Lecture 18

P, NP and reductions

CS 161 Design and Analysis of Algorithms

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Different time complexities

Different algorithms can have different time complexities.

Some common complexity classes	Notation (input size $= n$)
Constant	$O(1)$
Logarithmic	$O(\log n)$
Linear	$O(n)$
Log-linear	$O(n \log n)$
Quadratic	$O(n^2)$
Cubic	$O(n^3)$
Exponential	$O(e^n)$
Factorial	$O(n!)$
Doubly-exponential	$O(e^{e^n})$

Polynomial time (Constant through Cubic)

We say an algorithm runs in **polynomial time** if its time complexity is $\boldsymbol{O(n^c)}$ for some constant c.

P and NP

Given a decision problem A (output yes/no), there could be many possible solutions, with possibly different time complexities.

The class P: We say **can be solved in polynomial time** or belongs in P if there exist at least one algorithm that solves the problem and runs in **polynomial** time.

P and NP

Given a decision problem $\boldsymbol{A}$ (output yes/no), there could be many possible solutions, with possibly different time complexities.

The class P: We say **can be solved in polynomial time** or belongs in $\boldsymbol{P}$ if there exist at least one algorithm that solves the problem and runs in **polynomial** time.

The class NP: It stands for Non-deterministic polynomial time.

In high level, if the answer is "yes", it can be verified in polynomial time.

Example: "Given a number x, is it composite?"

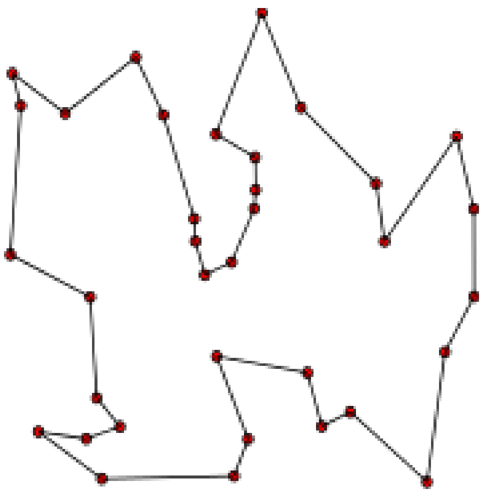
Example: "Given a graph $G(V, E)$, does it contain a cycle?".

Optimization Problems

Problem: The traveling salesman problem

Given a list of cities and the distances between each pair of cities, what is a shortest possible route that visits each city exactly once and returns to the origin city?

- If there are n cities, then the "best" known solution uses dynamic programming and has time complexity $O(n^2 2^n)$.
- "best" solution ≈ brute-force search + dynamic programming



This problem is <u>suspected</u> to be not solvable in polynomial time.

- We still do not know…

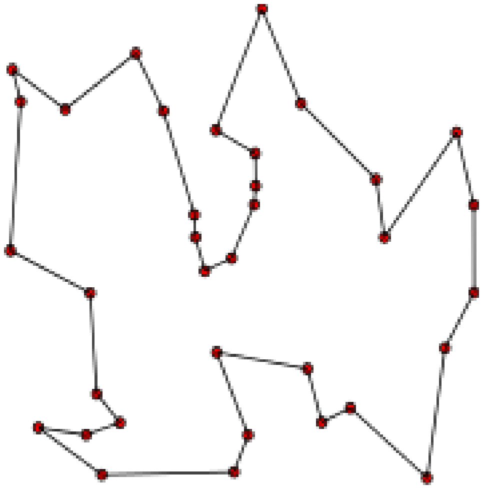
Other example: 0/1 Knapsack problem.

Convert optimization to decision problems

Problem: The traveling salesman problem

Given a list of cities and the distances between each pair of cities, is there a route of length at most k that visits each city exactly once and returns to the origin city?

- If there are n cities, then the "best" known solution uses dynamic programming and has time complexity $O(n^2 2^n)$.
- "best" solution ≈ brute-force search + dynamic programming



This problem belongs to NP. Why?

Unsolvable problems?

Question: Are there unsolvable computational problems?

