# Lecture 10

# Dynamic Programming II: Bellman-Ford (cont), Interval Scheduling, Longest Common Subsequence

CS 161 Design and Analysis of Algorithms

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# Case study III: Bellman-Ford

In words: $d[s] = 0, d[u] = +\infty$ for $u \neq s$.

For $n - 1$ times, relax all the edges $(u, v)$.

**Relaxation of** $(u, v)$

**if** $\mathrm{d}[v] > \mathrm{d}[u] + w(u, v)$ **then**

$\mathrm{d}[v] \leftarrow \mathrm{d}[u] + w(u, v)$

**Running time** $\Theta(|V| \cdot |E|)$

# Case study III: Bellman-Ford

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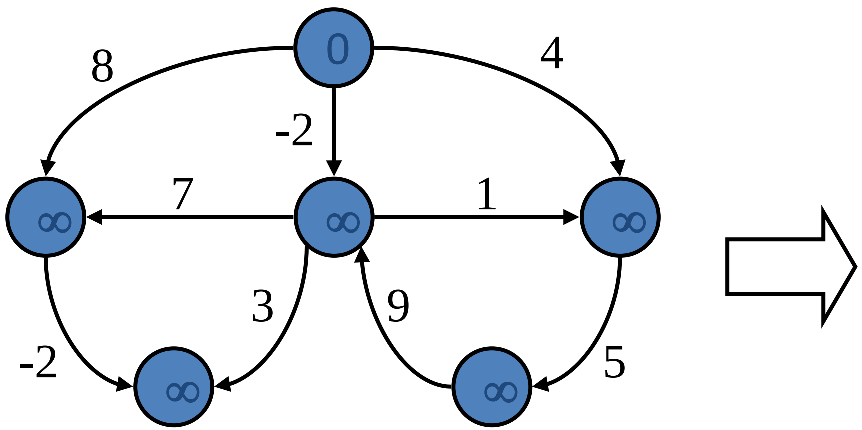
**If** $\mathrm{d}[v] > \mathrm{d}[u] + w(u, v)$ **then**
$\quad \mathrm{d}[v] \leftarrow \mathrm{d}[u] + w(u, v)$

**Running time** $\Theta(|V| \cdot |E|)$

Property: Suppose we relax all edges one more time. If $d[]$ decreases for a vertex then there is a negative cycle. If $d[]$ remains the same, no negative cycle.

# Case study III: Bellman-Ford

Find the shortest weight path from node 0.



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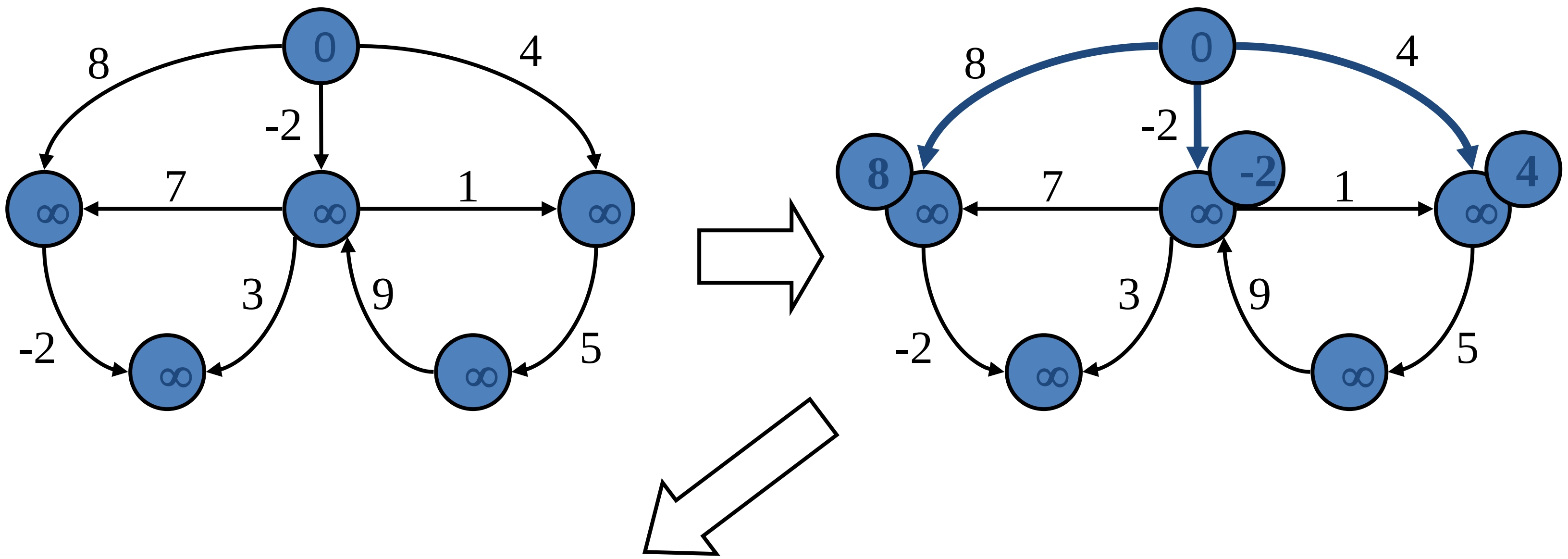

