Lecture 15

Minimum Spanning Trees

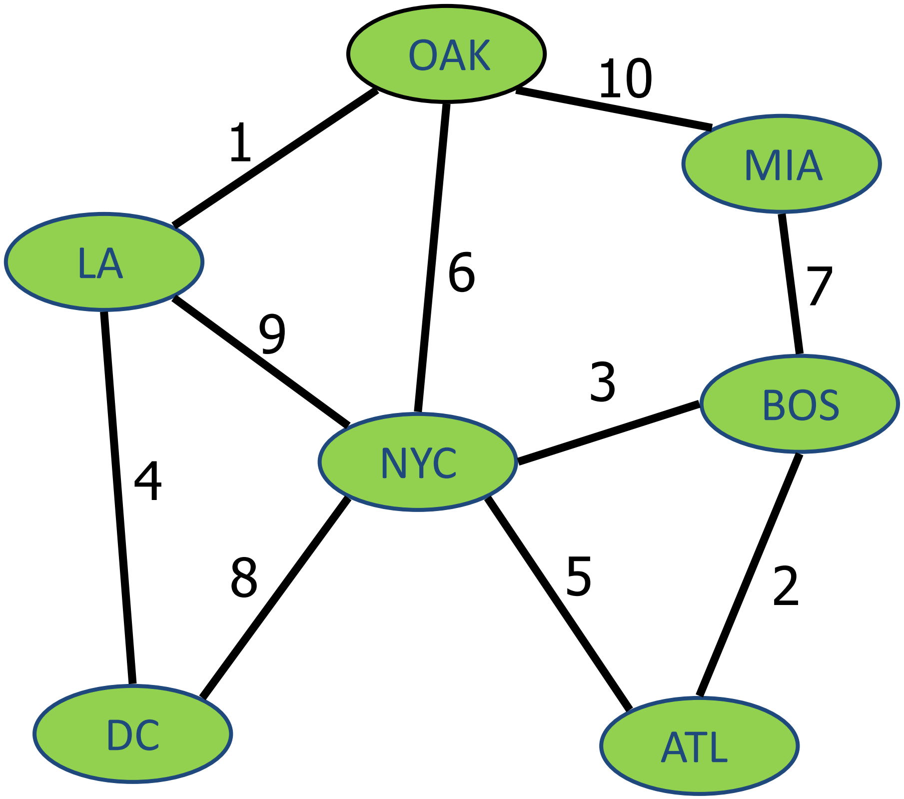
CS 161 Design and Analysis of Algorithms

Ioannis Panageas

Spanning Tree

Definition: We are given an undirected, weighted graph G. A spanning tree of G is a connected acyclic (tree) subgraph of G that includes all the vertices of G (spanning).

Example:



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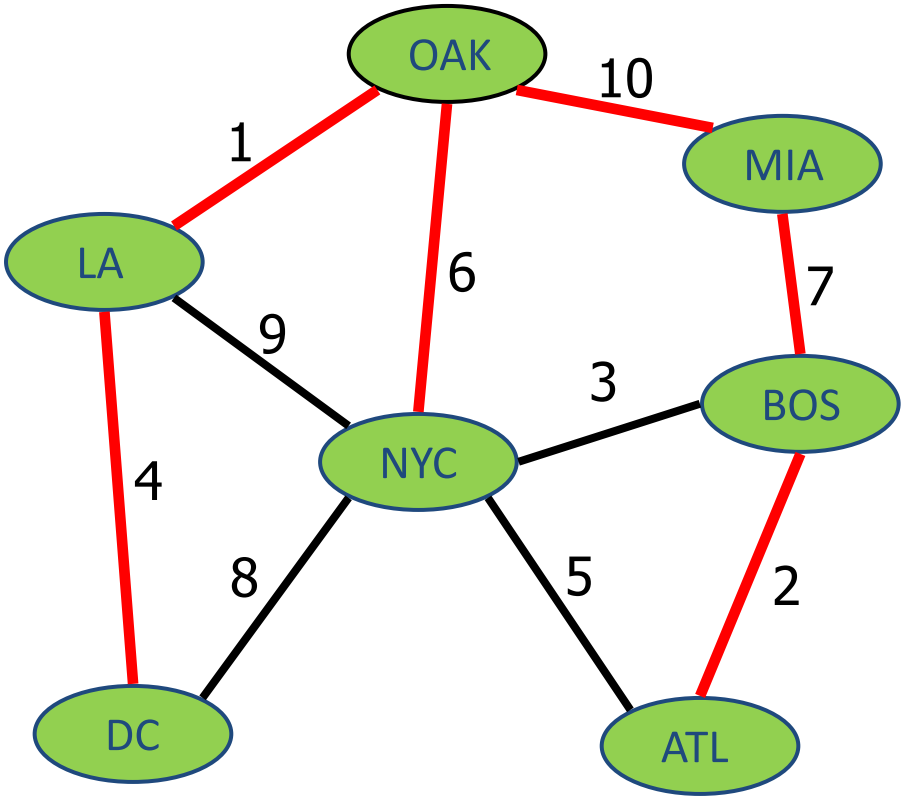

Total cost
$4+1+10+6+7+2 = 30$

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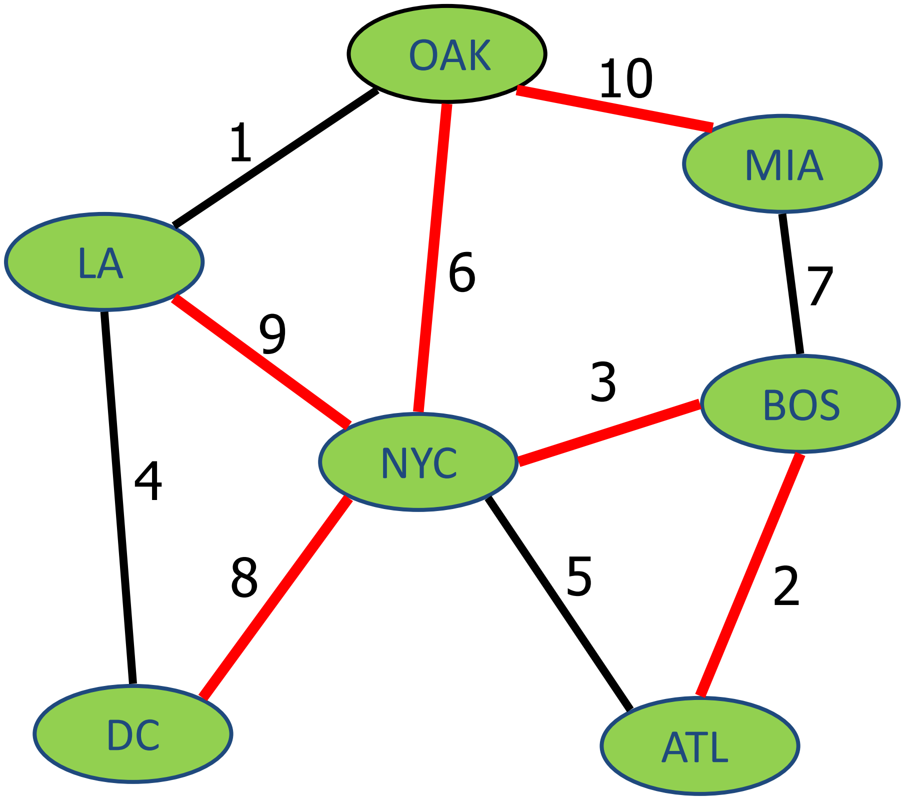