There are examples of unsolvable problems.

- The most famous one is called the **halting problem**.

The Halting Problem:

Given a computer program $\mathbf{\Pi}$ and some input I, determine whether $\mathbf{\Pi}$ will terminate when executed with input I.

- This is a decision (yes/no) problem. The answer to the halting problem is either yes or no.
  - Yes, if $\mathbf{\Pi}$ terminates.
  - No, if $\mathbf{\Pi}$ runs forever (e.g. enters an infinite loop).
- If I is not a valid input for $\mathbf{\Pi}$, then $\mathbf{\Pi}$ executed with input I will terminate with an error message.

How do we show a problem is not in P?

Question: How can we prove that a problem is not in P?

- **Short answer:** For many problems, we don't know how!

Current Status: We do not know of any general method that works on all problems, that can prove that a problem is not in $\boldsymbol{P}$.

- In fact, we do not even know of any general method that can prove that a problem is not solvable in linear time.

- We can characterize their computational difficulty using **reductions**.

The idea of reductions

There are so many different computational problems that we may want to solve.

- Do we have to solve every one of these problems from scratch?

Key Idea of reductions

Given a Problem $\boldsymbol{A}$ that we want to solve, and suppose there is another Problem $\boldsymbol{B}$ that we already know how to solve.

- If we can reformulate Problem $\boldsymbol{A}$ to "look like" Problem $\boldsymbol{B}$, so that by solving Problem $\boldsymbol{B}$, we are able to solve Problem $\boldsymbol{A}$.

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Example: $\boldsymbol{A}$ = maximum matching and $\boldsymbol{B}$ = Maxflow.

- Then we say that we have **reduced** Problem $\boldsymbol{A}$ to Problem $\boldsymbol{B}$.
- Problem $\boldsymbol{B}$ is at least as hard as Problem $\boldsymbol{A}$.

NP-complete problems

NP-complete: A problem ***A*** is NP-complete if

1. Belongs in NP
2. Any other problem in NP reduces in poly-time to ***A***. In other words, ***A*** is **NP-hard**.

What does this mean? ***A*** is the **"hardest"** problem in class NP.

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In 1971, the **first** NP-complete problem appears.

Theorem: The **3-SAT** problem is NP-complete. (Cook–Levin's Thm, 1971)

3-SAT is NP-complete

Problem: 3-SAT

Given a Boolean expression E, such that E is a conjunction of clauses, where each clause is a disjunction of exactly 3 literals, is E satisfiable?

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A **literal** is a Boolean expression consisting of just a single Boolean variable, or the negation of a Boolean variable.

- Example: "$\bar{x}_1$" and "x_2" are literals.

A **clause** is a Boolean expression of the form "$\ell_1 \vee \ell_2 \vee \cdots \vee \ell_k$", i.e. a **disjunction** of some literals $\ell_1, \ell_2, \ldots, \ell_k$. In 3-SAT $k = 3$.

- Example: "$\mathrm{C}_1 \equiv x_1 \vee \bar{x}_2 \vee x_3$" is a clause.

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$(x_1 \vee \bar{x}_2 \vee \bar{x}_3 \vee F)$

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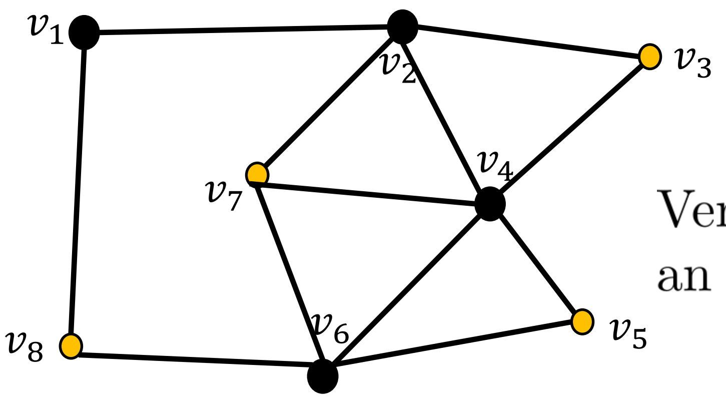
A Boolean expression is a conjunction of clauses.

