Lecture 7

Dynamic Programming I: Introduction, Memoization, Rod Cutting, Knapsack

CS 161 Design and Analysis of Algorithms

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Dynamic Programming

Technique for solving optimization problems.

Solve problem by solving **sub**-problems and combine:

This is called Optimal substructure property.

Dynamic Programming

Technique for solving optimization problems.

Solve problem by solving **sub**-problems and combine:

This is called Optimal substructure property.

- Similar to divide-and-conquer: recursion (for solving sub-problems)
- Sub-problems overlap: solve them only once!

DP = recursion + re-use (Memoization)

Dynamic Programming

Example: Given a positive integer numbers n, compute Fibonacci F_n. Definition: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

Recursion (slow):

$$\begin{array}{l} \mathrm{Fib}(n) \\ \quad \textbf{If } n \leq 2 \textbf{ then return } 1 \\ \quad \textbf{return } \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2) \end{array}$$

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Recursion (slow):

Why is it slow?

$\mathrm{Fib}(n)$
  If $n \leq 2$ **then return** 1
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Recursion (slow):

Why is it slow? F(6)

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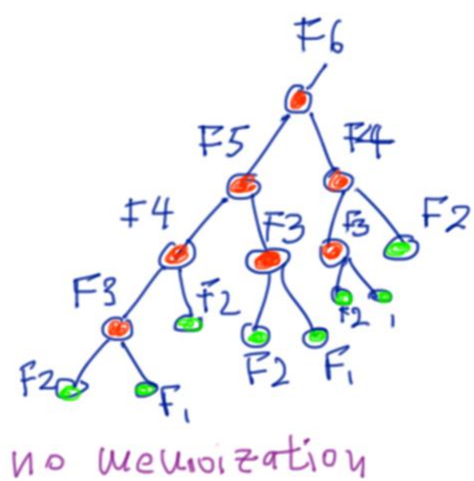

Red nodes: Recursive calls.

Green nodes: Bases cases.

F(5) is computed once, F(4) **twice**,

F(3) **three times**, F(2) **five times**, F(1) **three times**

Dynamic Programming

Example: Given a positive integer numbers n, compute Fibonacci F_n. Definition: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

Recursion (slow):

Exponential time

$$\begin{array}{l}\mathrm{Fib}(n)\\ \quad \textbf{If } n \le 2 \textbf{ then return } 1\\ \quad \textbf{return } \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2)\end{array}$$

Running time
$T(n) = T(n-1) + T(n-2)$
which is $\Omega(2^{n/2})$

Red nodes: Recursive calls.

Green nodes: Bases cases.

F(5) is computed once, F(4) **twice**, F(3) **three times**, F(2) **five times**, F(1) **three times**

Dynamic Programming

Example: Given a positive integer numbers n, compute Fibonacci F_n. Definition: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

Memoization (fast):

Array mem[]
Fib(n)
  If mem[n] non-empty **then**
    return mem[n]
  If $n \leq 2$ **then** mem[n] $= 1$
    mem[n] $=$ Fib($n-1$) $+$ Fib($n-2$)
  return mem[n]

Dynamic Programming

Example: Given a positive integer numbers n, compute Fibonacci F_n. Definition: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

Linear time: Let's see F(6)

Memoization (fast):