Total cost

$8+9+6+10+3+2 = 38$

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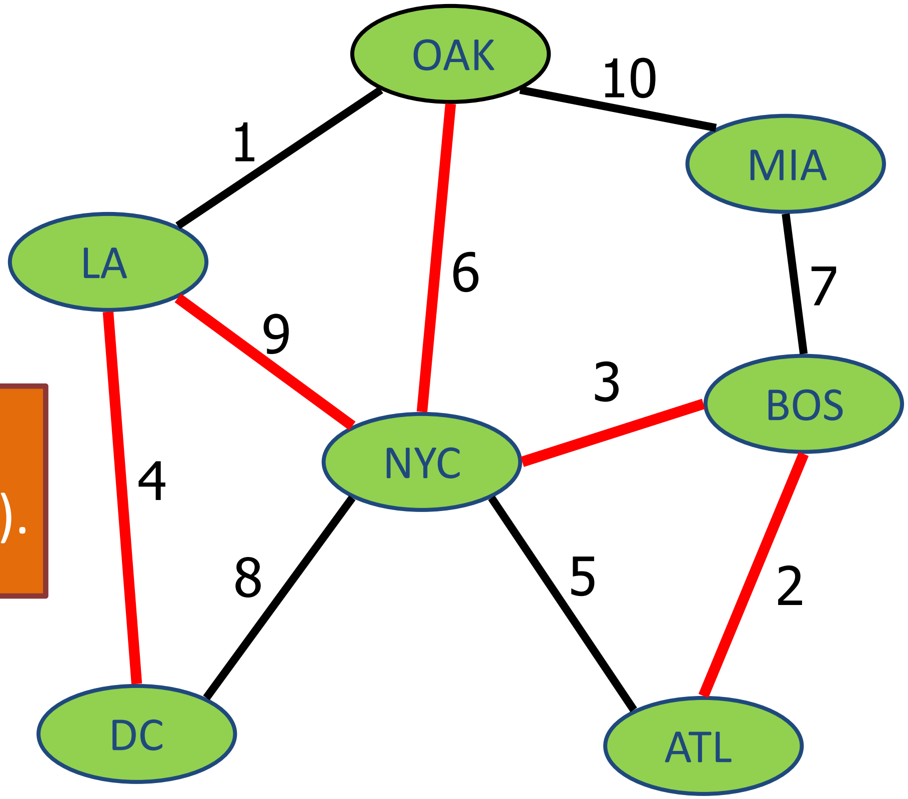

Not a spanning tree.
It is not **spanning** (MIA).

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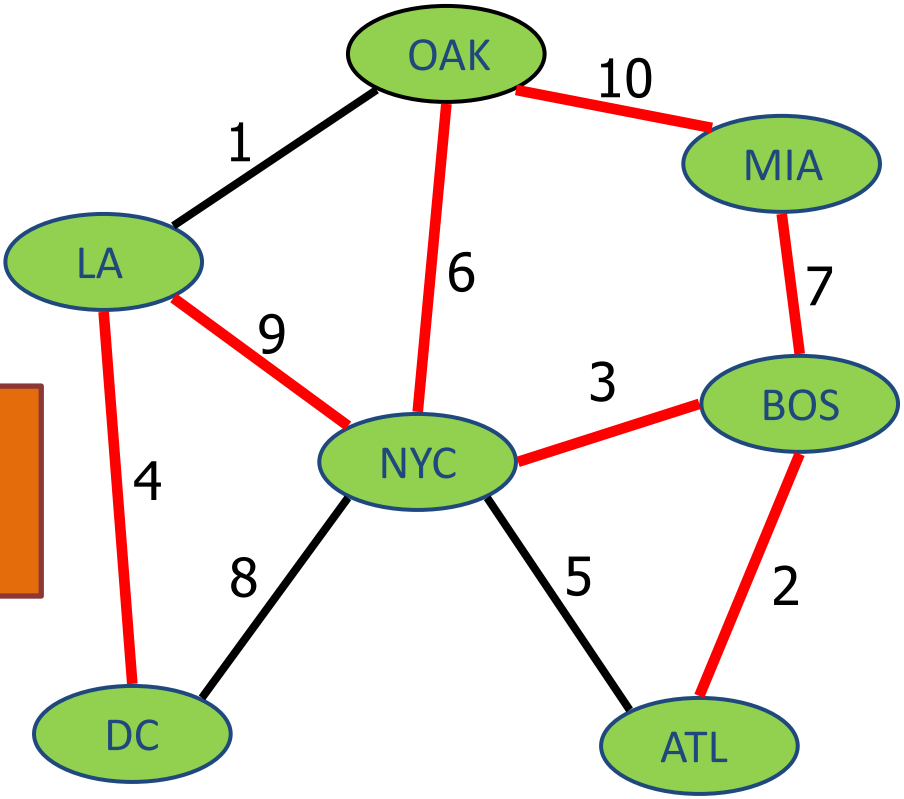
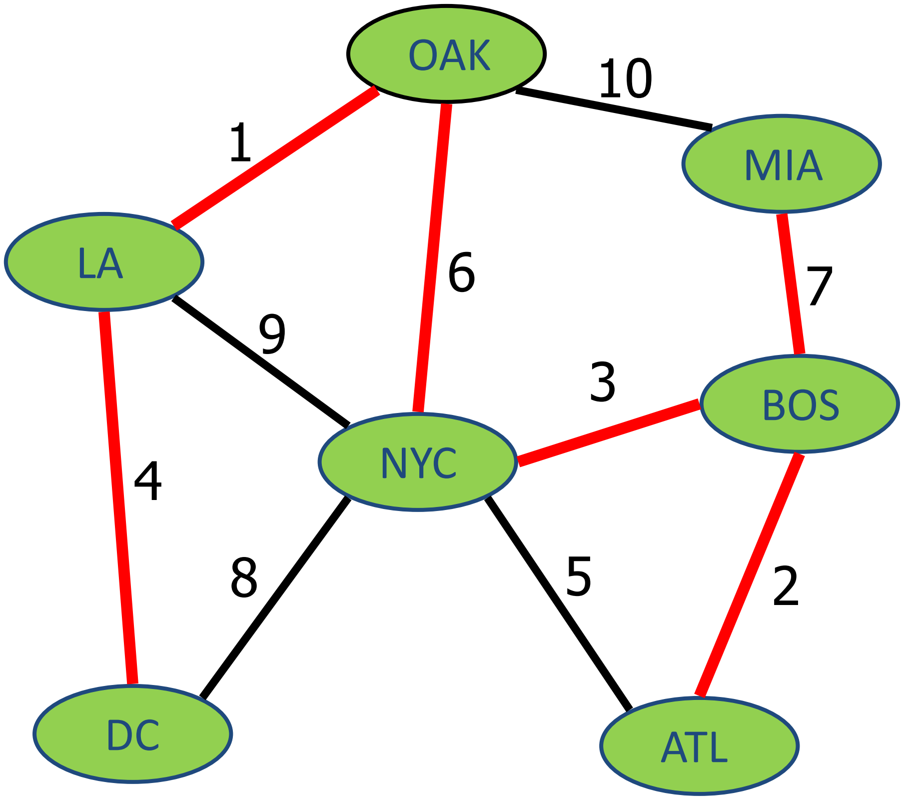


Not a spanning tree.
It is not **a tree** (cycle).

Minimum Spanning Tree

Problem: We are given an undirected, weighted graph G, find the **minimum spanning tree (MST).**

Example:



Total cost

$1{+}4{+}6{+}7{+}3{+}2 = 23$

Minimum Spanning Tree

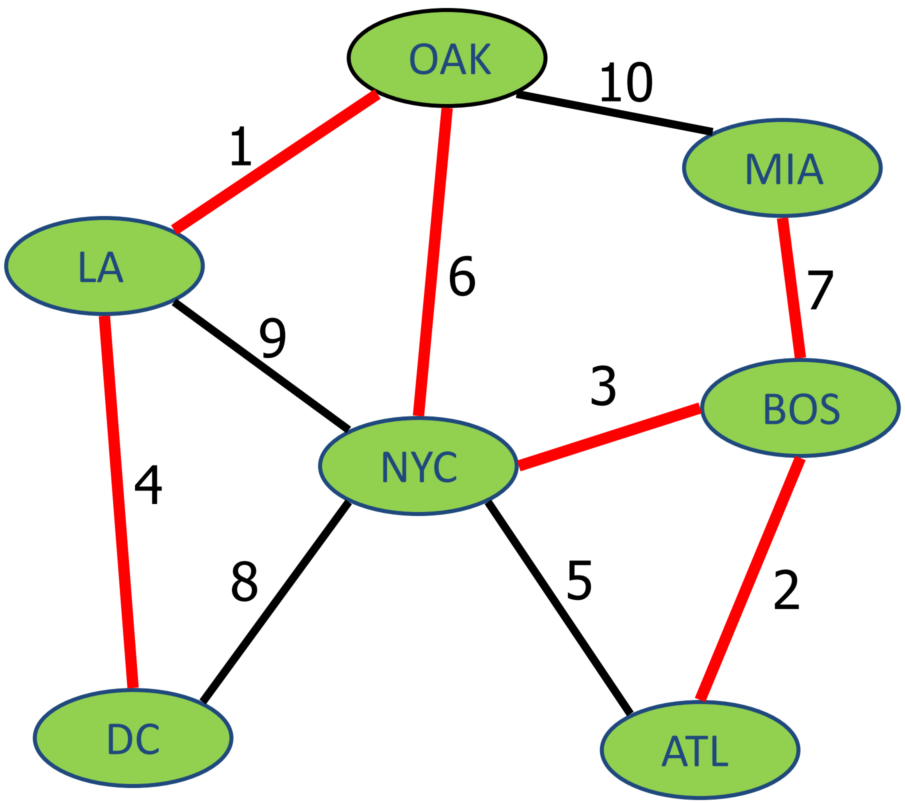
Cycle Property

Let T be a minimum spanning tree of a weighted graph G.