Example: $(\boldsymbol{x_1} \vee \bar{\boldsymbol{x}}_2 \vee \bar{\boldsymbol{x}}_3) \wedge (\bar{\boldsymbol{x}}_1 \vee \boldsymbol{x_2} \vee \boldsymbol{x_3}) \wedge (\bar{\boldsymbol{x}}_1 \vee \boldsymbol{x_2} \vee \bar{\boldsymbol{x}}_3) \wedge (\boldsymbol{x_1} \vee \bar{\boldsymbol{x}}_2 \vee \boldsymbol{x_3})$

Reductions in NP

Example: **INDEPENDENT SET (IS) Problem**

Given a simple undirected graph $G(V, E)$ and k, is there an independent set in G of size $\geq k$? Independent set is called a set $I \subset V$ of vertices such that pairwise the vertices in I do not share an edge.



Graph G.

Vertices v_3, v_5, v_7, v_8 form an independent set.

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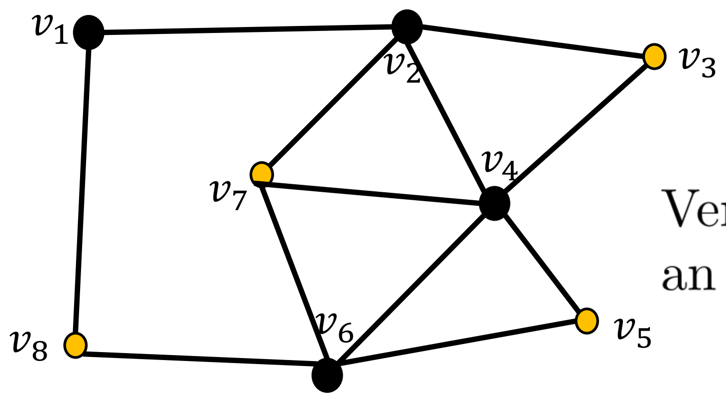

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Vertices v_3, v_5, v_7, v_8 form an independent set.

Claim: INDEPENDENT SET is **NP-complete**.

Proof: (1) INDEPENDENT SET **belongs** to NP (why?).
(2) Reduce 3-SAT to INDEPENDENT SET. Since 3-SAT is NP-hard, INDEPENDENT SET is NP-hard.

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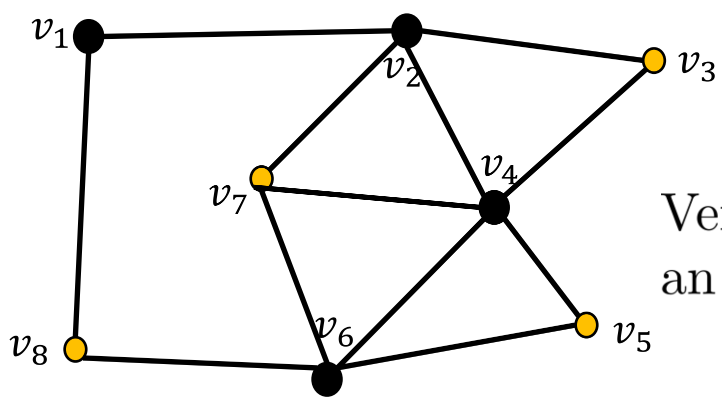

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(1), (2) imply IND. SET is NP-complete!

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3-SAT reduction to IS

3-SAT instance: Can you assign True, False to the variables of the formula below so that the expression is True?

$$E = (x_1 \lor \bar{x}_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_2 \lor x_3) \land (\bar{x}_1 \lor x_2 \lor \bar{x}_3) \land (x_1 \lor \bar{x}_2 \lor x_3)$$

Let's reduce the above to an IS instance. We need a graph!

