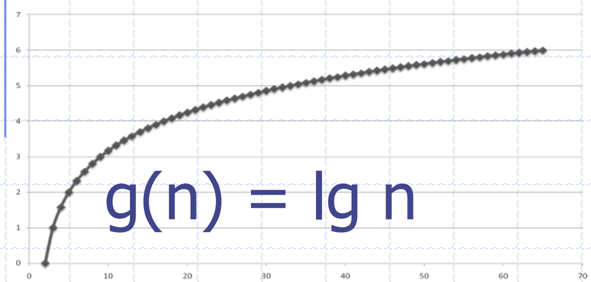
Functions Graphed Using "Normal" Scale

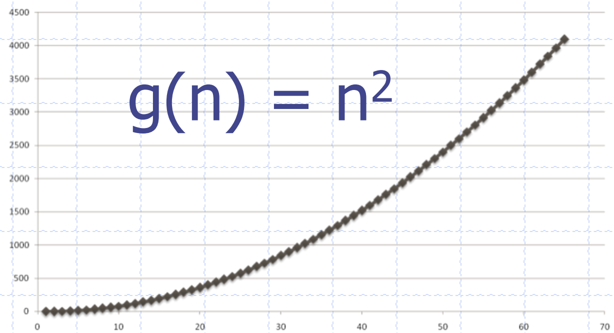
Slide by Matt Stallmann included with permission.

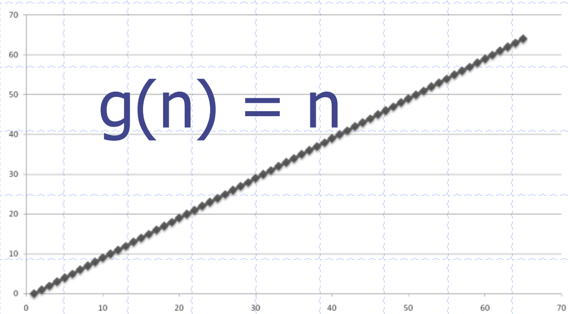


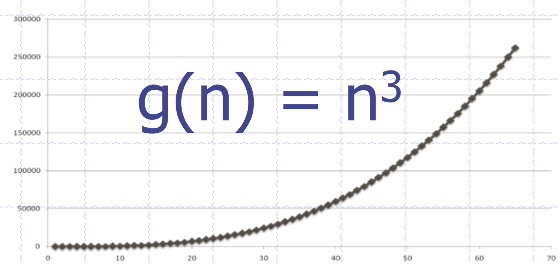












Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important
- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method

Counting Primitive Operations

- Example: By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

Algorithm arrayMax(A, n):

  Input: An array A storing $n \geq 1$ integers.

  Output: The maximum element in A.