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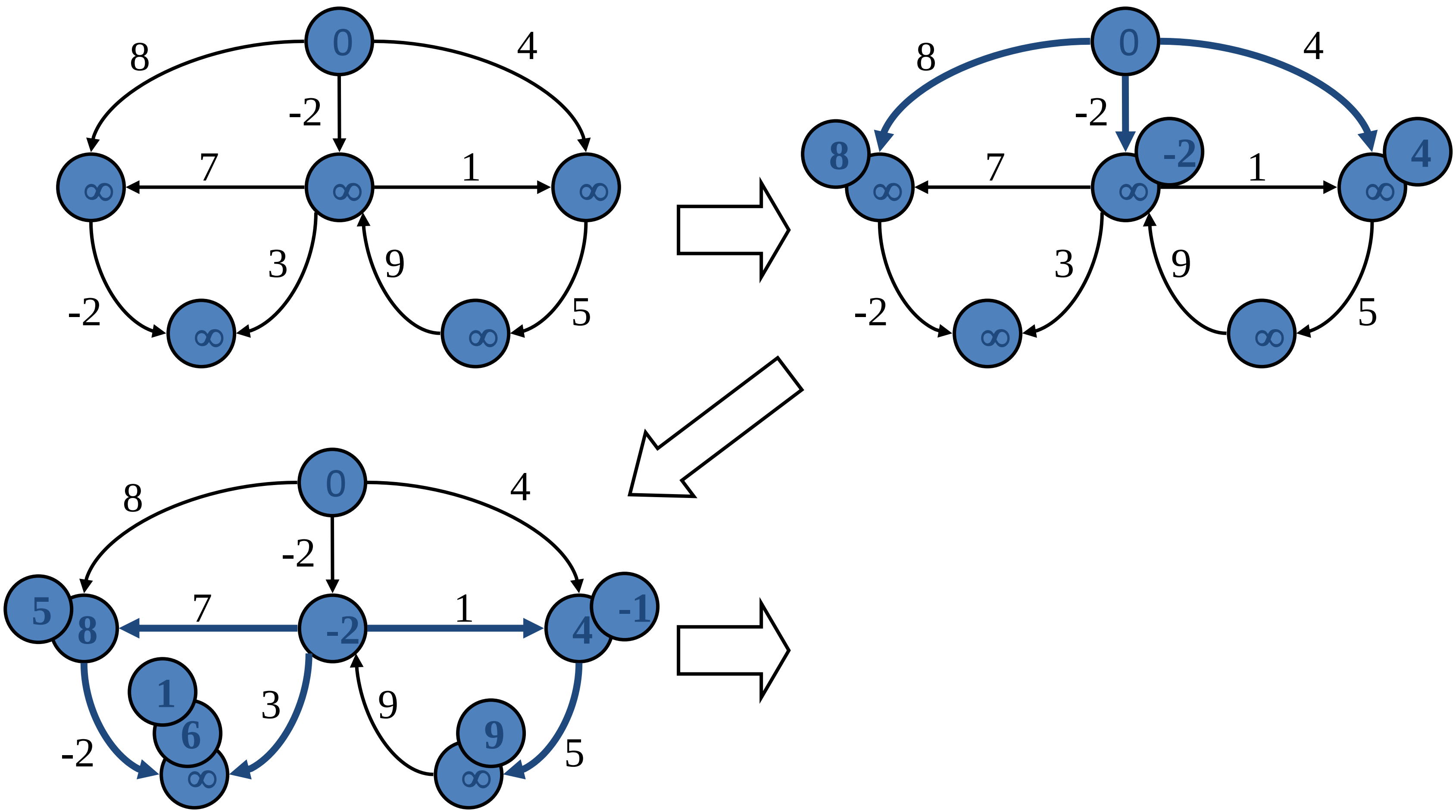