Running time: $\Theta(n)$

Fib(n) will be invoked twice: (a) first recursion and (b) second memoization

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  If mem[n] non-empty **then**
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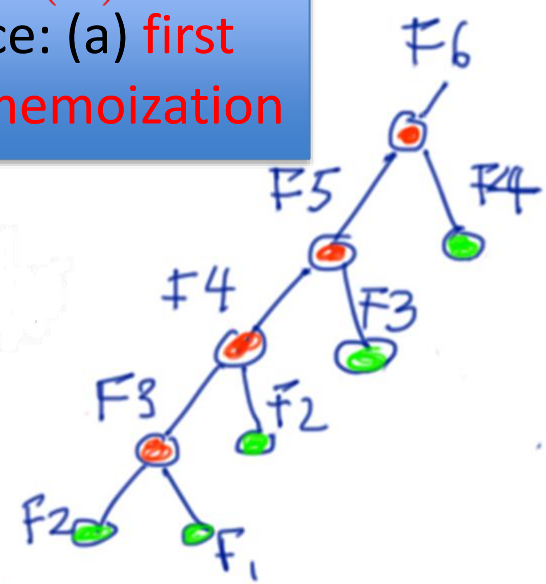


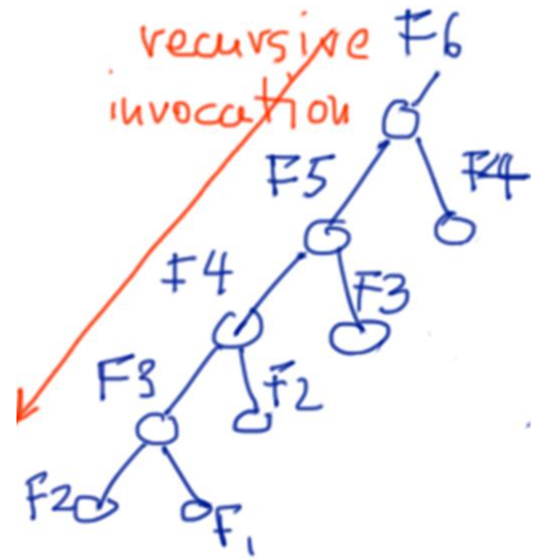
Dynamic Programming

DP = recursion + re-use (Memoization)

Two approaches in Dynamic Programming

1. Top-down approach:

If solution is stored in the array, return it (memoization). Otherwise solves subproblems **recursively**



Dynamic Programming

DP = recursion + re-use (Memoization)

Two approaches in Dynamic Programming

2. Bottom-up approach:

Solves subproblems **iteratively** in the order of smallest to largest subproblems.

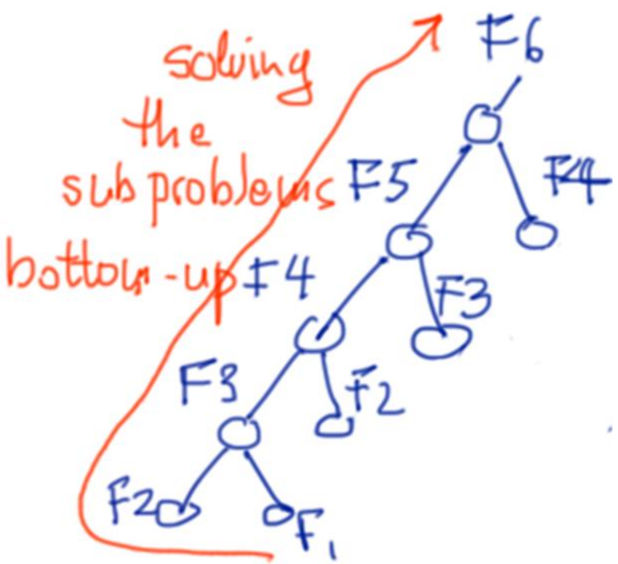
Array fib[]

$\text{fib}[1] \leftarrow 1, \text{fib}[2] \leftarrow 1$

For $i = 3$ to n **do**

  $\text{fib}[i] = \text{fib}(i-1) + \text{fib}(i-2)$

return $\text{fib}[n]$



Case study I: Rod cutting problem

Problem: You are given a rod of size n and a table of prices $p_1, \dots, p_n$ where p_i is the price in the market of a rod of size i. Determine the maximum revenue obtained by cutting the rode into pieces and selling these to the market

Example: $n = 9$,

length i	1	2	3	4	5	6	7	8	9
price p_i	1	5	8	9	10	17	17	20	24

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Answer: Cut the rod in two pieces, 3 and 6 and get revenue $p_3 + p_6 = 25$.