$currentMax \leftarrow A[0]$

for $i \leftarrow 1$ **to** $n - 1$ **do**

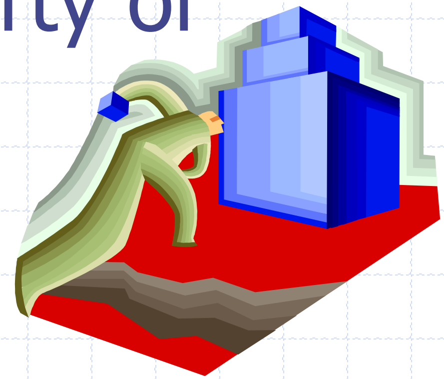
  if $currentMax < A[i]$ **then**

    $currentMax \leftarrow A[i]$

return $currentMax$

Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm arrayMax



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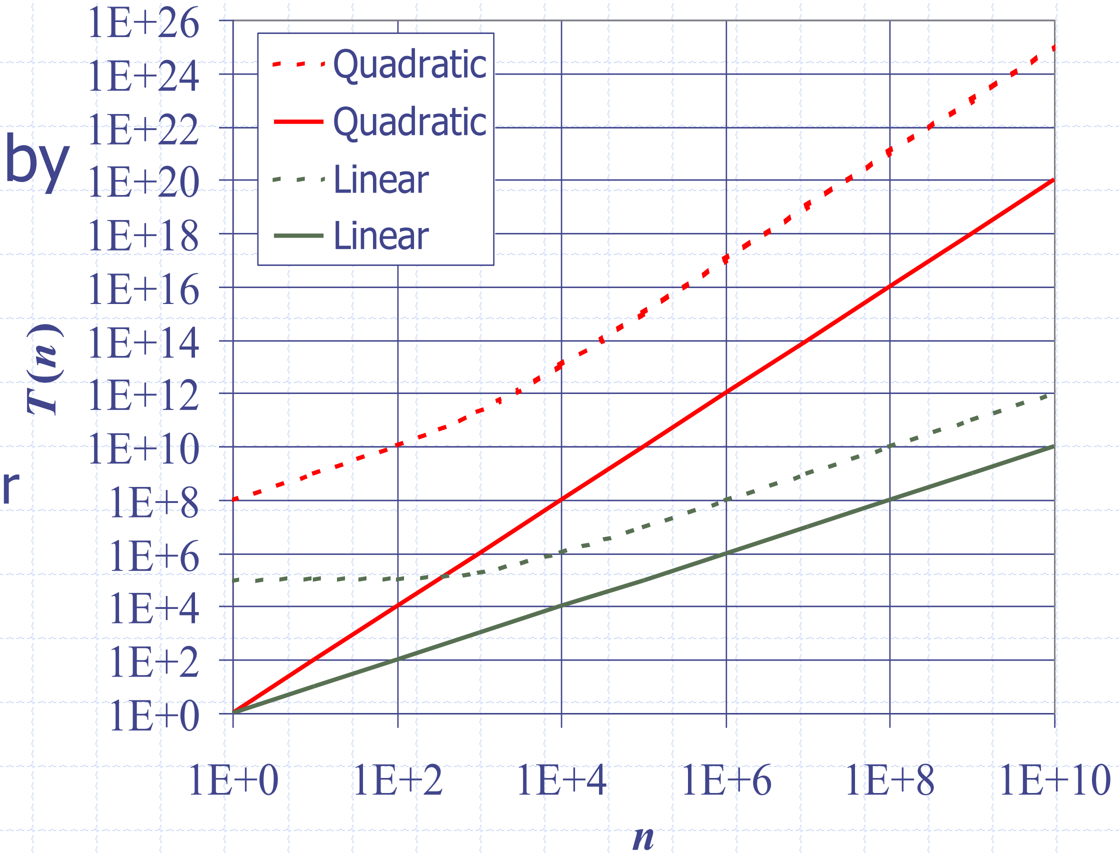
Why Growth Rate Matters

if runtime is...	time for n + 1	time for 2 n	time for 4 n
$c \lg n$	$c \lg (n + 1)$	$c (\lg n + 1)$	$c(\lg n + 2)$
$c\, n$	$c\, (n + 1)$	$2c\, n$	$4c\, n$
$c\, n \lg n$	$\sim c\, n \lg n + c\, n$	$2c\, n \lg n + 2cn$	$4c\, n \lg n + 4cn$
$c\, n^2$	$\sim c\, n^2 + 2c\, n$	**$4c\, n^2$**	$16c\, n^2$
$c\, n^3$	$\sim c\, n^3 + 3c\, n^2$	$8c\, n^3$	$64c\, n^3$
$c\, 2^n$	$c\, 2^{n+1}$	$c\, 2^{2n}$	$c\, 2^{4n}$

runtime quadruples when problem size doubles

Constant Factors

- The growth rate is minimally affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function



Asymptotic Algorithm Analysis

- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We say that algorithm arrayMax "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Big-Oh Rules

- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Analyzing Recursive Algorithms