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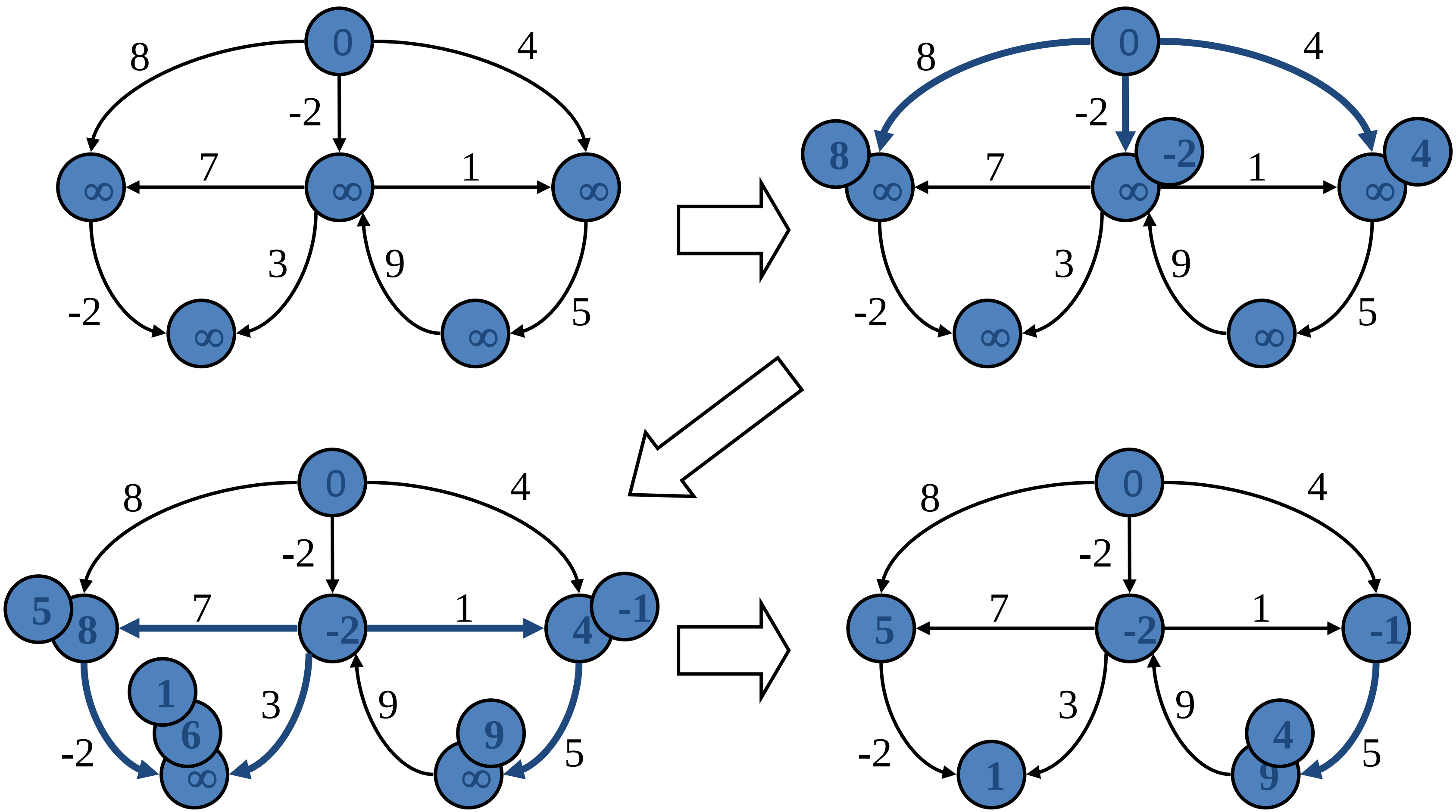
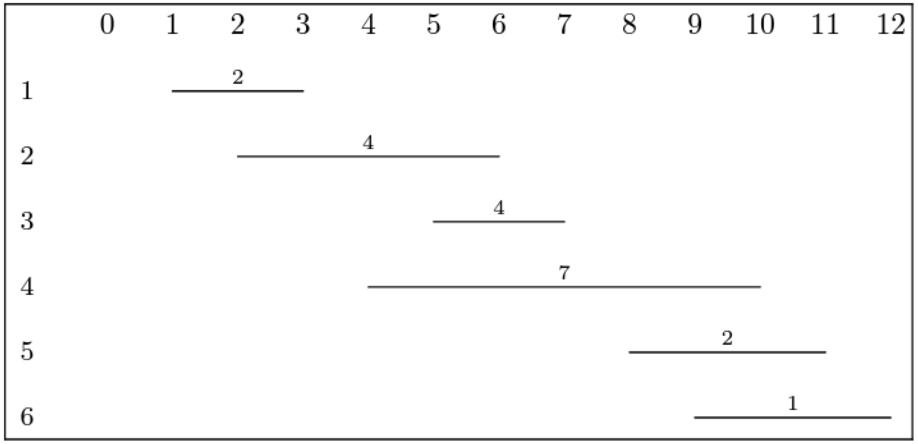