Brute force (slow): For each possible cut, compute the revenue and keep the maximum. How many possibilities? For $n = 4$, we have

1+1+1+1, 1+1+2, 1+2+1, 2+1+1, 2+2, 1+3, 3+1, 4.

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1+1+1+1, 1+1+2, 1+2+1, 2+1+1, 2+2, 1+3, 3+1, 4.

Exponential many 2^{n-1}
Hence exponential time

Case study I: Rod cutting problem

General Approach

Step 1: Define the problem and subproblems.

Answer: Let $DP[k]$ be the maximum value I can get from rod with size k.

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It is $\boldsymbol{DP[n]}$.

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It is $\boldsymbol{DP[n]}$.

Step 3: Define the base cases

It is $DP[0] = 0$.

Step 4: Define the recurrence

Case study I: Rod cutting problem

General Approach

Step 4: Define the **recurrence**.

Create a recursive relationship between the subproblems (the tricky part).

Question: Given a rod of size k, where should I cut it first?

length k

Case study I: Rod cutting problem

General Approach

Step 4: Define the recurrence.

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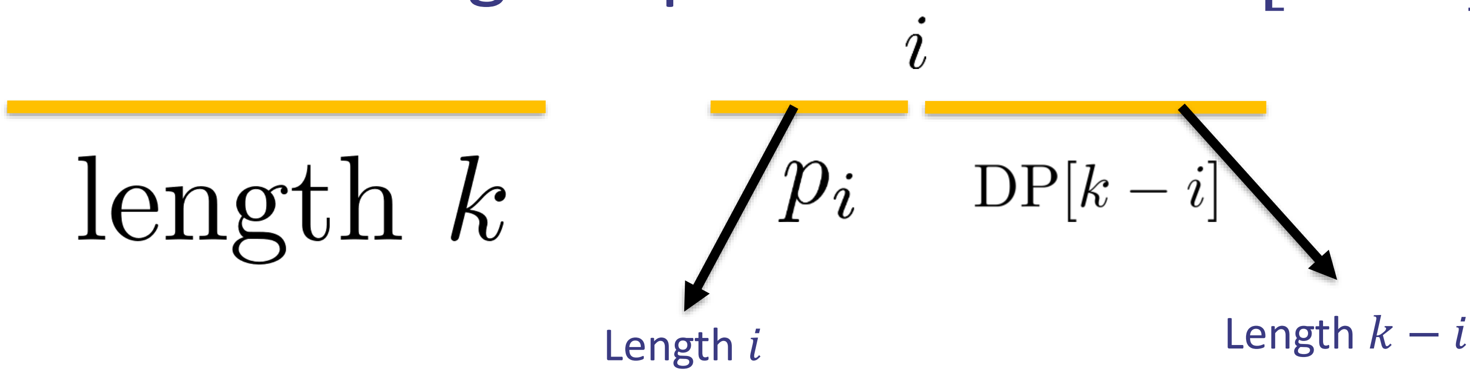
Question: Given a rod of size k, where should I cut it first? Cut at index i gives price of i and $DP[k-i]$



Case study I: Rod cutting problem

General Approach

Step 4: Define the recurrence.

Create a recursive relationship between the subproblems (the tricky part).