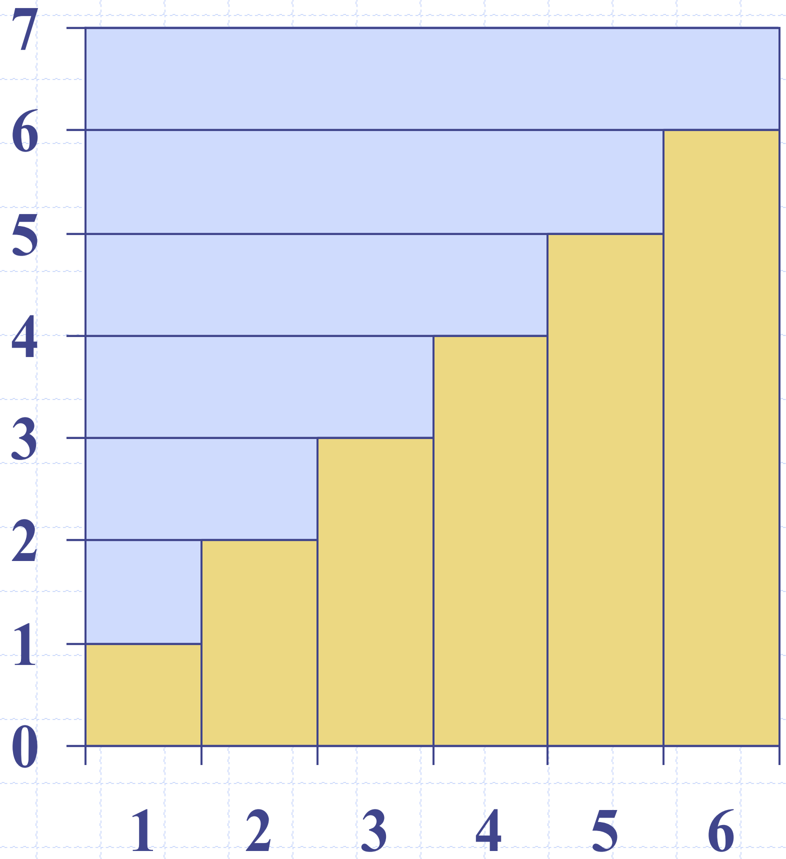
- Use a function, T(*n*), to derive a **recurrence relation** that characterizes the running time of the algorithm in terms of smaller values of *n*.

Algorithm recursiveMax(A, n):
    Input: An array A storing $n \geq 1$ integers.
    Output: The maximum element in A.
  if $n = 1$ **then**
      return $A[0]$
  return $\max\{$recursiveMax$(A, n-1),\ A[n-1]\}$

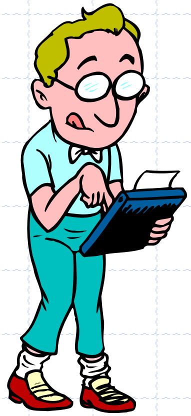
$$T(n) = \begin{cases} 3 & \text{if } n = 1 \\ T(n-1) + 7 & \text{otherwise,} \end{cases}$$

Arithmetic Progression

- Assume the running time of P is $O(1 + 2 + \ldots + n)$
- The sum of the first n integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm P runs in $O(n^2)$ time



Math you need to Review



- Summations
- Powers
- Logarithms
- Proof techniques
- Basic probability

- Properties of powers:

  $a^{(b+c)} = a^b a^c$

  $a^{bc} = (a^b)^c$

  $a^b / a^c = a^{(b-c)}$

  $b = a^{\log_a b}$

  $b^c = a^{c*\log_a b}$

- Properties of logarithms:

  $\log_b(xy) = \log_b x + \log_b y$

  $\log_b (x/y) = \log_b x - \log_b y$

  $\log_b xa = a\log_b x$

  $\log_b a = \log_x a / \log_x b$

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- $g \in O(f)$ if "g grows no faster than (a constant multiple of) f."
- $g \in O(f)$ if the ratio g/f is bounded above by a constant (for sufficiently values of n).

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- Sometimes we write: $g = O(f)$ rather than $g \in O(f)$

Examples of O-notation:

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Proof: Let $C = 1000$. Then $g(n) \leq C \cdot f(n)$ for all n.

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Hence we can choose $C = 1$ and $n_0 = 1$.

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Proof: $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$.
Hence there is no $C > 0$ such that $g(n) \leq C \cdot f(n)$ for sufficiently large n.

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So we can take $C = 30$, $n_0 = 1$.

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To see that $g \in \Theta(f)$, we could argue that $g \in O(f)$ (shown earlier) and $g \in \Omega(f)$ (shown above).

Or we can take $C_1 = 1$, $C_2 = 1000$. Then

$$C_1 \leq \frac{g(n)}{f(n} \leq C_2.$$

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So we can take $C = 4$, $n_0 = 23$.