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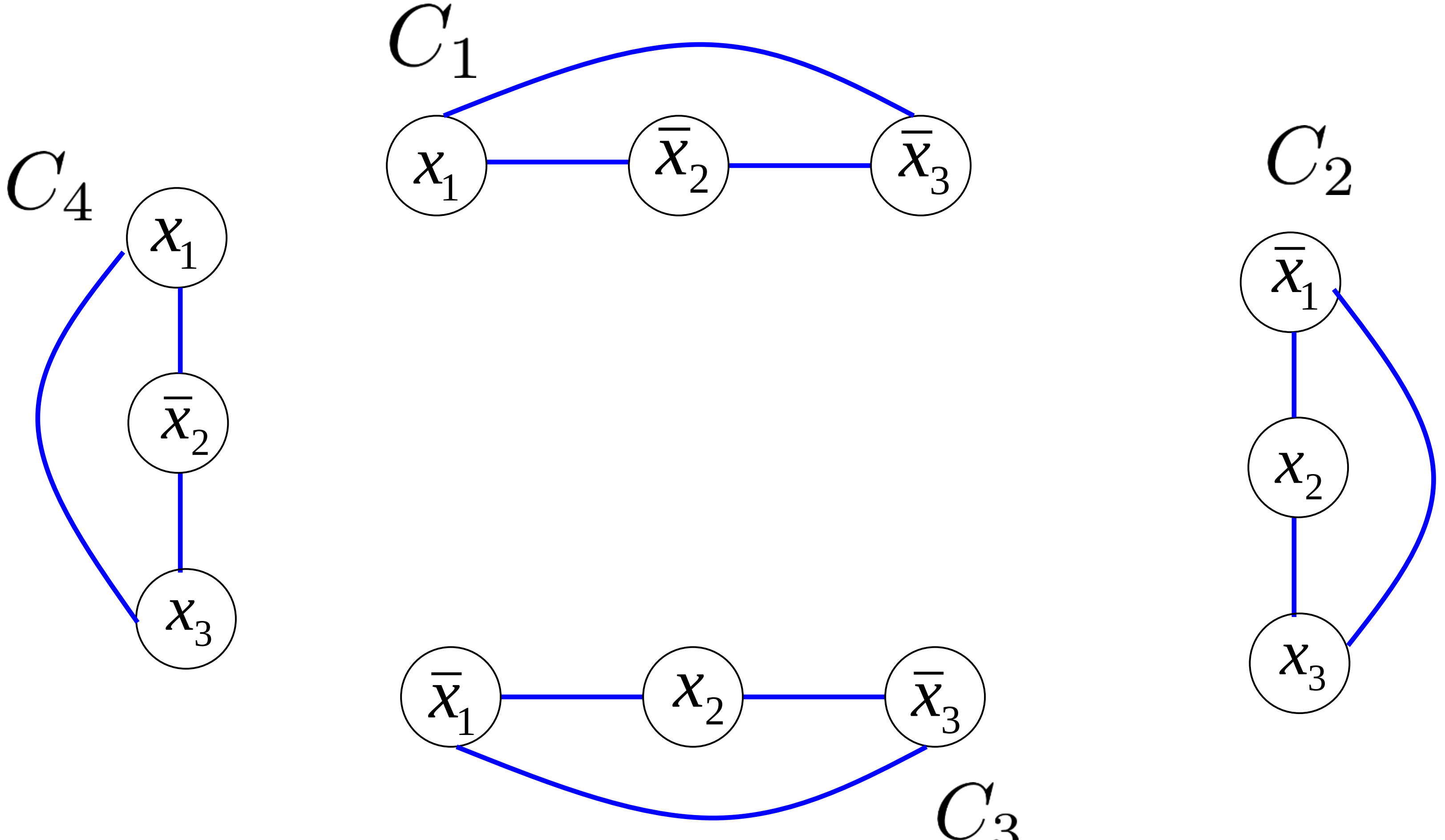