# Case study IV: Interval Scheduling

Problem: You are given a collection of $n$ intervals represented by start time, finish time, and value: $(s_j, f_j, v_j)$, sorted w.r.t $f_j$. Find a non-overlapping set of intervals with maximum total value.

Example:

| $j$ | $s(j)$ | $f(j)$ | $v(j)$ |
|---|---|---|---|
| 1 | 1 | 3 | 2 |
| 2 | 2 | 6 | 4 |
| 3 | 5 | 7 | 4 |
| 4 | 4 | 10 | 7 |
| 5 | 8 | 11 | 2 |
| 6 | 9 | 12 | 1 |



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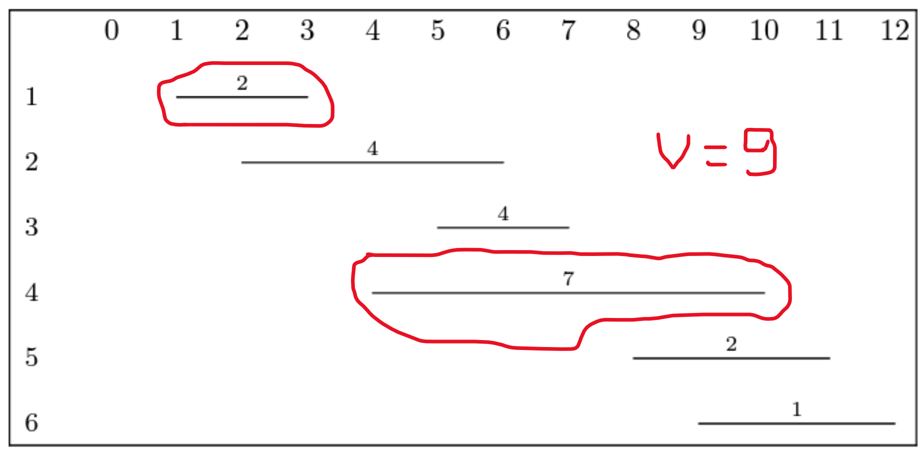

# Case study IV: Interval Scheduling

Step 1: Define the problem and subproblems.

Answer: Let $DP[j]$ be the maximum value that can be obtained from a set of non-overlapping intervals with indices in the range $\{1, \ldots, j\}$

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Step 2: Define the goal/output given Step 1.

It is $\boldsymbol{DP[n]}$.

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Step 2: Define the goal/output given Step 1.

It is $\boldsymbol{DP[n]}$.

Step 3: Define the base cases

It is $DP[0] = 0$.

Step 4: Define the recurrence

# Case study IV: Interval Scheduling

## Step 4: Define the recurrence

Interval $j$ belongs to the optimal solution or **not**.

$$DP[j] \; = \; \max(\mathbf{DP}[\$] + \mathbf{v_j}, \mathbf{DP}[\mathbf{j-1}])$$

What is **$** ?

# Case study IV: Interval Scheduling

## Step 4: Define the recurrence

Interval $j$ belongs to the optimal solution or **not**.

$$DP[j] = \max(\mathbf{DP[\$] + v_j, DP[j-1]})$$

**$** should be the interval with highest index in $\{1, \ldots, j-1\}$ that does not intersect with $j$ (since $j$ is chosen).