- Let $\boldsymbol{e}$ be an edge of G that is not in T and C let be the cycle formed by $\boldsymbol{e}$ with T.

It holds that:

For every edge $\boldsymbol{f}$ of C, $weight(f) \leq weight(e)$.



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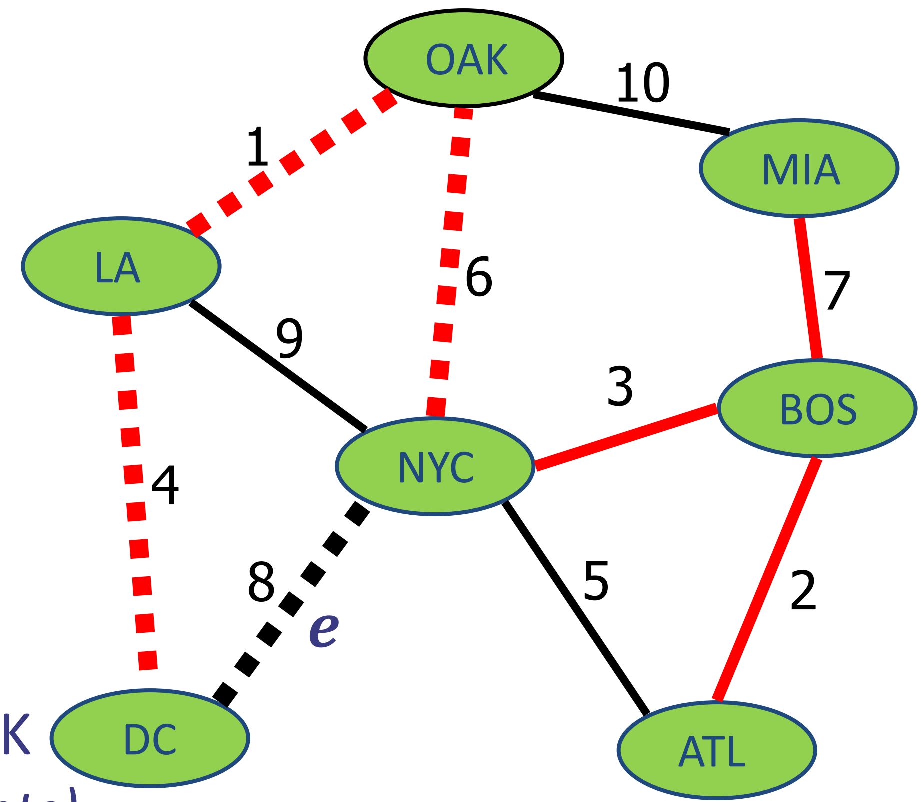


It holds that:

For every edge $\boldsymbol{f}$ of C,

$weight(f) \leq weight(e)$.

Example 1: Cycle LA, DC, NYC, OAK

$w(e) = 8 \geq 1, 6, 4$ *(rest of weights)*

Minimum Spanning Tree

Cycle Property

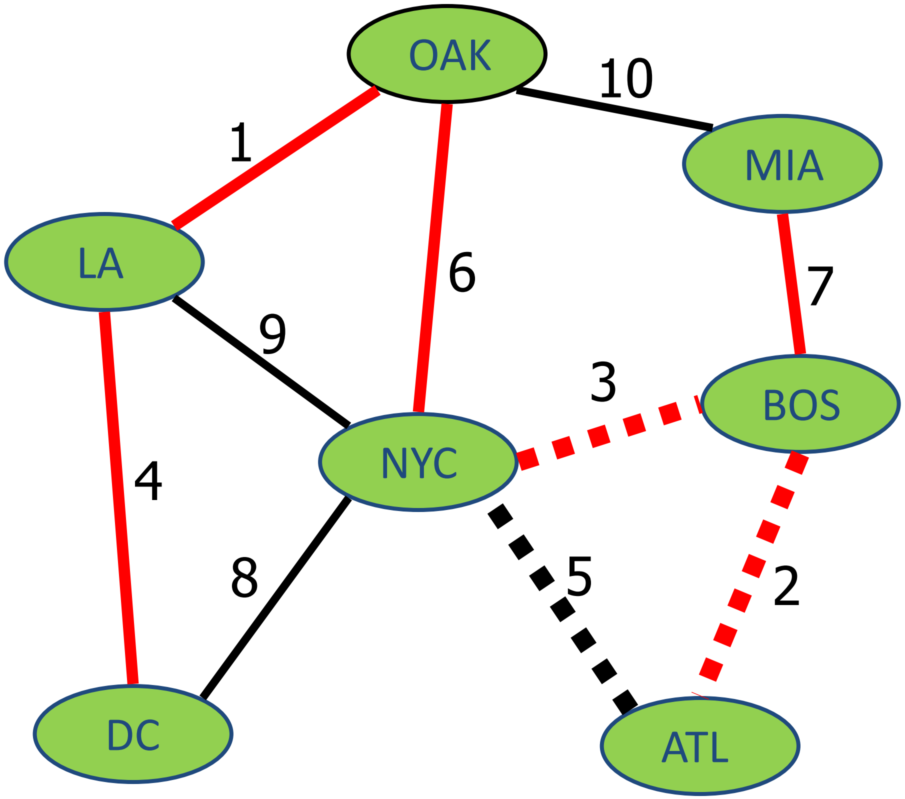
Let T be a minimum spanning tree of a weighted graph G.

- Let $\boldsymbol{e}$ be an edge of G that is not in T and C let be the cycle formed by $\boldsymbol{e}$ with T.

It holds that:

For every edge $\boldsymbol{f}$ of C, $weight(f) \leq weight(e)$.

Example 2: Cycle BOS, ATL, NYC
$w(e) = 5 \geq 2, 3$ *(rest of weights)*



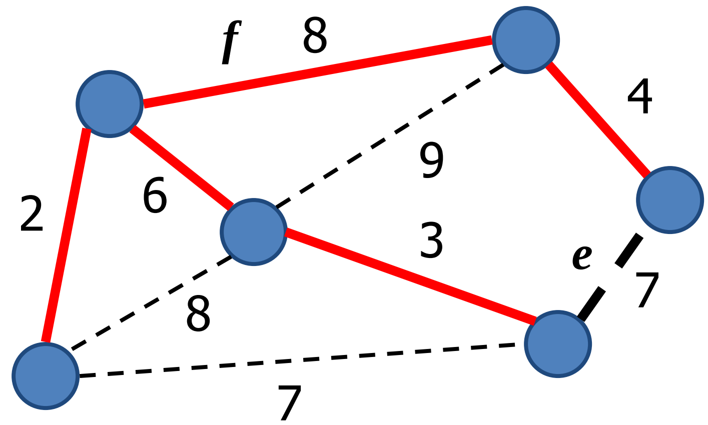
Minimum Spanning Tree

Cycle Property

For the sake of contradiction:

Assume there exist f, e so that

$$weight(f) > weight(e)$$



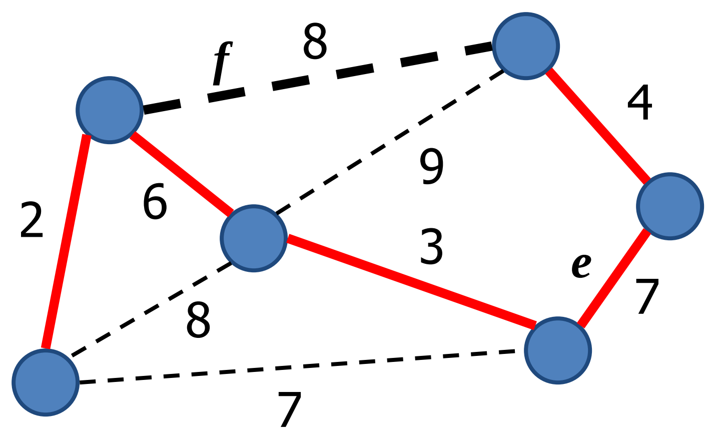
Total cost

$2+3+4+6+8 = 23$

Replacing f with e yields a better spanning tree

Total cost

$2+3+4+6+7 = 22$



Kruskal's Algorithm for MSTs

Idea 1: Greedy approach. Consider the edges from smaller weight to larger. Include each edge in the current solution as long as it does not create a cycle, otherwise discard it.