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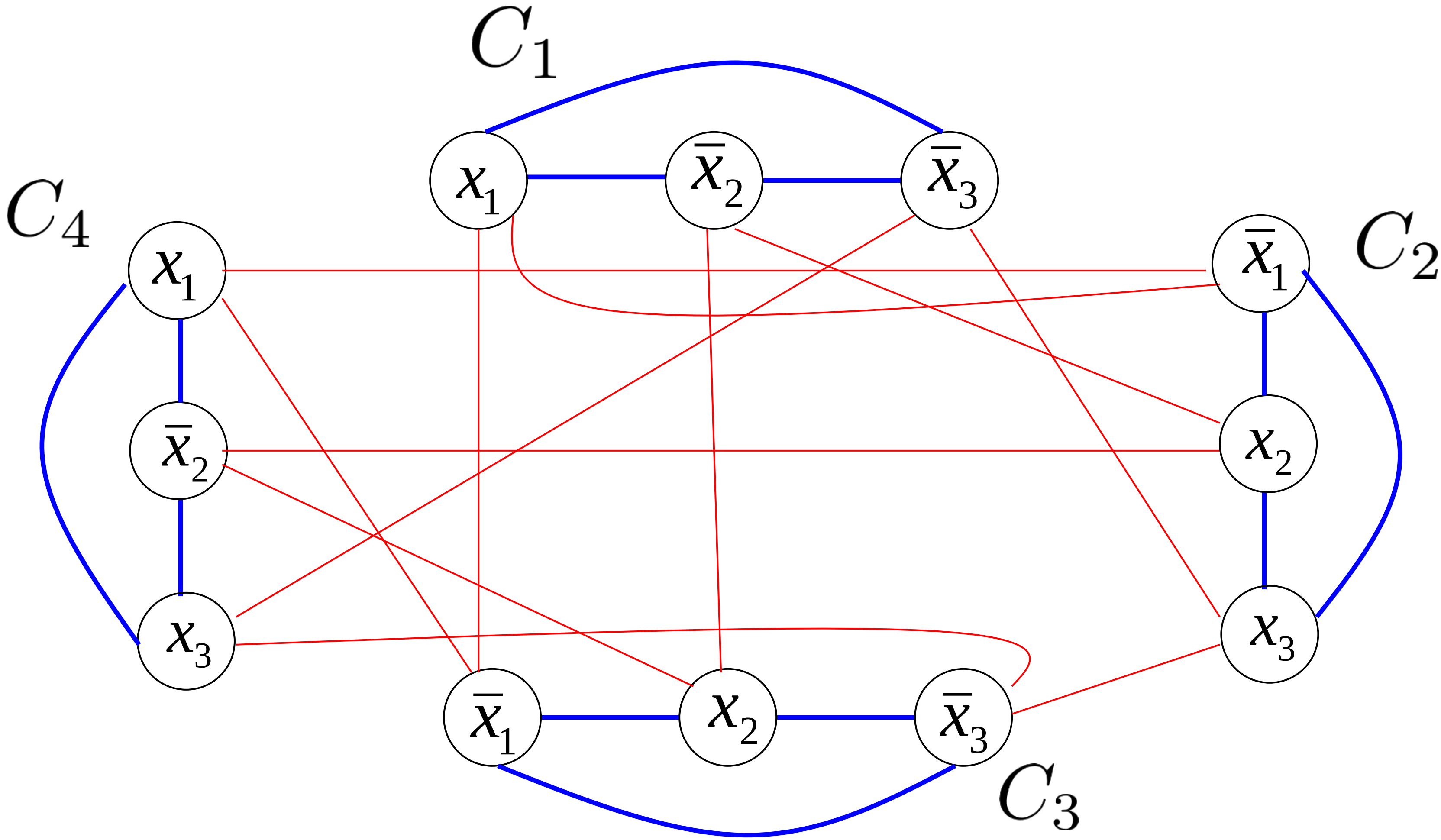

