Lecture 6

Divide and Conquer IV: integer multiplication, further examples

CS 161 Design and Analysis of Algorithms

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Divide and Conquer (recap)

Steps of method:

- **Divide** input into parts (smaller problems)
- **Conquer** (solve) each part recursively
- **Combine** results to obtain solution of original

$$T(n) = \text{divide time} + T(n_1) + T(n_2) + \ldots + T(n_k) + \text{combine time}$$

Case study VI: Integer Multiplication

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Example: $a = 101$, $b = 111$. Answer: 100011.

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Standard Algorithm: $\Theta(n^2)$ time. Summing two n-bit numbers takes $\Theta(n)$ time.

Addition

	1	1	1	1	1	1	0	1	
		1	1	0	1	0	1	0	1
+		0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0	

Multiplication

				1	1	0	1	0	1	0	1				
			*	0	1	1	1	1	1	0	1				
			1	1	0	1	0	1	0	1	0				
		0	0	0	0	0	0	0	0	0					
		1	1	0	1	0	1	0	1	0					
	1	1	0	1	0	1	0	1	0						
	1	1	0	1	0	1	0	1	0						
1	1	0	1	0	1	0	1	0							
	1	1	0	1	0	1	0	1	0						
0	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	1	0

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	1	1	0	1	0	1	0	1	0						
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0	1	1	0	1	0	0	0	0	0	0	0	0	0	1	0

Case study VI: Integer Multiplication

Idea: Divide and conquer.

$$a = \underbrace{a_1 a_2 \dots a_{n/2}}_{a_L} \underbrace{a_{n/2+1} \dots a_n}_{a_R}$$

$$b = \underbrace{b_1 b_2 \dots b_{n/2}}_{b_L} \underbrace{b_{n/2+1} \dots b_n}_{b_R}$$

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Recursively

$$a \cdot b = a_R \cdot b_R + 2^{\frac{n}{2}} a_L \cdot b_R + a_R \cdot 2^{\frac{n}{2}} b_L + 2^{\frac{n}{2}} a_L \cdot 2^{\frac{n}{2}} b_L$$

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Running time: $T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n) \to \Theta(n^2)$ by Master thm

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Idea (modified): Divide and conquer.

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$$a \cdot b = 2^{\frac{n}{2}} a_L \cdot 2^{\frac{n}{2}} b_L + a_R \cdot b_R +$$

$$2^{\frac{n}{2}}((a_L - a_R) \cdot (b_R - b_L) + a_L \cdot b_L + a_R \cdot b_R)$$

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Recursively compute

1. $(a_L - a_R)(b_R - b_L)$
2. $a_L \cdot b_L$
3. $a_R \cdot b_R$

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Recursively compute

1. $(a_L - a_R)(b_R - b_L)$
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$\Theta(n^{1.585})$

Running time: $T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n) \rightarrow \Theta(n^{\log_2 3})$ by Master thm

Case study VII: Computing powers

Problem: Given two positive integers numbers a, n compute a^n .

Example: $a = 3, n = 4$. Answer: 81.

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Obvious approach:

$\text{ans} \leftarrow 1$
For $i = 1$ to n **do**
  $\text{ans} \leftarrow a \cdot \text{ans}$
return ans

$\Theta(n)$ operations

Can we do better?

Case study VII: Computing powers

Problem: Given two positive integers numbers a, n compute a^n .

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Idea: Divide and Conquer.

Divide n in $n/2$ and $n/2$. Compute x = $a^{n/2}$ recursively. Return x^2.

Be careful on the parity of n.

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Idea: Divide and Conquer.

$\text{Power}(a, n)$
If $n == 1$ **then return** a
$x \leftarrow \text{Pow}(a, \lfloor n/2 \rfloor)$
If $n \bmod 2 == 0$ **then**
  return $x \cdot x$
else return $a \cdot x \cdot x$

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Base case

Divide + Conquer

Combine

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Running time: $T(n) = T\left(\frac{n}{2}\right) + \Theta(1) \rightarrow \Theta(\log n)$ by Master thm

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Remark: Same works for powers of Matrices.

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Case study VIII: Fibonacci sequence

Problem: Given a positive integer numbers n, compute Fibonacci F_n .