Question: Given a rod of size k, where should I cut it first? Cut at index i gives price of i and $DP[k-i]$

i

length k $\qquad p_i \quad \mathrm{DP}[k-i]$

$\mathrm{DP}[k]$ is the max of $p_i + \mathrm{DP}[k-i]$ for all $1 \leq i \leq k$

$$\boxed{\mathrm{DP}[k] = \max_{1 \leq i \leq k} \; p_i + \mathrm{DP}[k-i]}$$

Case study I: Rod cutting problem

size of piece	1	2	3	4
market price	2	5	7	8

Rod of size $n = 4$

$DP[k]$ = maximum value from rod with size k.

$$DP[k] = \max_{1 \le i \le k} p_i + DP[k-i]$$

$$DP[0] = 0$$

$DP[0]$	$DP[1]$	$DP[2]$	$DP[3]$	$DP[4]$
0				

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Rod of size $n = 4$

$DP[k]$ = maximum value from rod with size k.

$$DP[k] = \max_{1 \le i \le k} \{p_i + DP[k-i]\}$$

$$DP[0] = 0$$

$$DP[1] = p_1 + DP[0] = 2$$

+2

$DP[0]$	$DP[1]$	$DP[2]$	$DP[3]$	$DP[4]$
0	2			

Case study I: Rod cutting problem

size of piece	1	2	3	4
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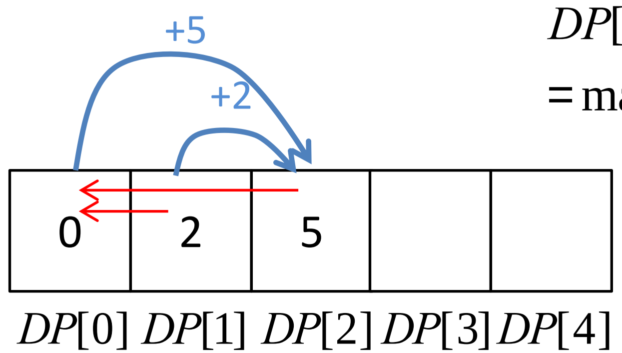
Rod of size $n = 4$

$DP[k]$ = maximum value from rod with size k.

$$DP[k] = \max_{1 \le i \le k} \{p_i + DP[k-i]\}$$

$$DP[0] = 0$$

$$DP[2] = \max\{p_2 + DP[0], p_1 + DP[1]\}$$
$$= \max\{5+0, 2+2\}$$



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size of piece	1	2	3	4
market price	2	5	7	8

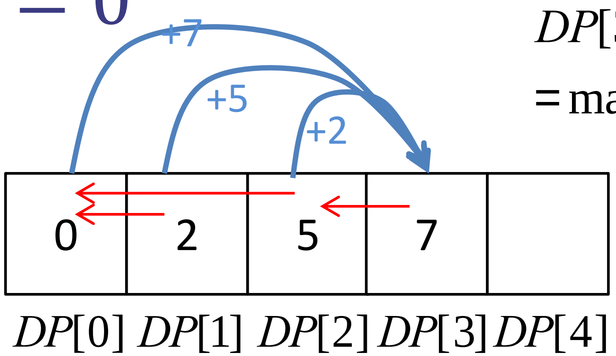
Rod of size $n = 4$

$DP[k]$ = maximum value from rod with size k.

$$DP[k] = \max_{1 \le i \le k} \{p_i + DP[k-i]\}$$

$$DP[0] = 0$$

$$DP[3] = \max\{p_3 + DP[0], p_2 + DP[1], p_1 + DP[2]\}$$
$$= \max\{7+0, 5+2, 2+5\}$$



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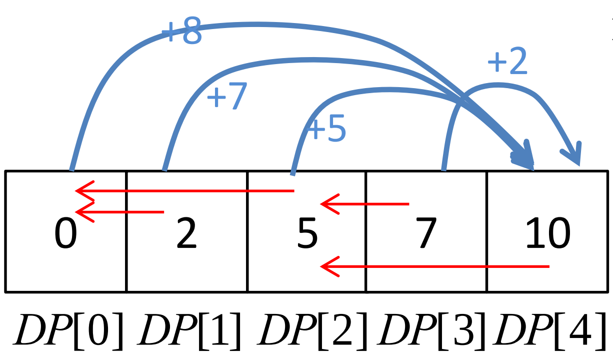
Rod of size $n = 4$