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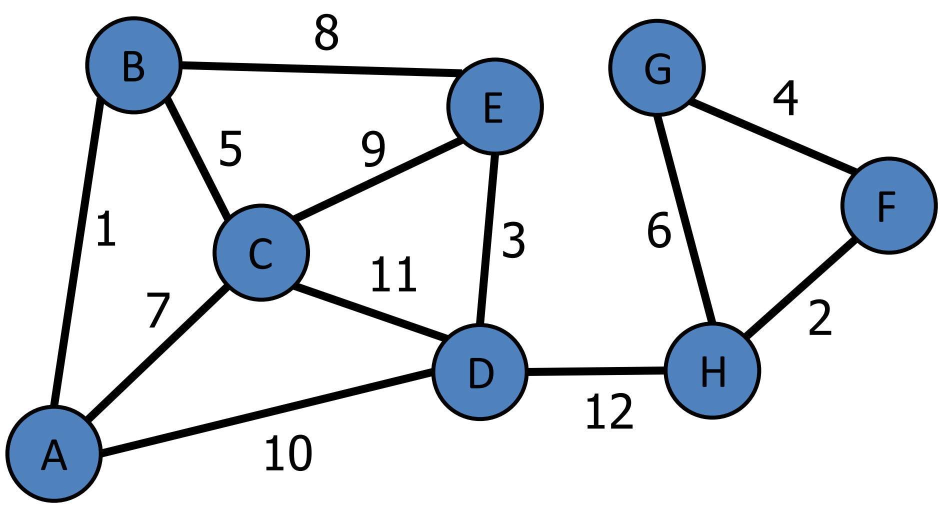

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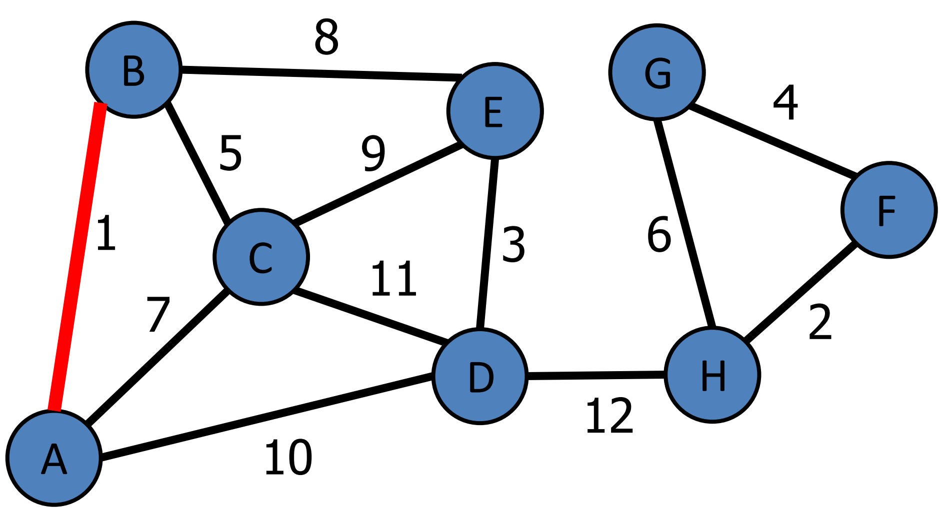

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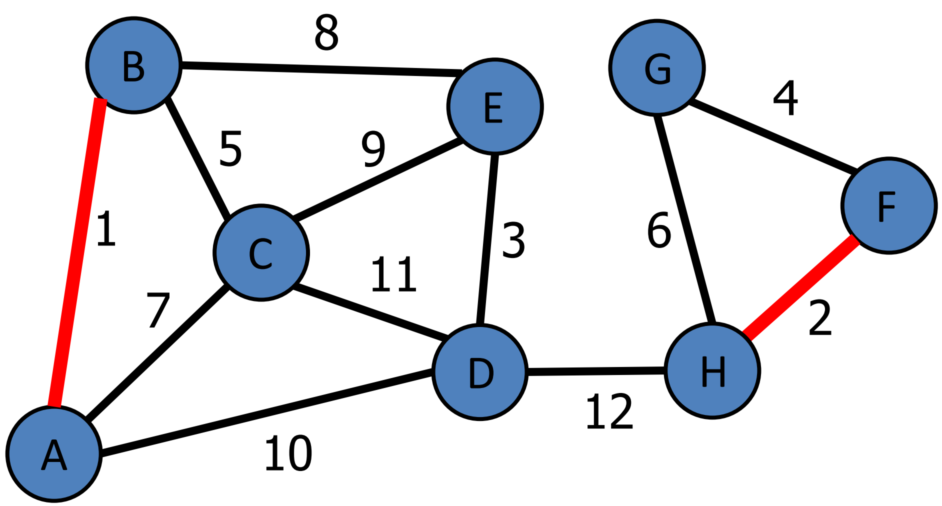

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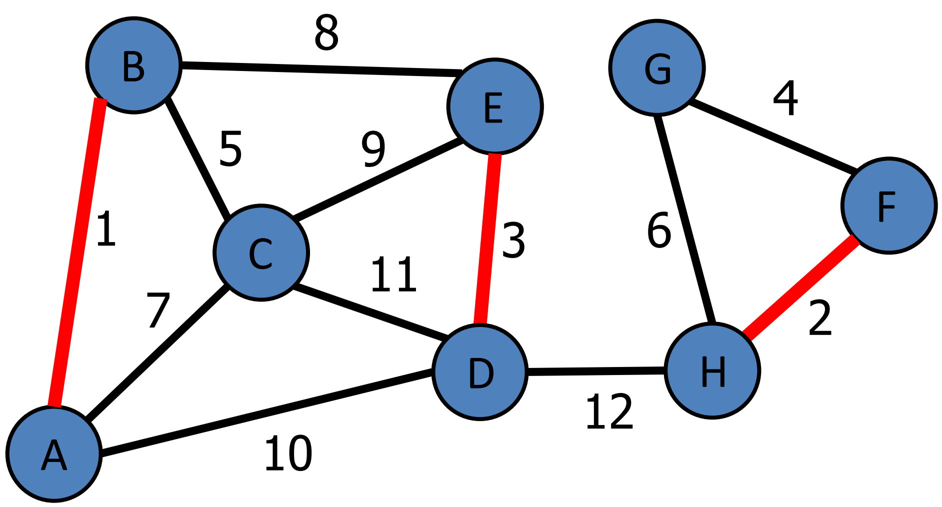

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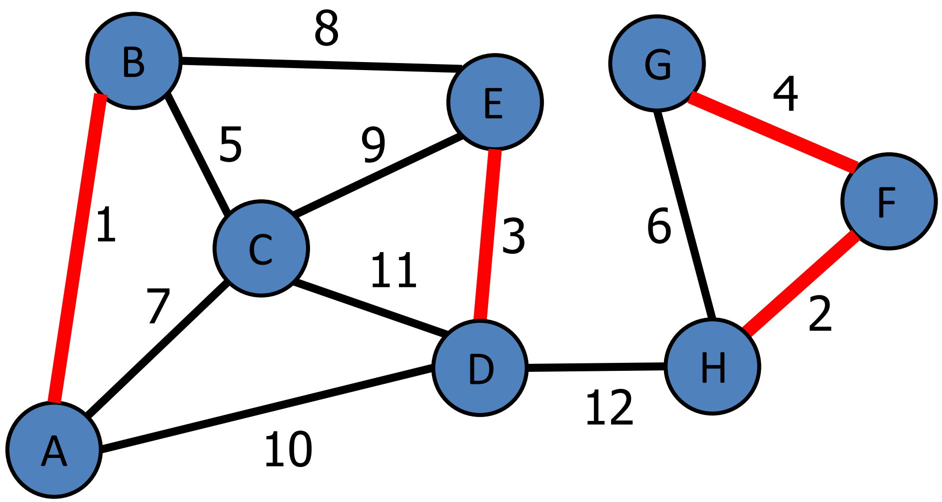