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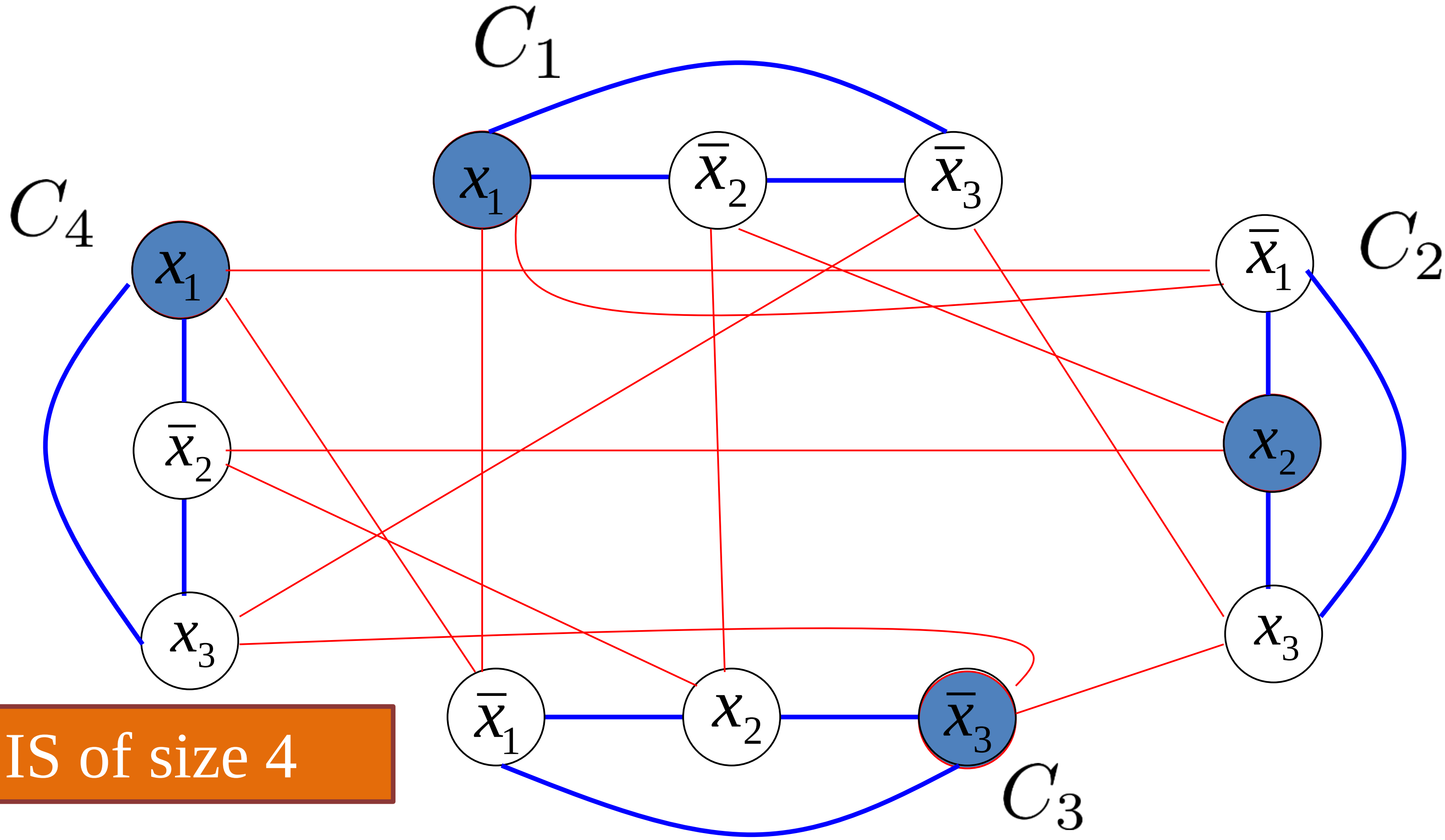


IS of size 4

3-SAT reduction to IS

Claim: Expression E with k clauses is satisfiable if and only if the induced graph G has an IS of size k.

Therefore, given a **graph *G* and a *k***, if we can identify in **poly-time** if there exists an **Independent Set of size at least k**, then we can solve **in poly-time 3-SAT**.

3-SAT reduction to IS

Claim: Expression E with k clauses is satisfiable if and only if the induced graph G has an IS of size k.