$DP[k]$ = maximum value from rod with size k.

$$DP[k] = \max_{1 \le i \le k} \{p_i + DP[k-i]\}$$

$$DP[0] = 0$$

$$DP[4] =$$
$$\max\{p_4 + DP[0], p_3 + DP[1], p_2 + DP[2], p_1 + DP[3]\}$$
$$= \max\{8+0, 7+2, 5+5, 2+7\}$$



Case study I: Rod cutting problem

Pseudocode:

$$
\begin{array}{l}
\text{Array DP[], S[]} \\
\text{DP}[0] \leftarrow 0 \\
\textbf{For } k = 1 \text{ to } n \textbf{ do} \\
\quad \max \leftarrow 0 \\
\quad \textbf{For } i = 1 \text{ to } k \textbf{ do} \\
\quad\quad \textbf{If } \max < p[i] + \text{DP}[k-i] \ \textbf{ then} \\
\quad\quad\quad \max \leftarrow p[i] + \text{DP}[k-i] \\
\\
\quad \text{DP}[k] \leftarrow \max \\
\textbf{return } \text{DP}[n]
\end{array}
$$

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Pseudocode:

Array DP[], S[]

$DP[0] \leftarrow 0$

For $k = 1$ to n **do**

  $\max \leftarrow 0$

  For $i = 1$ to k **do**

    If $\max < p[i] + DP[k-i]$ **then**

      $\max \leftarrow p[i] + DP[k-i]$

  $DP[k] \leftarrow \max$

return $DP[n]$

Base case

Implement recursive formula with double for-loop

GOAL

Running time: $\Theta(n^2)$

Case study I: Rod cutting problem

Pseudocode:

Array DP[], S[]

DP[0] $\leftarrow$ 0

For $k = 1$ to n **do**

  max $\leftarrow$ 0

  For $i = 1$ to k **do**

    If max $< p[i]+$ DP$[k-i]$ **then**

      max $\leftarrow p[i]+$ DP$[k-i]$

  DP$[k]$ $\leftarrow$ max

return DP$[n]$

Base case

Implement recursive formula with double for-loop

GOAL

Question: What is the cut that gives maximum revenue?

Case study I: Rod cutting problem

Pseudocode:

Array DP[], S[]

$\text{DP}[0] \leftarrow 0$

For $k = 1$ to n **do**

  $\max \leftarrow 0$

  For $i = 1$ to k **do**

    If $\max < p[i] + \text{DP}[k-i]$ **then**

      $\max \leftarrow p[i] + \text{DP}[k-i]$

      $S[k] \leftarrow i$

  $\text{DP}[k] \leftarrow \max$

return $\text{DP}[n]$

Answer: Use pointer S

Base case

Implement recursive formula with double for-loop

GOAL

Case study I: Rod cutting problem

Example: $n = 9$

length i	1	2	3	4	5	6	7	8	9
price p_i	1	5	8	9	10	17	17	20	24

len	0	1	2	3	4	5	6	7	8	9
DP[]	0	1	5	8	10	13	17	18	22	25
S[]	0	0	0	0	2	2	0	1	2	3

Solution for $n = 9$:

Need to cut at $S[9] = 3$. Then remaining length is 9-3=6.

Need to cut at $S[6] = 0$. The solution is 3+6 which give 8+17=25

Case study I: Rod cutting problem

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DP[]	0	1	5	8	10	13	17	18	22	25
S[]	0	0	0	0	2	2	0	1	2	3

Solution for $n = 9$:
Need to cut at $S[9] = 3$. Then remaining length is 9-3=6.
Need to cut at $S[6] = 0$. The solution is 3+6 which give 8+17=25

Solution for $n = 5$:
Need to cut at $S[5] = 2$. Then remaining length is 5-2=3.
Need to cut at $S[3] = 0$. The solution is 2+3 which give 5+8=13