Let $p[j]$ be the highest index in $\{1, \ldots, j-1\}$ that does not intersect with $j$. Then the recurrence becomes

$$DP[j] = \max(\mathbf{DP[p[j]] + v_j, DP[j-1]})$$

# Case study IV: Interval Scheduling

## Pseudocode:

$$\text{Array DP}[]$$
$$\text{DP}[0] \leftarrow 0$$
$$\textbf{For } k = 1 \text{ to } n \textbf{ do}$$
$$\quad \text{DP}[k] \leftarrow \max(\text{DP}[k-1], \text{DP}[p[k]] + v[k])$$
$$\textbf{return } \text{DP}[n]$$

Initialization

Bottom up filliing DP

Goal

# Case study IV: Interval Scheduling

## Pseudocode:

Array DP[]

DP[0] $\leftarrow 0$ — Initialization

**For** $k = 1$ to $n$ **do**

  $\mathrm{DP}[k] \leftarrow \max(\mathrm{DP}[k-1], \mathrm{DP}[p[k]] + v[k])$ — Bottom up filliing DP

**return** DP[$n$] — Goal

Question: How can we compute $p[j]$ for $1 \leq j \leq n$ in $\Theta(n \log n)$ time?

# Case study IV: Interval Scheduling

Question: How can we compute $p[j]$ for $1 \leq j \leq n$ in $\Theta(n \log n)$ time?

Answer:

- Sort first the intervals in increasing order of finishing times.

# Case study IV: Interval Scheduling

**Question:** How can we compute $p[j]$ for $1 \le j \le n$ in $\Theta(n \log n)$ time?

$\log(j)$

1, 2, ... | $j-1$

$f_{j/2} \le s_j$

**Answer:**

- Sort first the intervals in increasing order of finishing times.
- For every $j$, do binary search to find the interval before $j$ with finishing time at most $s_j$

$$\sum_{j=1}^{n} \log(j) = \log(n!) \in \Theta(n \log n)$$

# Case study V: Longest Common Subsequence

Problem: You are given two strings $x = X_1 \ldots X_n$ and $y = Y_1 \ldots Y_m$ of sizes $n, m$ and you are asked to find the size of a longest common substring $z$ of $x$ and $y$.

Example:

$x$ = H I E R O G L Y P H O L O G Y

$y$ = M I C H E L A N G E L O

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Example:

$x$ = H I E R O G L Y P H O L O G Y

$y$ = M I C H E L A N G E L O

$z$ = H E G L O

# Case study V: LCS

Step 1: Define the problem and subproblems.

Answer: Let $DP[i, j]$ be the longest common substring that can be obtained from substrings $X_1 X_2 \ldots X_i$ and $Y_1 Y_2 \ldots Y_j$.

# Case study V: LCS

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Step 2: Define the goal/output given Step 1.

It is $\boldsymbol{DP[n, m]}$.

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Step 2: Define the goal/output given Step 1.

It is $\boldsymbol{DP[n,m]}$.

Step 3: Define the base cases. "One of two strings is empty". $DP[0,j] = 0$ for all $j$, $DP[i,0] = 0$ for all $i$.

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Answer: Let $DP[i, j]$ be the longest common substring that can be obtained from substrings $X_1X_2 \dots X_i$ and $Y_1Y_2 \dots Y_j$.

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Step 4: Define the recurrence

# Case study V: LCS

## Step 4: Define the recurrence

Case 1: $x_i = X_1 X_2 \dots X_{i-1}$ "$A$"

$y_j = Y_1 Y_2 \dots Y_{j-1}$ "$A$"

Question: What is the LCS of $x_i, y_j$?

# Case study V: LCS

**Step 4**: Define the recurrence

Case 1: $x_i = X_1 X_2 \dots X_{i-1} A$

$y_j = Y_1 Y_2 \dots Y_{j-1} A$

Question: What is the LCS of $x_i, y_j$?

Answer: **1** + the LCS of $\boldsymbol{x_{i-1}, y_{j-1}}$

# Case study V: LCS

## Step 4: Define the recurrence

Case 2: $x_i = X_1 X_2 \ldots X_{i-1}\, A$

$y_j = Y_1 Y_2 \ldots Y_{j-1}\, B$

$LCS(x_{i-1}, y_{j-1})$

Question: What is the LCS of $x_i$, $y_j$?