Definition: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

First 10 numbers of sequence: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55$

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Obvious approach:

```
ans1 ← 1
ans2 ← 1
If n ≤ 2 then return 1
For i = 3 to n do
  temp ← ans1
  ans1 ← ans1 + ans2
  ans2 ← temp
return ans
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Idea: Express F_n as a power of a Matrix.

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} F_{n-1} + F_{n-2} \\ F_{n-1} \end{pmatrix}$$

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$$\vdots$$

$$\begin{pmatrix} F_3 \\ F_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} F_2 \\ F_1 \end{pmatrix}$$

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$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

F_n is $a + b$ and F_{n-1} is $c + d$!

Case study VIII: Fibonacci sequence

Problem: Given a positive integer numbers n, compute Fibonacci F_n .

Solution:

Compute matrix $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^{\boldsymbol{n-2}}$ in $\boldsymbol{\Theta(\log n)}$ time.

Return the sum of the entries of first row.

Case study IX: From practice problems

Problem: Suppose you have an array A of n intervals $(x_1, y_1), \ldots, (x_n, y_n)$, where x_i, y_i are positive integers such that $x_i \leq y_i$. The interval (x_i, y_i) represents the set of integers between x_i and y_i. For example, the interval $(3, 8)$ represents the set $\{3, 4, 5, 6, 7, 8\}$.

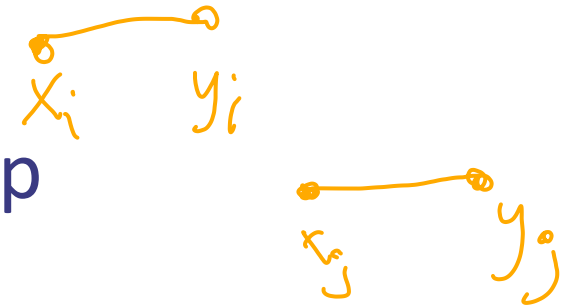
Define the **overlap** of two intervals to be the number of integers that are members of both intervals. For example $(3, 8)$ and $(4, 9)$ have overlap 5 (numbers $4, 5, 6, 7, 8$) and $(1, 2)$ and $(3, 4)$ have overlap 0. Find the size of maximum overlap among all possible pairs of intervals.

Example: $(1, 2), (3, 4), (3, 8), (4, 9)$. Answer: 5.

Case study IX: From practice problems

Obvious approach: For every pair i, j of intervals, find the overlap. Keep the maximum.

Suppose $x_i \leq x_j$.
(x_i, y_i) and (x_j, y_j) have overlap



$$\max(\min(y_i, y_j) - x_j + 1, 0).$$

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$\Theta(n^2)$ running time

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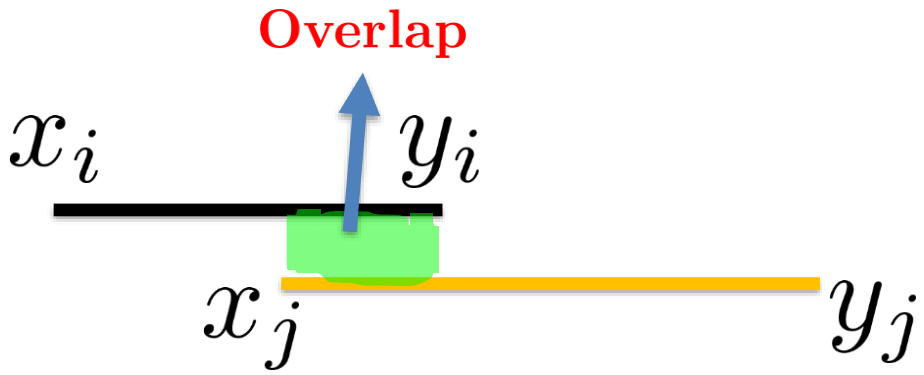

Can we do better?

