Lecture 6

Divide and Conquer IV: integer multiplication, further examples

CS 161 Design and Analysis of Algorithms

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Divide and Conquer (recap)

Steps of method:

- **Divide** input into parts (smaller problems)
- **Conquer** (solve) each part recursively
- **Combine** results to obtain solution of original

$$T(n) = \text{divide time}$$
$$+ T(n_1) + T(n_2) + \ldots + T(n_k)$$
$$+ \text{combine time}$$

Case study VII: Computing powers

Problem: Given two positive integers numbers a, n compute a^n .

Example: $a = 3, n = 4$. Answer: 81.

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Obvious approach:

$$\text{ans} \leftarrow 1$$
$$\textbf{For } i = 1 \text{ to } n \textbf{ do}$$
$$\quad \text{ans} \leftarrow a \cdot \text{ans}$$
$$\textbf{return } \text{ans}$$

$\Theta(n)$ operations

Can we do better?

Case study VII: Computing powers

Problem: Given two positive integers numbers a, n compute a^n .

Example: $a = 3, n = 4$. Answer: 81.

Idea: Divide and Conquer.

Divide n in $n/2$ and $n/2$. Compute x = $a^{n/2}$ recursively. Return x^2.
Be careful on the parity of n.

Case study VII: Computing powers

Problem: Given two positive integers numbers a, n compute a^n .

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Idea: Divide and Conquer.

$$
\begin{array}{l}
\text{Power}(a, n) \\
\textbf{If } n == 1 \textbf{ then return } \text{a} \\
x \leftarrow \text{Pow}(a, \lfloor n/2 \rfloor) \\
\textbf{If } n \bmod 2 == 0 \textbf{ then} \\
\quad \textbf{return } x \cdot x \\
\textbf{else return } \; a \cdot x \cdot x
\end{array}
$$

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else return $a \cdot x \cdot x$

Base case

Divide + Conquer

Combine

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Running time: $T(n) = T\left(\frac{n}{2}\right) + \Theta(1) \rightarrow \Theta(\log n)$ by Master thm

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Idea: Divide and Conquer.

Remark: Same works for powers of Matrices.

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Case study VIII: Fibonacci sequence

Problem: Given a positive integer numbers n, compute Fibonacci F_n .

Definition: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

First 10 numbers of sequence: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55$

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Obvious approach:

$\text{ans1} \leftarrow 1$
$\text{ans2} \leftarrow 1$
If $n \leq 2$ **then return** 1
For $i = 3$ to n **do**
  $\text{temp} \leftarrow \text{ans1}$
  $\text{ans1} \leftarrow \text{ans1} + \text{ans2}$
  $\text{ans2} \leftarrow \text{temp}$
return ans

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Idea: Express F_n as a power of a Matrix.

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

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$$\vdots$$

$$\begin{pmatrix} F_3 \\ F_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} F_2 \\ F_1 \end{pmatrix}$$

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$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

F_n is $a + b$ and F_{n-1} is $c + d$!

Case study VIII: Fibonacci sequence

Problem: Given a positive integer numbers n, compute Fibonacci F_n .

Solution:

Compute matrix $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^{\boldsymbol{n-2}}$ in $\boldsymbol{\Theta}(\log \boldsymbol{n})$ time.

Return the sum of the entries of first row.

Case study IX: From practice problems

Problem: Suppose you have an array A of n intervals $(x_1, y_1), \ldots, (x_n, y_n)$, where x_i, y_i are positive integers such that $x_i \leq y_i$. The interval (x_i, y_i) represents the set of integers between x_i and y_i. For example, the interval $(3, 8)$ represents the set $\{3, 4, 5, 6,7,8\}$.

Define the **overlap** of two intervals to be the number of integers that are members of both intervals. For example $(3, 8)$ and $(4, 9)$ have overlap 5 (numbers $4, 5, 6, 7, 8$) and $(1, 2)$ and $(3, 4)$ have overlap 0. Find the size of maximum overlap among all possible pairs of intervals.

Example: $(1, 2), (3, 4), (3, 8), (4, 9)$. Answer: 5.