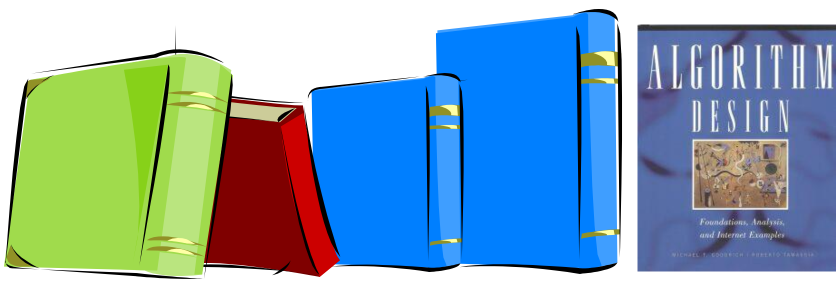
Case study II: 0/1 Knapsack

Problem: A set of n items, with each item i having positive weight w_i and positive benefit v_i. You are asked to choose items with **maximum total benefit** so that the **total weight** is **at most W**

Example:

"knapsack" with 9 lbs capacity

Items:



Weight:	4 lbs	2 lbs	2 lbs	6 lbs	2 lbs
Benefit:	\$20	\$3	\$6	\$25	\$80

Solution:

- item 5 (\$80, 2 lbs)
- item 3 (\$6, 2lbs)
- item 1 (\$20, 4lbs)

Case study II: 0/1 Knapsack

Idea: Dynamic Programming (first attempt).

Step 1: Define the problem and subproblems.

Answer: Let $DP[k]$ be the maximum value I can get from items $\{1, \dots, k\}$ without exceeding W.

Step 2: Define the goal/output given Step 1.

It is $\boldsymbol{DP[n]}$.

Step 3: Define the base cases

It is $DP[0] = 0$.

Step 4: Define the recurrence

Case study II: 0/1 Knapsack

Idea: Dynamic Programming (first attempt).

Step 4: Define the recurrence

Item k will be used or not.

$$DP[k] \; = \; \max(DP[k-1], DP[k-1] + v_k)$$

But how do we know that DP[k-1] does not exceed $W - w_k$ in weight so we can use k?

Case study II: 0/1 Knapsack

Idea: Dynamic Programming (correct attempt).

Step 1: Define the problem and subproblems.

Answer: Let $DP[k, j]$ be the maximum value I can get from items $\{1, \dots, k\}$ without exceeding j.

Case study II: 0/1 Knapsack

Idea: Dynamic Programming (correct attempt).

Step 1: Define the problem and subproblems.

Answer: Let $DP[k, j]$ be the maximum value I can get from items $\{1, \ldots, k\}$ without exceeding j.

Step 2: Define the goal/output given Step 1.

It is $\boldsymbol{DP[n, W]}$.

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Answer: Let $DP[k, j]$ be the maximum value I can get from items $\{1, \dots, k\}$ without exceeding j.

Step 2: Define the goal/output given Step 1.

It is $\boldsymbol{DP[n, W]}$.

Step 3: Define the base cases

It is $DP[0, j] = 0$ for all j and $DP[i, 0] = 0$ for all i.

Step 4: Define the recurrence

Case study II: 0/1 Knapsack

Idea: Dynamic Programming (correct attempt).

Step 4: Define the recurrence

Item k will be **used** or **not**.

$$DP[k][j] \; = \; \max(\mathbf{DP[k-1][j-w_k] + v_k}, \mathbf{DP[k-1][j]})$$

Case study II: 0/1 Knapsack

Idea: Dynamic Programming (correct attempt).

Step 4: Define the recurrence

Item k will be **used** or **not**.

$$DP[k][j] = \max(\mathbf{DP[k-1][j-w_k] + v_k}, \mathbf{DP[k-1][j]})$$

Question: How do we know that item k does not have weight more than j?