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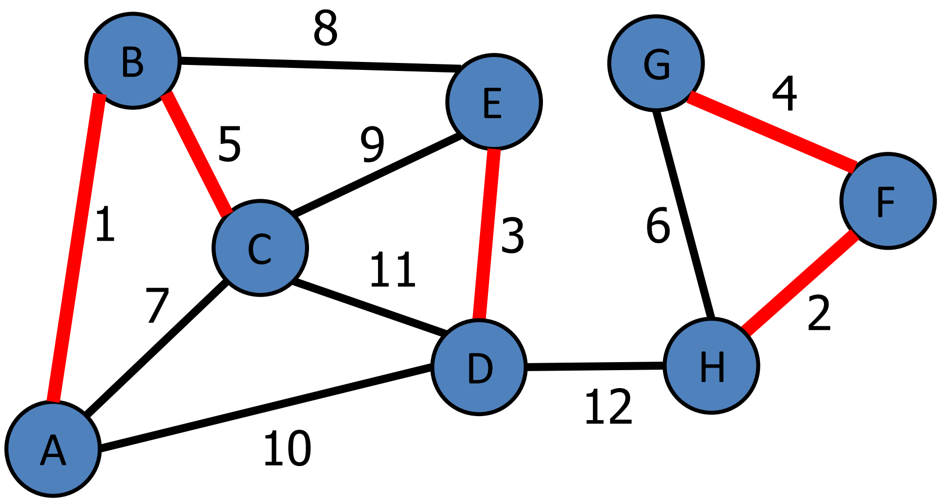

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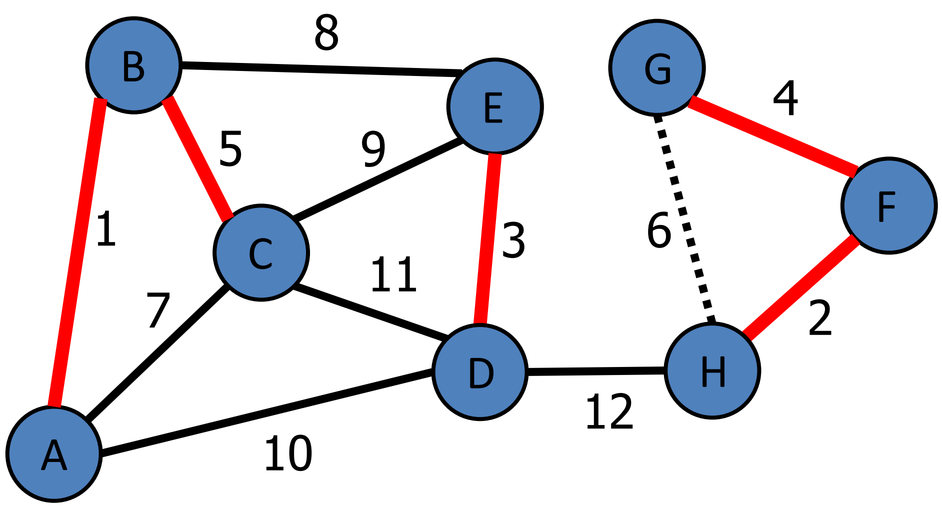

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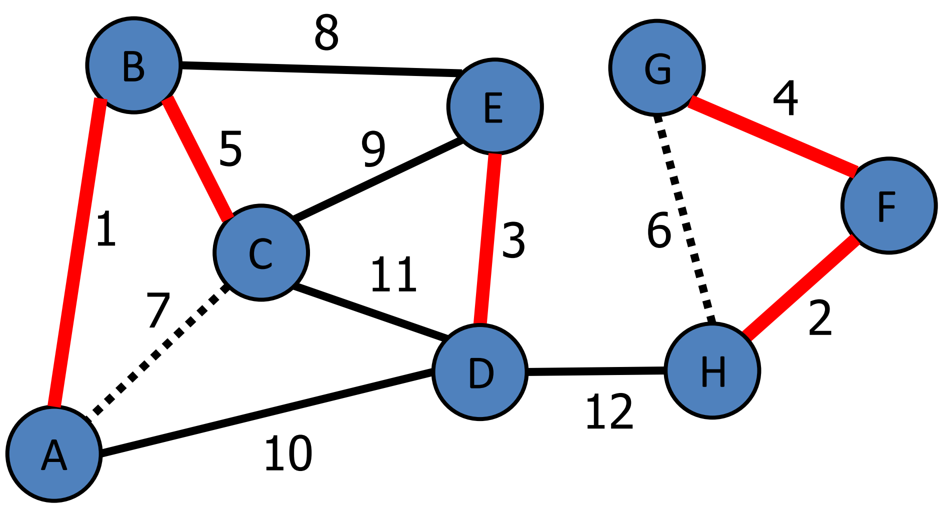

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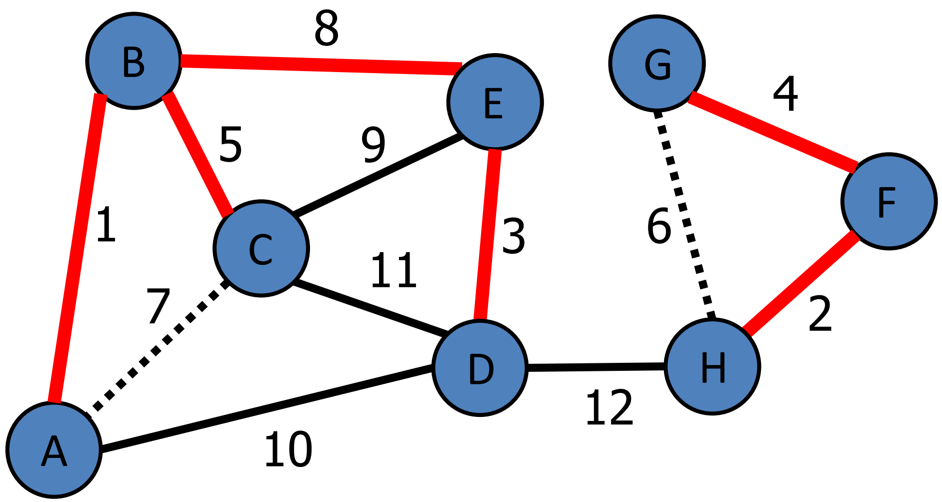

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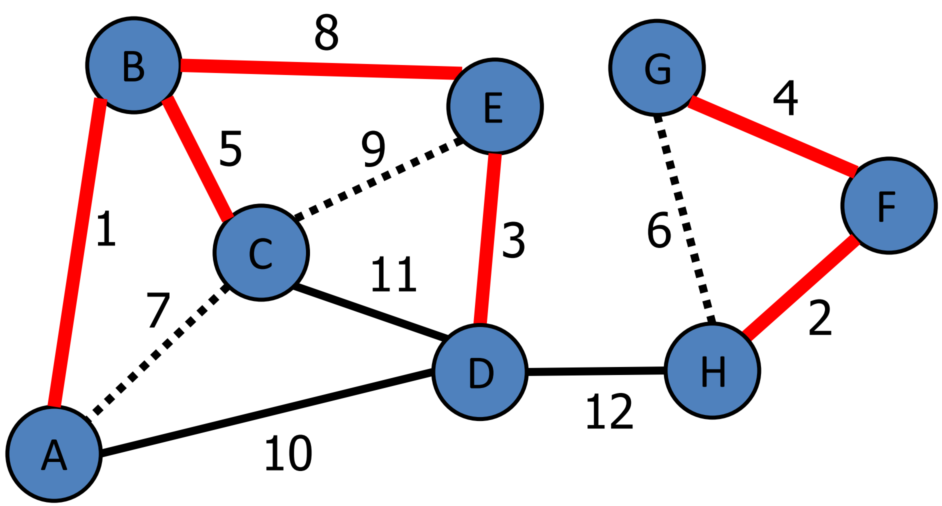