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(Continued on next slide)

Example 5, continued: $\ln n = o(n)$

L'Hôpital's rule: If the ratio of limits

$$\frac{\lim_{n\to\infty} g(n)}{\lim_{n\to\infty} f(n)}$$

is an indeterminate form (i.e., ∞/∞ or $0/0$), then

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{g'(n)}{f'(n)}$$

where f' and g' are, respectively, the derivatives of f and g.

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Hence $g(n) = o(f(n))$.

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4. $\log_b(x^a) = a \log_b x$.
5. $x^{\log_b y} = y^{\log_b x}$.
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Exercise: Prove the above properties.

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Special cases: $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = \frac{n(n-1)}{2}$

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These numbers go to infinity:

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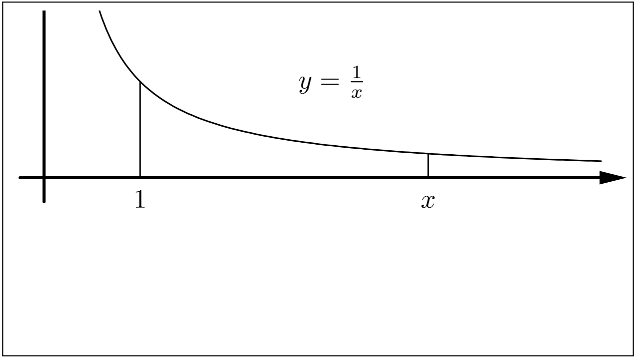

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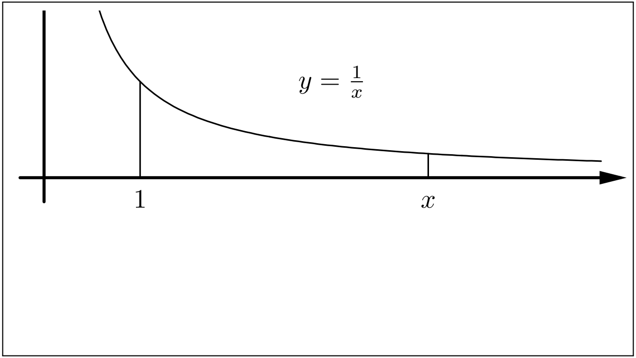

We will show that $H_n = \Theta(\log n)$.

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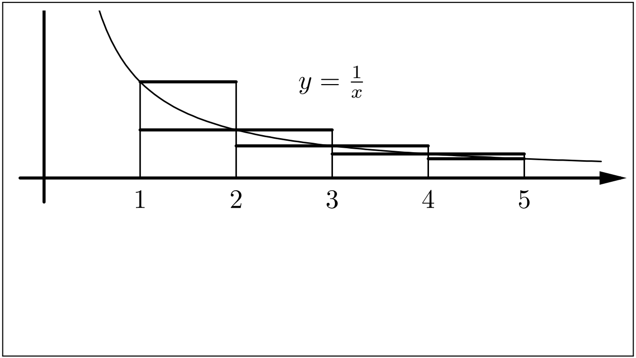

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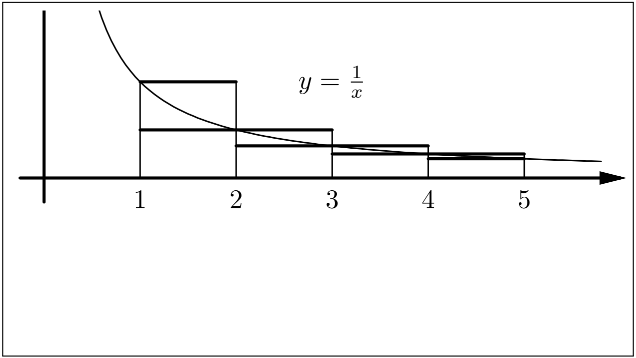

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \quad < \quad \ln n \quad < \quad 1 + \frac{1}{2} + \ldots + \frac{1}{n-1}$$