Case study II: 0/1 Knapsack

Idea: Dynamic Programming (correct attempt).

Step 4: Define the recurrence

Item k will be **used** or **not**.

$$DP[k][j] = \begin{cases} \text{if } w_k \leq j & \max(\mathbf{DP[k-1][j-w_k]+v_k}, \mathbf{DP[k-1][j]}) \\ \text{If } w_k > j & \boldsymbol{DP[k-1][j]} \end{cases}$$

Answer: Add an if statement in the recurrence.

Case study II: 0/1 Knapsack

Example: 3 items, $W = 4$
$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

Initialization:

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0				
2	0				
3	0				

Case study II: 0/1 Knapsack

Example: 3 items, $W = 4$
$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0	0 ($j < w_1$)			
2	0	0 ($j < w_2$)			
3	0	0 ($j < w_3$)			

Case study II: 0/1 Knapsack

Example: 3 items, $W = 4$
$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0	0	max(0,v_1+0)		
2	0	0			
3	0	0			

Case study II: 0/1 Knapsack

Example: 3 items, $W = 4$
$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0	0	1		
2	0	0	max(1,v_2+0)		
3	0	0			

Case study II: 0/1 Knapsack

Example: 3 items, $W = 4$
$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0	0	1		
2	0	0	1		
3	0	0	1 $(j < w_3)$		

Case study II: 0/1 Knapsack

Example: 3 items, $W = 4$
$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0	0	1	max(0,v_1+0)	
2	0	0	1		
3	0	0	1		

Case study II: 0/1 Knapsack

Example: $3 \text{ items}, W = 4$
$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0	0	1	1	
2	0	0	1	max(1,v_2+0)	
3	0	0	1		

Case study II: 0/1 Knapsack

Example: 3 items, $W = 4$
$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0	0	1	1	
2	0	0	1	1	
3	0	0	1	max(1,v_3+0)	

Case study II: 0/1 Knapsack

Example: 3 items, $W = 4$
$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0	0	1	1	max(0,v_1+0)
2	0	0	1	1	
3	0	0	1	5	

Case study II: 0/1 Knapsack

Example: 3 items, $W = 4$
$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0	0	1	1	1
2	0	0	1	1	max(1,v_2+1)
3	0	0	1	5	

Case study II: 0/1 Knapsack

Example: 3 items, $W = 4$
$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0	0	1	1	1
2	0	0	1	1	2
3	0	0	1	5	max(2,0+v_3)

Case study II: 0/1 Knapsack

Example: 3 items, $W = 4$

$w_1 = 2, v_1 = 1, w_2 = 2, v_2 = 1, w_3 = 3, v_3 = 5$

	j=0	1	2	3	4
i=0	0	0	0	0	0
1	0	0	1	1	1
2	0	0	1	1	2
3	0	0	1	5	**5**

Case study II: 0/1 Knapsack

Pseudocode:

Array DP[][]
For $i = 0$ to n **do**
  $\mathrm{DP}[i, 0] \leftarrow 0$
For $j = 1$ to W **do**
  $\mathrm{DP}[0, j] \leftarrow 0$
For $i = 1$ to n **do**
  For $j = 1$ to W **do**
    If $j < w_i$ **then**
      $\mathrm{DP}[i][j] \leftarrow \mathrm{DP}[i-1][j]$
    else $\mathrm{DP}[i][j] \leftarrow \max(\mathrm{DP}[i-1][j], \mathrm{DP}[i-1][j-w_i] + v_i$
return $\mathrm{DP}[n][W]$

Initialization

Bottom up filliing DP

Goal

Case study II: 0/1 Knapsack

Pseudocode:

Array DP[][]
For $i = 0$ to n **do**
  $\text{DP}[i, 0] \leftarrow 0$
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return $\text{DP}[n][W]$

Initialization

Bottom up filliing DP

Goal

Running time: $\Theta(nW)$