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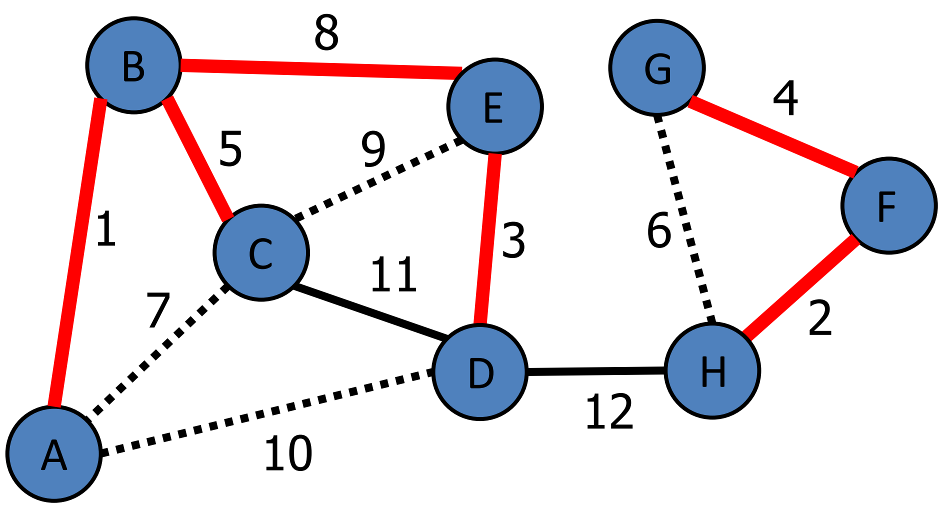

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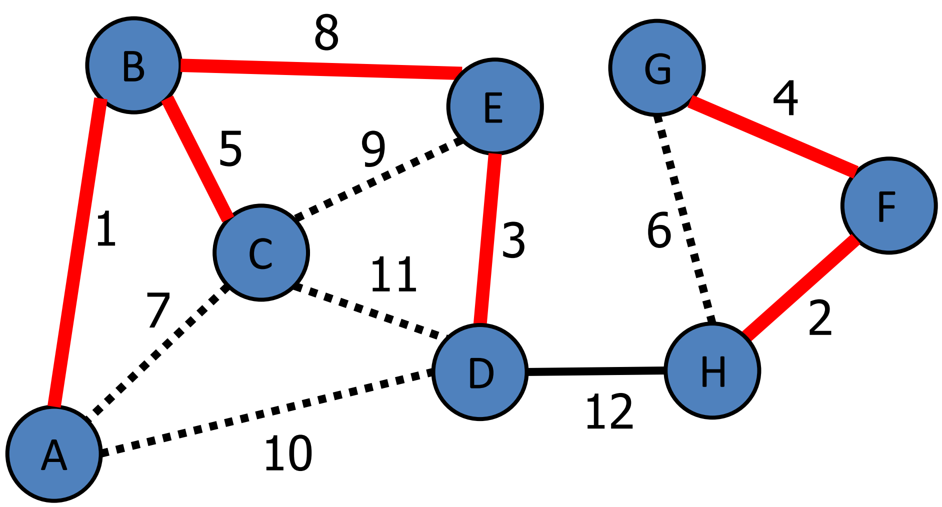

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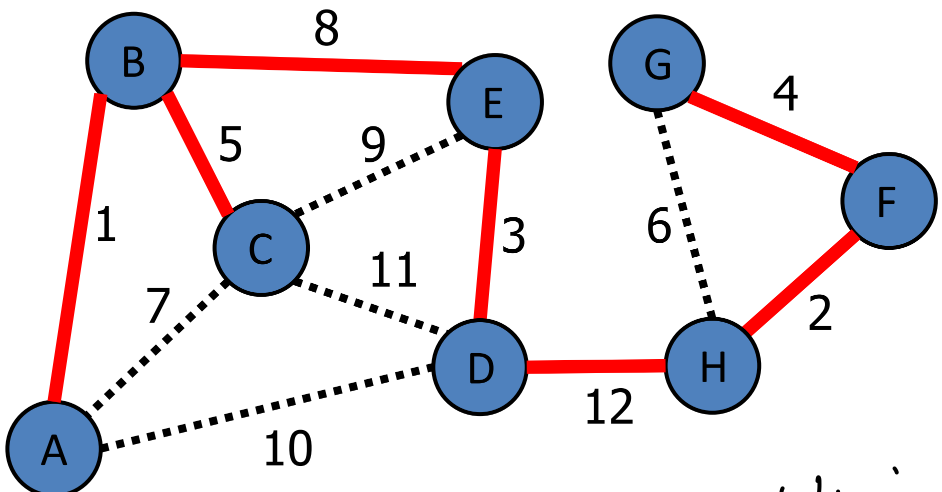
$\Theta(|V|+|E|)$

Example:



$|E| \cdot$

Union-find

Total cost

$1+2+3+4+5+8+12 = 35$

Kruskal's Algorithm for MSTs

Why Kruskal's algo works: General argument. Suppose there is a better solution. Assume the m edges of G are ordered in increasing order of weights, i.e., $w_1 \leq w_2 \ldots \leq w_m$. G has also n vertices.

- Let $x_1, \ldots, x_{n-1}$ be the weight values of the edges in increasing order of the minimum spanning tree T'.
- Let $y_1, \ldots, y_{n-1}$ be the weight values of the edges in increasing order of Kruskal's spanning tree T.

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- There is an index i, so that $y_i < x_i$. We add edge with value y_i in T', we create a cycle C.
  - If x_i is in C, we remove it and create a spanning tree smaller than T' (contradiction).

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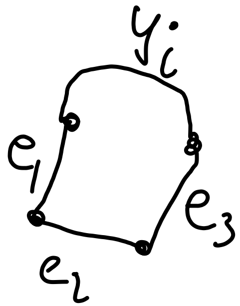
  - If x_i is in C, we remove it and create a spanning tree smaller than T' (contradiction).
  - If x_i not in C, by cycle property, y_i is the largest value from edges in C. Kruskal would not have chosen y_i (contradiction).



Prim's Algorithm for MSTs

Idea 2: Similar to Dijkstra's algorithm. We pick an arbitrary vertex s. We build the **tree** by adding one new vertex at a time. Each vertex v has label $d[v] := $ smallest weight of an edge connecting v to a vertex in the built tree.

At each step:

- We add to the current tree the vertex u with the smallest $d[u]$ and the corresponding incident to u edge.
- We update the labels of the vertices adjacent to u.

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$d[C] = \infty$
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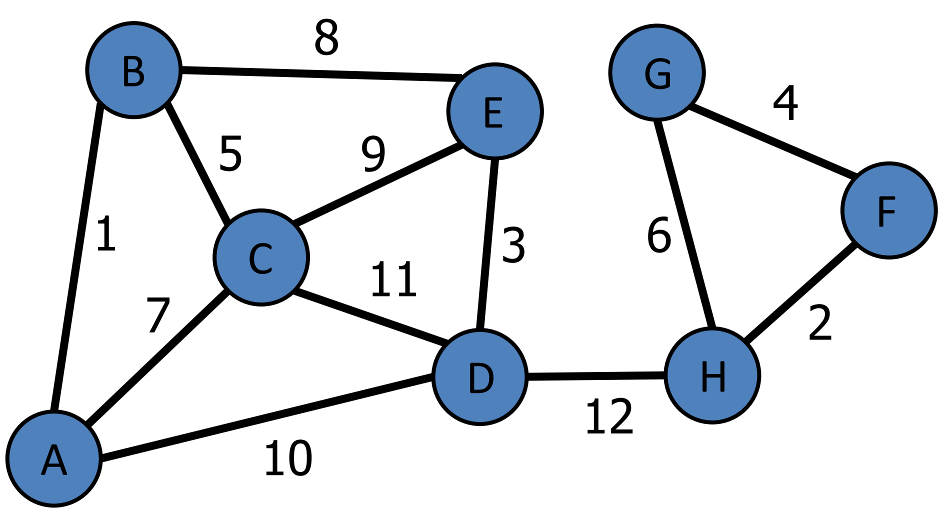