# Case study V: LCS

Step 4: Define the recurrence

Case 2: $x_i = X_1 X_2 \dots X_{i-1}\, A$

$y_j = Y_1 Y_2 \dots Y_{j-1}\, B$

Question: What is the LCS of $x_i, y_j$?

Answer: the maximum of the LCS of $\boldsymbol{x_{i-1}, y_j}$ and LCS of $\boldsymbol{x_i, y_{j-1}}$

$LCS(x_i, y_j)$

$\geq LCS(x_{i-1}, y_j)$

$LCS(x_i, y_j)$

$\geq LCS(x_i, y_{j-1})$

$LCS(x_i, y_j) \geq LCS(x_{i-1}, y_{j-1})$

# Case study V: LCS

## Step 4: Define the recurrence

(case 1)

$$DP[i,j] = \begin{cases} \text{if } \mathrm{X}_{\mathrm{i}} == \mathrm{Y}_j \text{ then } DP[i-1, j-1] + 1 \\ \text{if } \mathrm{X}_{\mathrm{i}} \neq \mathrm{Y}_j \text{ then } \max(DP[i-1, j], DP[i, j-1]) \end{cases}$$

(case 2)

# Case study V: LCS

$S[n, m]$

$S[n-1, m]$

$S[n-1, m-1]$

## Pseudocode:

Array DP[][], X[], Y[], S[][] = l, u, d

**For** $i = 1$ to $n$ **do**
  $\mathrm{DP}[i, 0] \leftarrow 0$
**For** $j = 1$ to $m$ **do**
  $\mathrm{DP}[0, j] \leftarrow 0$

Initialization

**For** $i = 1$ to $n$ **do**
  **For** $j = 1$ to $m$ **do**
    **If** $X[i] == Y[j]$ **then**
      $\mathrm{DP}[i, j] \leftarrow \mathrm{DP}[i-1, j-1] + 1$ . S[i][j] = d
    **else** $\mathrm{DP}[i, j] \leftarrow \max(\underbrace{\mathrm{DP}[i-1, j]}_{S[i][j] = u}, \underbrace{\mathrm{DP}[i, j-1]}_{S[i][j] = l})$

Bottom up filliing DP

**return** $\mathrm{DP}[n, m]$

Goal

# Case study V: LCS

Example: $x$ is the string "ABCBDAB" and $y$ is the string "BDCABA".

# Case study V: LCS

Example: $x$ is the string "ABCBDAB" and $y$ is the string "BDCABA".

| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | | | | | | |
| 2 | B | 0 | | | | | | |
| 3 | C | 0 | | | | | | |
| 4 | B | 0 | | | | | | |
| 5 | D | 0 | | | | | | |
| 6 | A | 0 | | | | | | |
| 7 | B | 0 | | | | | | |

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| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1+0=1 | | |
| 2 | B | 0 | | | | | | |
| 3 | C | 0 | | | | | | |
| 4 | B | 0 | | | | | | |
| 5 | D | 0 | | | | | | |
| 6 | A | 0 | | | | | | |
| 7 | B | 0 | | | | | | |

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| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1+0=1 |
| 2 | B | 0 | | | | | | |
| 3 | C | 0 | | | | | | |
| 4 | B | 0 | | | | | | |
| 5 | D | 0 | | | | | | |
| 6 | A | 0 | | | | | | |
| 7 | B | 0 | | | | | | |

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| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | |
| 2 | B | 0 | | | | | | |
| 3 | C | 0 | | | | | | |
| 4 | B | 0 | | | | | | |
| 5 | D | 0 | | | | | | |
| 6 | A | 0 | | | | | | |
| 7 | B | 0 | | | | | | |

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Example: $x$ is the string "ABCBDAB" and $y$ is the string "BDCABA".

| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1+0=1 | | | | | |
| 3 | C | 0 | | | | | | |
| 4 | B | 0 | | | | | | |
| 5 | D | 0 | | | | | | |
| 6 | A | 0 | | | | | | |
| 7 | B | 0 | | | | | | |

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Example: $x$ is the string "ABCBDAB" and $y$ is the string "BDCABA".

| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1 | | | | | |
| 3 | C | 0 | | | | | | |
| 4 | B | 0 | | | | | | |
| 5 | D | 0 | | | | | | |
| 6 | A | 0 | | | | | | |
| 7 | B | 0 | | | | | | |