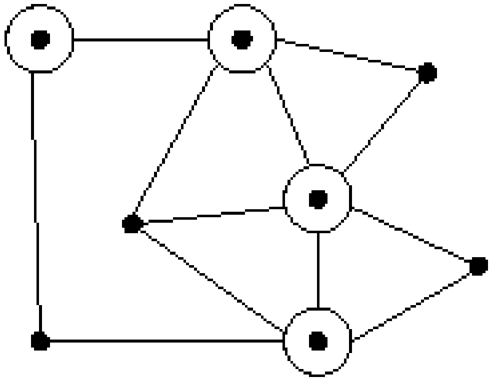
Therefore, given a **graph G and a k**, if we can identify in **poly-time** if there exists an **Independent Set of size at least k**, then we can solve **in poly-time 3-SAT**.

3-SAT $\leq_p$ INDEPENDENT SET $\Rightarrow$ INDEPENDENT SET is NP-complete!

Vertex Cover (VC)

Problem: **Vertex Cover (VC):**

Given a simple undirected graph $G(V, E)$ and k, is there an vertex cover in G of size $\geq k$? Vertex cover is called a set $I \subset V$ of vertices such that all edges are "covered"?



e.g., in this graph, 4 of the 8 vertices are enough to cover all edges.

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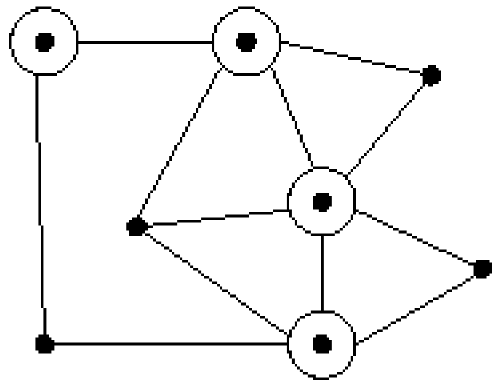

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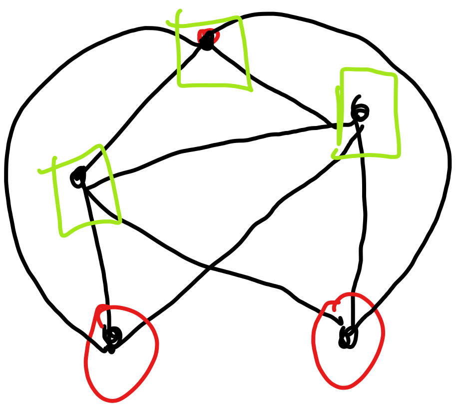
Question: VC is NP-Complete? Answer: YES

- First, it belongs in NP (why?)
- Reduce 3-SAT to VC (or there is something **simpler**?)

Reduction of IS to Vertex Cover (VC)

- Given a graph $G(V, E)$, with $|\mathrm{V}| = \mathrm{n}$, we want to know if there exists an Independent Set of size k.



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- **Lemma:** Given $G(V, E)$, the set of vertices S is an *independent set* if and only if $V - S$ (set of remaining vertices) is a *vertex cover*.

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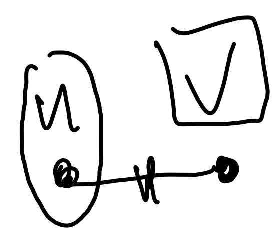
- **Lemma:** Given $G(V, E)$, the set of vertices S is an *independent set* if and only if $V - S$ is a *vertex cover*.

Reduction: Does G have a VC of size $n - k$?

Yes: Then it has an IS of size k.

No: Then it does not.



Reduction of IS to Vertex Cover (VC)

- Given a graph $G(V, E)$, with $|\mathrm{V}| = \mathrm{n}$, suppose there exists an Independent Set of size k.

- **Lemma:** Given $G(V, E)$, the set of vertices S is an *independent set* if and only if $V - S$ is a *vertex cover*.

Proof: Let S be an independent set, and $e = (u, v)$ be some edge. Only one of u, v can be in S. Hence, at least one of u, v is in $V - S$. So, $V - S$ is a vertex cover. The other direction is similar.