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$\mathbf{d[C]{=}7}$
$\mathbf{d[D]{=}10}$
$d[E] = \infty$
$d[F] = \infty$
$d[G] = \infty$
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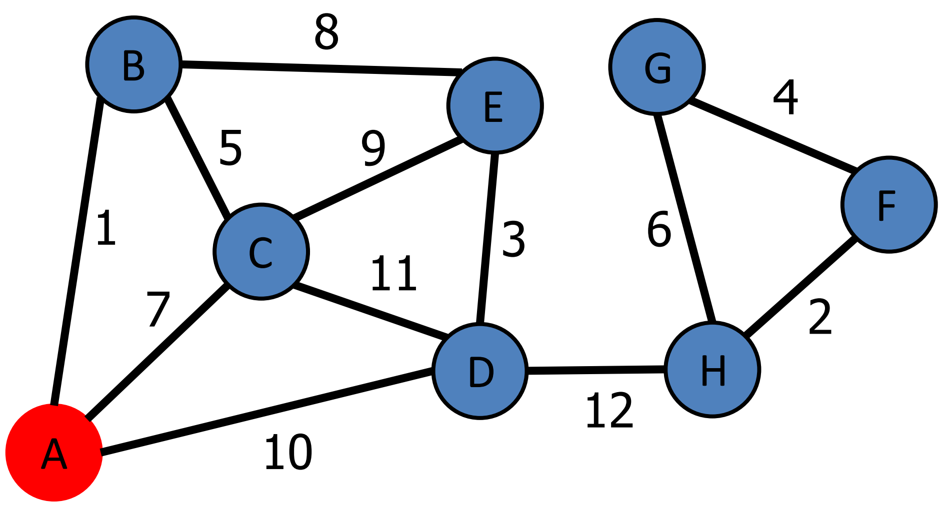

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$\mathbf{d[C]=5}$
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$d[G] = \infty$
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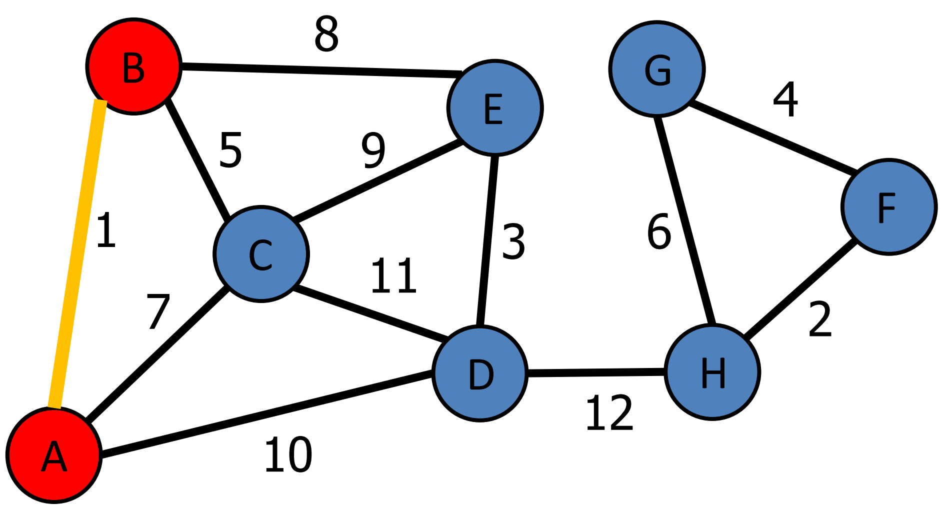

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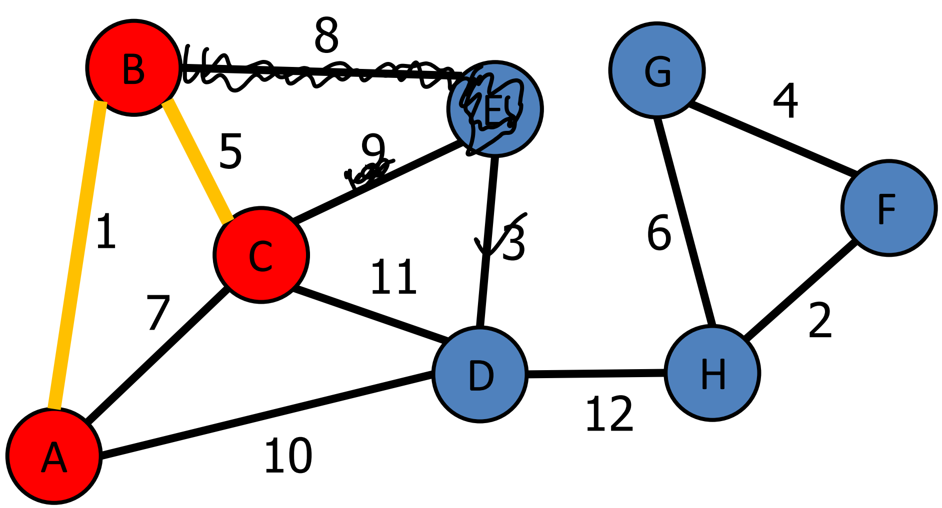

(E)

ED

$d[D] = 10$

v.s $w(ED) = 3$

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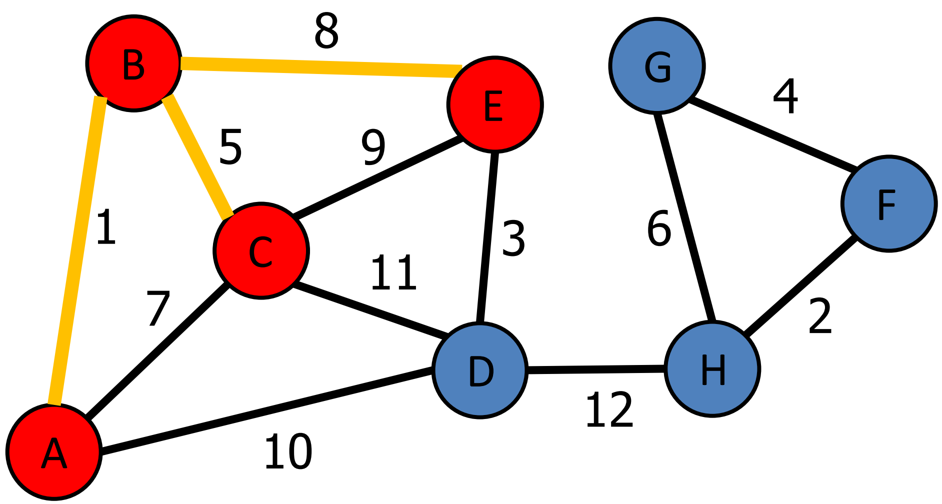

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$d[D] = 3$
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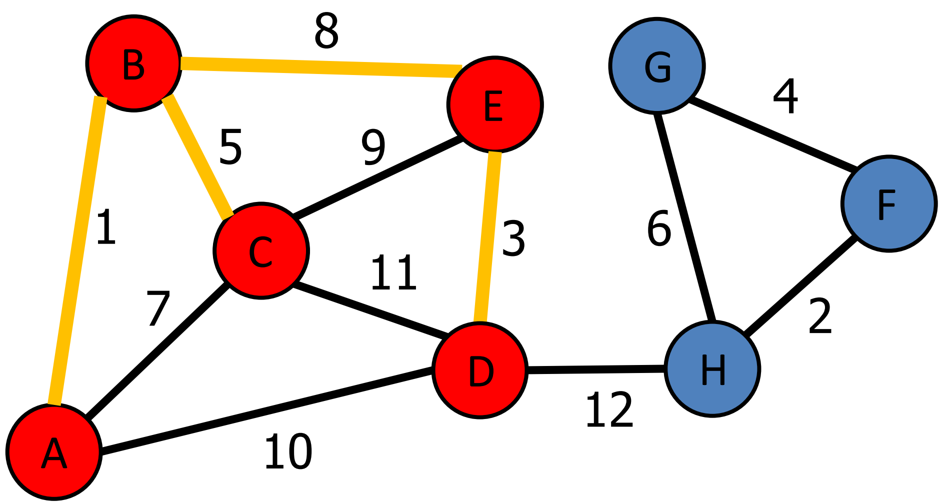
$d[F] = \infty$
$d[G] = \infty$
$\mathbf{d[H]=12}$