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Idea: Use divide and conquer. Suppose we first sort the intervals in increasing order of x-coordinate.

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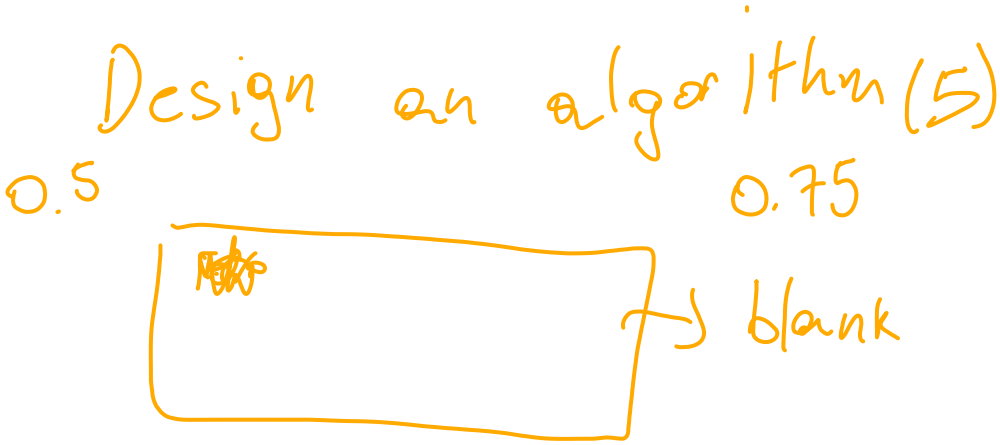
- Divide the intervals in two parts L and R.
- Recursively find max overlap for each part maxL and maxR.
- Combine step?



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- Divide the intervals in two parts L and R.
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- Combine step: maximum of maxL and maxR?

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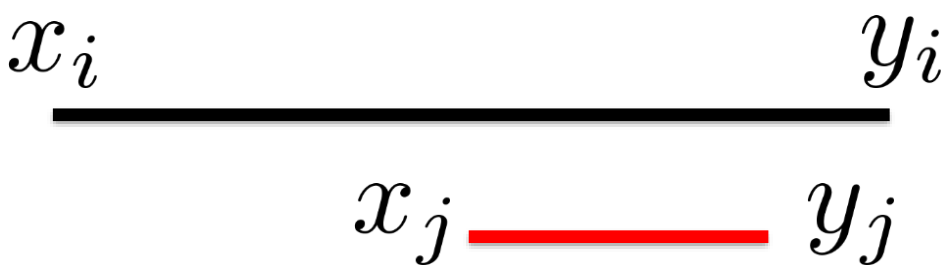
Idea: Use divide and conquer. Suppose we first sort the intervals in increasing order of x-coordinate.

- Divide the intervals in two parts L and R.
- Recursively find max overlap for each part maxL and maxR.
- Combine step: Check overlap between an interval in L and an interval in R. This should be in $\Theta(n)$.

We will scan the intervals once. One index for L and one index for R.

Case study IX: From practice problems

Combine step: Black is in L, red in R.



Overlap is $(y_j - x_j + 1)$. We can remove interval j from R.

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Combine step: Black is in L, red in R.

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Combine step: Black is in L, red in R.

Overlap is $(y_i - x_j + 1)$. We can remove interval i from L.

All intervals after j in R will not give larger overlap with interval i.

Case study IX: From practice problems

Pseudocode:

Maxoverlap($A[1:n]$)

If $n == 1$ **return** 0

maxL $\leftarrow$Maxoverlap($A[1:n/2]$)

maxR $\leftarrow$Maxoverlap($A[n/2+1:n]$)

maxComb $\leftarrow 0$

$i \leftarrow 1, j \leftarrow n/2+1$

While $i \leq n/2$ and $j \leq n$ **do**

If maxComb $<$ overlap(i,j) **then**

maxComb $=$ overlap(i,j)

If case 1 **then** $j \leftarrow j+1$

else If case 2 **then** $i \leftarrow i+1$

return maximum of maxL, maxR and maxComb

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maxL $\leftarrow$Maxoverlap($A[1:n/2]$) — $T(n/2)$ Running time

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$i \leftarrow 1, j \leftarrow n/2+1$

While $i \leq n/2$ and $j \leq n$ **do** — $\Theta(n)$ Running time

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$\Theta(n \log n)$ Running time

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$\Theta(n)$ Running time