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| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1 | 1 | | | | |
| 3 | C | 0 | | | | | | |
| 4 | B | 0 | | | | | | |
| 5 | D | 0 | | | | | | |
| 6 | A | 0 | | | | | | |
| 7 | B | 0 | | | | | | |

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| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1 | 1 | 1 | | | |
| 3 | C | 0 | | | | | | |
| 4 | B | 0 | | | | | | |
| 5 | D | 0 | | | | | | |
| 6 | A | 0 | | | | | | |
| 7 | B | 0 | | | | | | |

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| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1 | 1 | 1 | 1 | 1+1=2 | |
| 3 | C | 0 | | | | | | |
| 4 | B | 0 | | | | | | |
| 5 | D | 0 | | | | | | |
| 6 | A | 0 | | | | | | |
| 7 | B | 0 | | | | | | |

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| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1 | 1 | 1 | 1 | 2 | |
| 3 | C | 0 | | | | | | |
| 4 | B | 0 | | | | | | |
| 5 | D | 0 | | | | | | |
| 6 | A | 0 | | | | | | |
| 7 | B | 0 | | | | | | |

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| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1 | 1 | 1 | 1 | 2 | 2 |
| 3 | C | 0 | | | | | | |
| 4 | B | 0 | | | | | | |
| 5 | D | 0 | | | | | | |
| 6 | A | 0 | | | | | | |
| 7 | B | 0 | | | | | | |

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| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1 | 1 | 1 | 1 | 2 | 2 |
| 3 | C | 0 | 1 | 1 | 2 | 2 | 2 | 2 |
| 4 | B | 0 | 1 | 1 | 2 | 2 | 3 | 3 |
| 5 | D | 0 | 1 | 2 | 2 | 2 | 3 | 3 |
| 6 | A | 0 | 1 | 2 | 2 | 3 | 3 | 4 |
| 7 | B | 0 | 1 | 2 | 2 | 3 | 4 | 4 |

# Case study V: LCS

Example: $x$ is the string "ABCBDAB" and $y$ is the string "BDCABA".

| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1 | 1 | 1 | 1 | 2 | 2 |
| 3 | C | 0 | 1 | 1 | 2 | 2 | 2 | 2 |
| 4 | B | 0 | 1 | 1 | 2 | 2 | 3 | 3 |
| 5 | D | 0 | 1 | 2 | 2 | 2 | 3 | 3 |
| 6 | A | 0 | 1 | 2 | 2 | 3 | 3 | 4 |
| 7 | B | 0 | 1 | 2 | 2 | 3 | 4 | 4 |

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Example: $x$ is the string "ABCBDAB" and $y$ is the string "BDCABA".

| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1 | 1 | 1 | 1 | 2 | 2 |
| 3 | C | 0 | 1 | 1 | 2 | 2 | 2 | 2 |
| 4 | B | 0 | 1 | 1 | 2 | 2 | 3 | 3 |
| 5 | D | 0 | 1 | 2 | 2 | 2 | 3 | 3 |
| 6 | A | 0 | 1 | 2 | 2 | 3 | 3 | 4 |
| 7 | B | 0 | 1 | 2 | 2 | 3 | 4 | 4 |

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Example: $x$ is the string "ABCBDAB" and $y$ is the string "BDCABA".

| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1 | 1 | 1 | 1 | 2 | 2 |
| 3 | C | 0 | 1 | 1 | 2 | 2 | 2 | 2 |
| 4 | B | 0 | 1 | 1 | 2 | 2 | 3 | 3 |
| 5 | D | 0 | 1 | 2 | 2 | 2 | 3 | 3 |
| 6 | A | 0 | 1 | 2 | 2 | 3 | 3 | 4 |
| 7 | B | 0 | 1 | 2 | 2 | 3 | 4 | 4 |

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Example: $x$ is the string "ABCBDAB" and $y$ is the string "BDCABA".

| | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| i | | $y_j$ | B | D | C | A | B | A |
| 0 | $x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | A | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | B | 0 | 1 | 1 | 1 | 1 | 2 | 2 |
| 3 | C | 0 | 1 | 1 | 2 | 2 | 2 | 2 |
| 4 | B | 0 | 1 | 1 | 2 | 2 | 3 | 3 |
| 5 | D | 0 | 1 | 2 | 2 | 2 | 3 | 3 |
| 6 | A | 0 | 1 | 2 | 2 | 3 | 3 | 4 |
| 7 | B | 0 | 1 | 2 | 2 | 3 | 4 | 4 |