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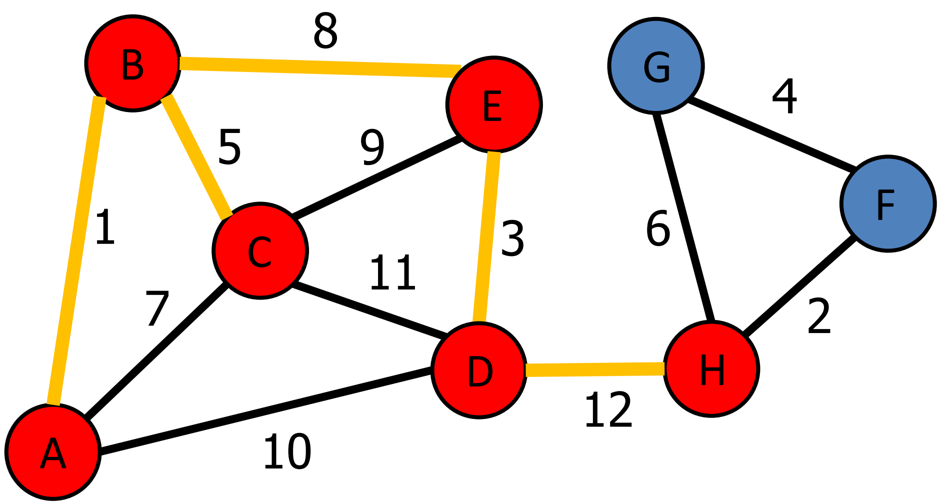
$\mathbf{d[G]=6}$
$d[H] = 12$



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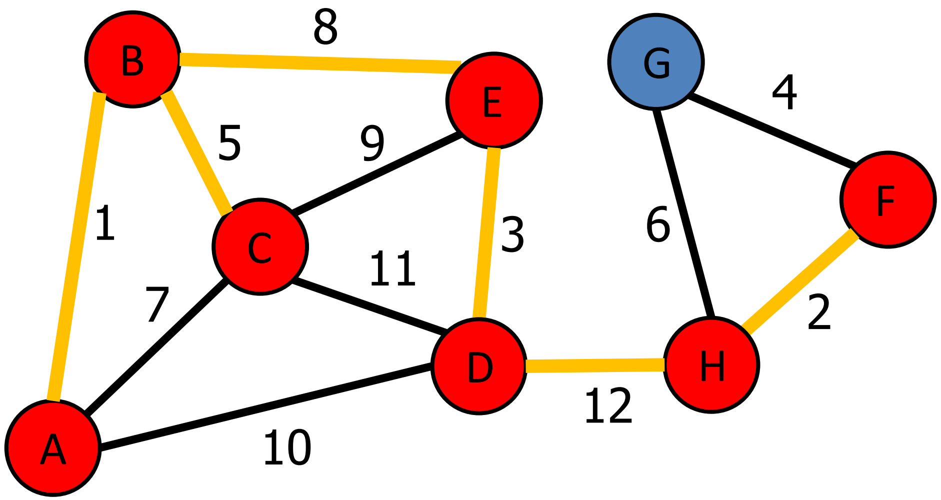

Prim's Algorithm for MSTs

Dijkstra: Relax(u,v) : d[v] vs d[u]+w(u,v)

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Prim: d[V] vs w(u,V)

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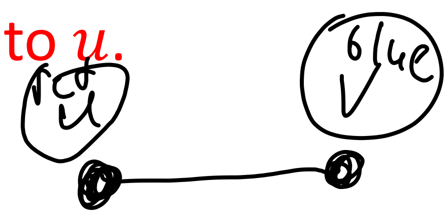

d[V] vs w(u,v)

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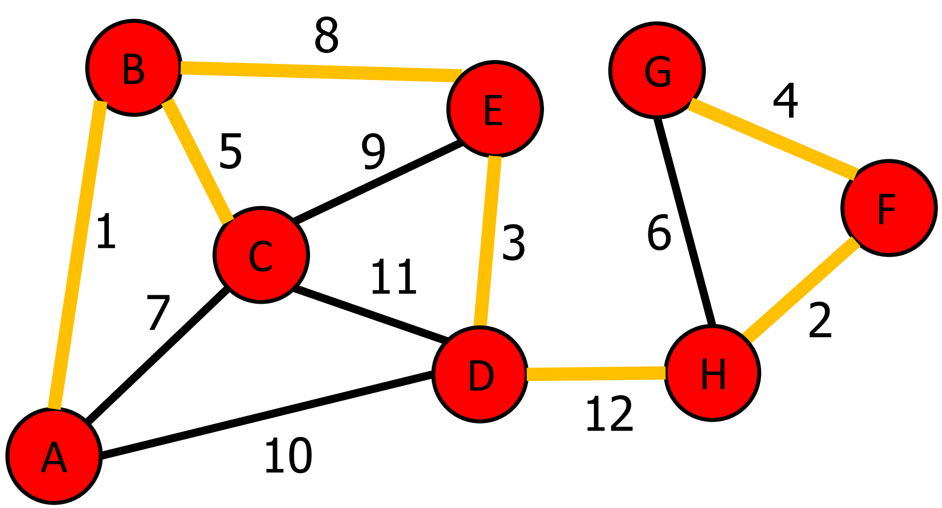

Prim's Algorithm for MSTs

Pseudocode:

Starting vertex

Pick any vertex v of G
$D[v] \leftarrow 0$

Initialization

for each vertex $u \neq v$ **do**
  $D[u] \leftarrow +\infty$
Initialize $T \leftarrow \emptyset$.
Initialize a priority queue Q with an item $((u, \text{null}), D[u])$ for each vertex u, where (u, null) is the element and $D[u]$ is the key.
while Q is not empty **do**
  $(u, e) \leftarrow Q$.removeMin()
  Add vertex u and edge e to T.
  for each vertex z adjacent to u such that z is in Q **do**
    // perform the relaxation procedure on edge (u, z)
    if $w((u, z)) < D[z]$ **then**
      $D[z] \leftarrow w((u, z))$
      Change to $(z, (u, z))$ the element of vertex z in Q.
      Change to $D[z]$ the key of vertex z in Q.
return the tree T

Relaxation

Prim's Algorithm for MSTs

Pseudocode:

Starting vertex

Initialization

Relaxation

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$D[v] \leftarrow 0$
for each vertex $u \neq v$ **do**
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    $(u, e) \leftarrow Q$.removeMin()
    Add vertex u and edge e to T.
    for each vertex z adjacent to u such that z is in Q **do**
        // perform the relaxation procedure on edge (u, z)
        if $w((u, z)) < D[z]$ **then**
            $D[z] \leftarrow w((u, z))$
            Change to $(z, (u, z))$ the element of vertex z in Q.
            Change to $D[z]$ the key of vertex z in Q.
return the tree T

Running time: If extractmin in $\Theta(|V|)$, update in $\Theta(1)$ then $|V|^2 + |E|$.

Prim's Algorithm for MSTs

Pseudocode:

Pick any vertex v of G
$D[v] \leftarrow 0$
for each vertex $u \neq v$ **do**
    $D[u] \leftarrow +\infty$
Initialize $T \leftarrow \emptyset$.
Initialize a priority queue Q with an item $((u, \text{null}), D[u])$ for each vertex u, where (u, null) is the element and $D[u]$ is the key.
while Q is not empty **do**
    $(u, e) \leftarrow Q$.removeMin()
    Add vertex u and edge e to T.
    for each vertex z adjacent to u such that z is in Q **do**
        // perform the relaxation procedure on edge (u, z)
        if $w((u, z)) < D[z]$ **then**
            $D[z] \leftarrow w((u, z))$
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Starting vertex

Initialization

Relaxation

Running time: If extractmin in $\Theta(|V|)$, update in $\Theta(1)$ t

$\Theta(|V|^2)$

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while Q is not empty **do**
    $(u, e) \leftarrow Q.\text{removeMin}()$
    Add vertex u and edge e to T.
    for each vertex z adjacent to u such that z is in Q **do**
        // perform the relaxation procedure on edge (u, z)
        if $w((u, z)) < D[z]$ **then**
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Running time: If extractmin, update in $\Theta(\log |V|)$ then $|E| \log |V|$.