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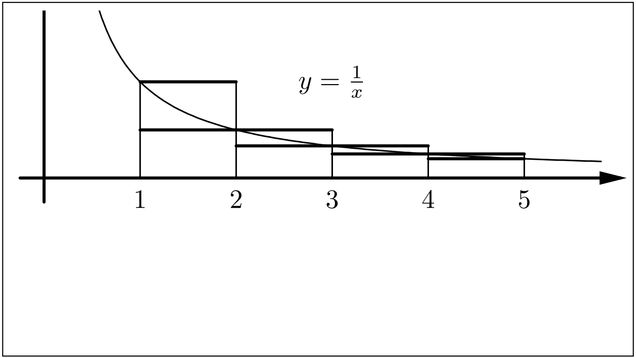

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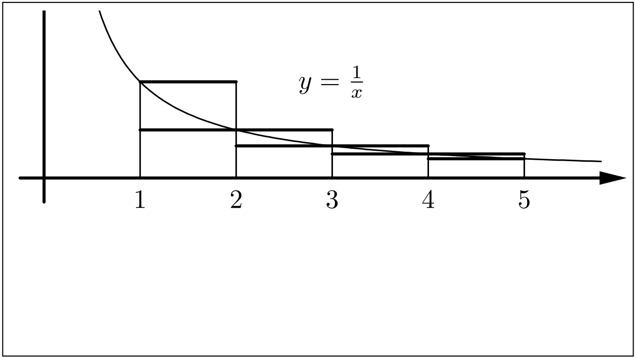

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Hence $\ln n + \frac{1}{n} < H_n < \ln n + 1$, so $H_n = \Theta(\log n)$.

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  - A statement of the form "There exists..." is true.
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  - A statement of the form "If P then Q" is false.
- Illustration: Consider the statement:

  All numbers of the form $2^k - 1$ *are prime.*

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  This statement is False: $2^4 - 1 = 15 = 3 \cdot 5$
- Note: The statement can be rewritten as:

  If n is an integer of the form $2^k - 1$, then n is prime.

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  See [GT] Section 1.3.3 for examples.

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  2. Inductive step: If $P(k)$ is true, then $P(k+1)$ is true.

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$$\sum_{i=1}^{n} i \cdot 2^i = (n-1) \cdot 2^{(n+1)} + 2$$

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- Note: Property 4 implies that if $A \subseteq B$ then $P(A) \leq P(B)$.

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Example: (2-coin example, continued). Let X be the number of heads when two coins are thrown. Then

$$\begin{aligned} E(X) &= 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) \\ &= 0 \cdot \left(\frac{1}{4}\right) + 1 \cdot \left(\frac{1}{2}\right) + 2 \cdot \left(\frac{1}{4}\right) \\ &= 1 \end{aligned}$$

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$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

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Example 1: Throw two six-sided dice. Let X be the sum of the values. Then

$$E(X)=E(X_1+X_2)=E(X_1)+E(X_2)=3.5+3.5=7,$$

where X_i is the value on die i ($i=1,2$).

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Example 2: Throw 100 six-sided dice. Let Y be the sum of the values. Then

$$E(Y) = 100 \cdot 3.5 = 350.$$

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Then $P(A_1) = \frac{1}{2}$, $P(A_2) = \frac{1}{2}$, and

$$P(A_1 \cap A_2) = P(\mathtt{HT}) = \frac{1}{4} = P(A_1) \cdot P(A_2).$$

So A_1 and A_2 are independent.

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A collection of n events $C = \{A_1, A_2, \ldots, A_n\}$ is mutually independent (or simply independent) if:

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Example: Suppose we flip 10 coins. Suppose the flips are fair ($P(\mathtt{H}) = P(\mathtt{T}) = 1/2$) and independent. Then the probability of any particular sequence of flips (e.g., `HHTTTHTHTH`) is $1/(2^{10})$.

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- The probability of each outcome is $1/(2^{10})$.
- The number of successful outcomes is $\binom{10}{7}$.
- Hence the probability of getting exactly 7 heads is:

$$\frac{\binom{10}{7}}{2^{10}} = \frac{120}{1024} = 0.117.$$

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- How many times is the running maximum updated?
  - In the worst case n.

An average-case result about finding the maximum

```
v = -∞
for i = 0 to n-1:
   if A[i] > v:
      v = A[i]
return v
```

- Worst-case number of comparisons is n.
- This can be reduced to $n - 1$
- How many times is the running maximum updated?
  - In the worst case n.
  - What about the average case? ...

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If there are 3,000,000,000 elements in the list, the expected update count is about 22.4