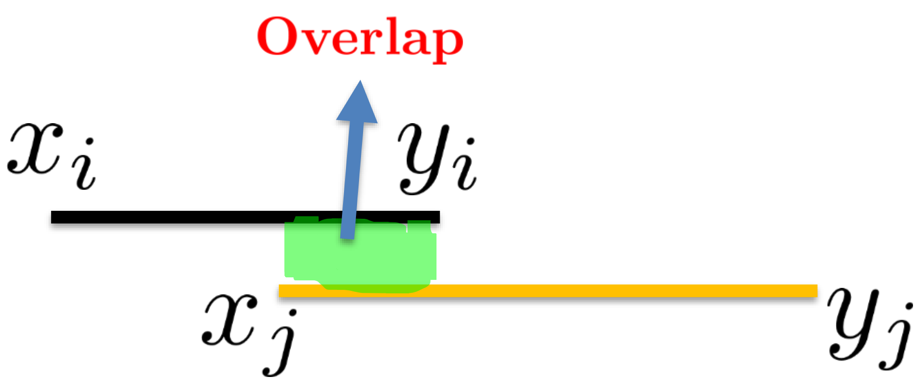
Case study IX: From practice problems

Obvious approach: For every pair i, j of intervals, find the overlap. Keep the maximum.

Suppose $x_i \leq x_j$.

(x_i, y_i) and (x_j, y_j) have overlap

$$\max\left(\min(y_i, y_j) - x_j + 1, 0\right).$$



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Suppose $x_i \leq x_j$.

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$\Theta(n^2)$ running time

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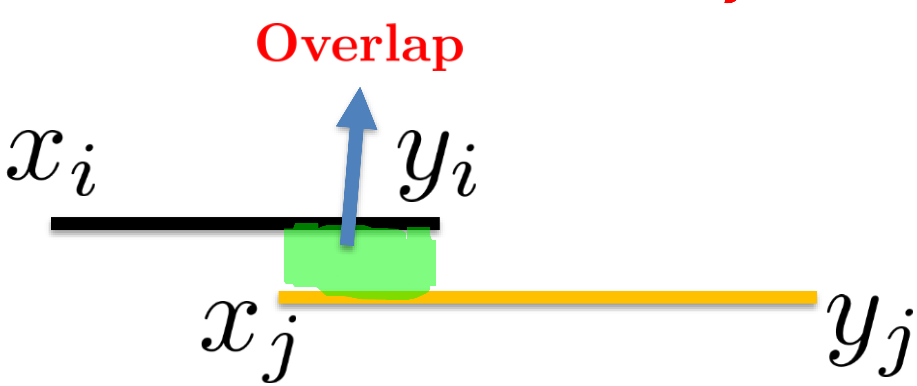

Can we do better?

Case study IX: From practice problems

Idea: Use divide and conquer. Suppose we first sort the intervals in increasing order of x-coordinate.

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- Divide the intervals in two parts L and R.
- Recursively find max overlap for each part maxL and maxR.
- Combine step?

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Idea: Use divide and conquer. Suppose we first sort the intervals in increasing order of x-coordinate.

- Divide the intervals in two parts L and R.
- Recursively find max overlap for each part maxL and maxR.
- Combine step: maximum of maxL and maxR?

Case study IX: From practice problems

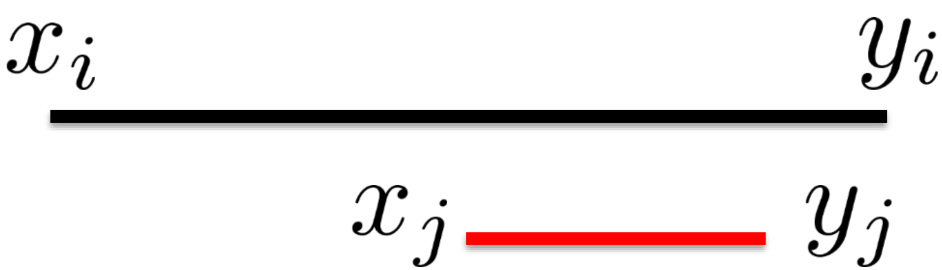
Idea: Use divide and conquer. Suppose we first sort the intervals in increasing order of x-coordinate.

- Divide the intervals in two parts L and R.
- Recursively find max overlap for each part maxL and maxR.
- Combine step: Check overlap between an interval in L and an interval in R. This should be in $\Theta(n)$.

We will scan the intervals once. One index for L and one index for R.

Case study IX: From practice problems

Combine step: Black is in L, red in R.



Overlap is $(y_j - x_j + 1)$. We can remove interval j from R.

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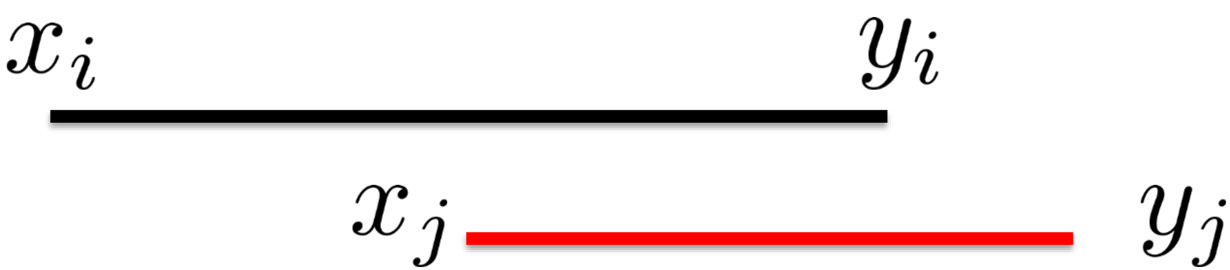

Overlap is $(y_i - x_j + 1)$. We can remove interval i from L.

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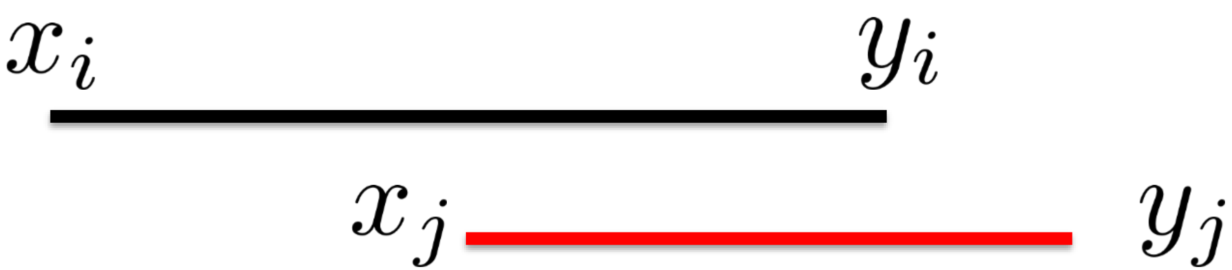

Overlap is $(y_i - x_j + 1)$. We can remove interval i from L.

All intervals after j in R will not give larger overlap with interval i.

Case study IX: From practice problems

Pseudocode:

$\text{Maxoverlap}(A[1:n])$

$\textbf{If } n == 1 \textbf{ return } 0$

$\textbf{maxL} \leftarrow \text{Maxoverlap}(A[1:n/2])$

$\textbf{maxR} \leftarrow \text{Maxoverlap}(A[n/2+1:n])$

$\textbf{maxComb} \leftarrow 0$

$i \leftarrow 1,\ j \leftarrow n/2+1$

$\textbf{While } i \leq n/2 \text{ and } j \leq n \textbf{ do}$

$\quad \textbf{If } \text{maxComb} < \text{overlap}(i,j) \textbf{ then}$

$\quad\quad \text{maxComb} = \text{overlap}(i,j)$

$\quad \textbf{If } \text{case } 1 \textbf{ then } j \leftarrow j+1$

$\quad \textbf{else If } \text{case } 2 \textbf{ then } i \leftarrow i+1$

$\textbf{return } \text{maximum of maxL, maxR and maxComb}$

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$\textbf{maxR} \leftarrow \text{Maxoverlap}(A[n/2+1:n])$ — $T(n/2)$ Running time

$\textbf{maxComb} \leftarrow 0$

$i \leftarrow 1,\ j \leftarrow n/2+1$

$\textbf{While } i \leq n/2 \text{ and } j \leq n \textbf{ do}$ — $\Theta(n)$ Running time

$\quad \textbf{If } \text{maxComb} < \text{overlap}(i,j) \textbf{ then}$

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Pseudocode: $\Theta(n \log n)$ Running time

$\text{Maxoverlap}(A[1:n])$

- **If** $n == 1$ **return** 0
- **maxL** $\leftarrow \text{Maxoverlap}(A[1:n/2])$ — $T(n/2)$ Running time
- **maxR** $\leftarrow \text{Maxoverlap}(A[n/2+1:n])$ — $T(n/2)$ Running time
- **maxComb** $\leftarrow 0$
- $i \leftarrow 1,\ j \leftarrow n/2+1$
- **While** $i \leq n/2$ and $j \leq n$ **do** — $\Theta(n)$ Running time
  - **If** maxComb $<$ overlap(i,j) **then**
    - maxComb $=$ overlap(i,j)
  - **If** case 1 **then** $j \leftarrow j+1$
  - **else If** case 2 **then** $i \leftarrow i+1$
- **return** maximum of maxL, maxR and maxComb

Case study X: Median from sorted

Exercise: Given two sorted arrays A, B of size n each, find the median of among the $2n$ numbers in $O(\log n)$ time.

Hint: Compare $A[n/2]$ with $B[n/2]$.