Lecture 3

Divide and Conquer I: Introduction, Merge-sort and Master Theorem

CS 161 Design and Analysis of Algorithms

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Recursion

- Definition:
  - Solving a task where the solution depends on solutions to smaller instances of the same problem, by using functions/algorithms that call themselves.

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- Example:

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$$\textbf{If } n == 0 \textbf{ then return } 1$$

$$\textbf{return } n \cdot \text{Factorial}(n-1)$$

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  – Solving a task where the solution depends on solutions to smaller instances of the same problem, by using functions/algorithms that call themselves.

Running time: $T(n) = T(n-1) + \Theta(1)$

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Since $T(n-2) = T(n-3) + 1$, by substitution we have

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Since $T(n-1) = T(n-2) + 1$, by substitution we have

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Since $T(n-2) = T(n-3) + 1$, by substitution we have

$$T(n) = T(n-3) + 3.$$

$$\vdots$$

Since $T(n-i) = T(n-i-1) + 1$, by substitution we have

$$T(n) = T(n-i) + i.$$

Recursion

Exercise: Let $T(n) = T(n-1) + 1$, with $T(1) = 1$.

Find $T(n)$.

Solution:

Since $T(n-i) = T(n-i-1) + 1$, by substitution we have

$$T(n) = T(n-i) + i.$$

By setting $i = n - 1$ we have

$$T(n) = T(1) + n - 1 = n \text{ which is } \Theta(n).$$

Mergesort - A fast sorting recursive Algorithm

Have seen algorithms like insertion sort that have running time (worst case) $\Theta(n^2)$.

- Key idea:

  Divide input into two parts of equal size

  Sort each part recursively

  Merge the two sorted parts to obtain the solution.

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Example: Sort the following 11 numbers

9 3 4 220 1 3 10 5 8 7 2

Divide

Recursion

Merge

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  Sort each part recursively
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9 3 4 220 1   3 10 5 8 7 2   **Divide**

1 3 4 9 220   2 3 5 7 8 10   **Recursion**

Merge

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- Key idea:
  Divide input into two parts of equal size
  Sort each part recursively
  Merge the two sorted parts to obtain the solution.

Example: Sort the following 11 numbers

9 3 4 220 1 3 10 5 8 7 2

9 3 4 220 1   3 10 5 8 7 2   Divide

1 3 4 9 220   2 3 5 7 8 10   Recursion

1 2 3 3 4 5 7 8 9 10 220   Merge

Mergesort - A fast sorting recursive Algorithm

- Tricky part: Merge

Problem: Given two sorted arrays A, B, merge them to a sorted array C.

Solution: Index i for A, index j for B, index k for C.

$$\begin{array}{l}
\textbf{While } k < \mathrm{len(A)} + \mathrm{len(B)} \textbf{ do} \\
\quad \textbf{If } A[i] <= B[j] \textbf{ then} \\
\quad\quad C[k] \leftarrow A[i] \\
\quad\quad i = i + 1, k = k + 1 \\
\quad \textbf{else} \\
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\end{aligned}$$

$A: 1\ 3\ 4\ 9\ 220$ $\quad$ $B: 2\ 3\ 5\ 7\ 8\ 10$

i $\quad$ j

$C:$

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    $C[k] \leftarrow A[i]$

    $i = i + 1, k = k + 1$

  else

    $C[k] \leftarrow B[j]$

    $j = j + 1, k = k + 1$

$A : 1\ 3\ 4\ 9\ 220$ $B : 2\ 3\ 5\ 7\ 8\ 10$

i j

$C : 1$

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$A : 1\ 3\ 4\ 9\ 220$ $\qquad$ $B : 2\ 3\ 5\ 7\ 8\ 10$

i $\qquad$ j

$C : 1\ 2$

Mergesort - A fast sorting recursive Algorithm

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$A : 1\ 3\ 4\ 9\ 220$ $\quad$ $B : 2\ 3\ 5\ 7\ 8\ 10$

i $\quad$ j

$C : 1\ 2\ 3$

Mergesort - A fast sorting recursive Algorithm

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$A : 1\ 3\ 4\ 9\ 220$ $B : 2\ 3\ 5\ 7\ 8\ 10$

i j

$C : 1\ 2\ 3\ 3$

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i $\quad$ j

$C : 1\ 2\ 3\ 3\ 4$

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$A : 1\ 3\ 4\ 9\ 220$ $\quad B : 2\ 3\ 5\ 7\ 8\ 10$

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$C : 1\ 2\ 3\ 3\ 4\ 5$

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$A : 1\ 3\ 4\ 9\ 220$ $\quad$ $B : 2\ 3\ 5\ 7\ 8\ 10$

$\qquad\quad\ \ i \qquad\qquad\qquad\qquad\qquad j$

$C : 1\ 2\ 3\ 3\ 4\ 5\ 7$

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$$

A : 1 3 4 9 220 $\qquad$ B : 2 3 5 7 8 10

i $\qquad$ j

C : 1 2 3 3 4 5 7 8 9

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\end{array}
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Running time: $\Theta(n)$

$A: 1\ 3\ 4\ 9\ 220$ $\quad$ $B: 2\ 3\ 5\ 7\ 8\ 10$

i $\quad$ j

$C: 1\ 2\ 3\ 3\ 4\ 5\ 7\ 8\ 9\ 10\ 220$

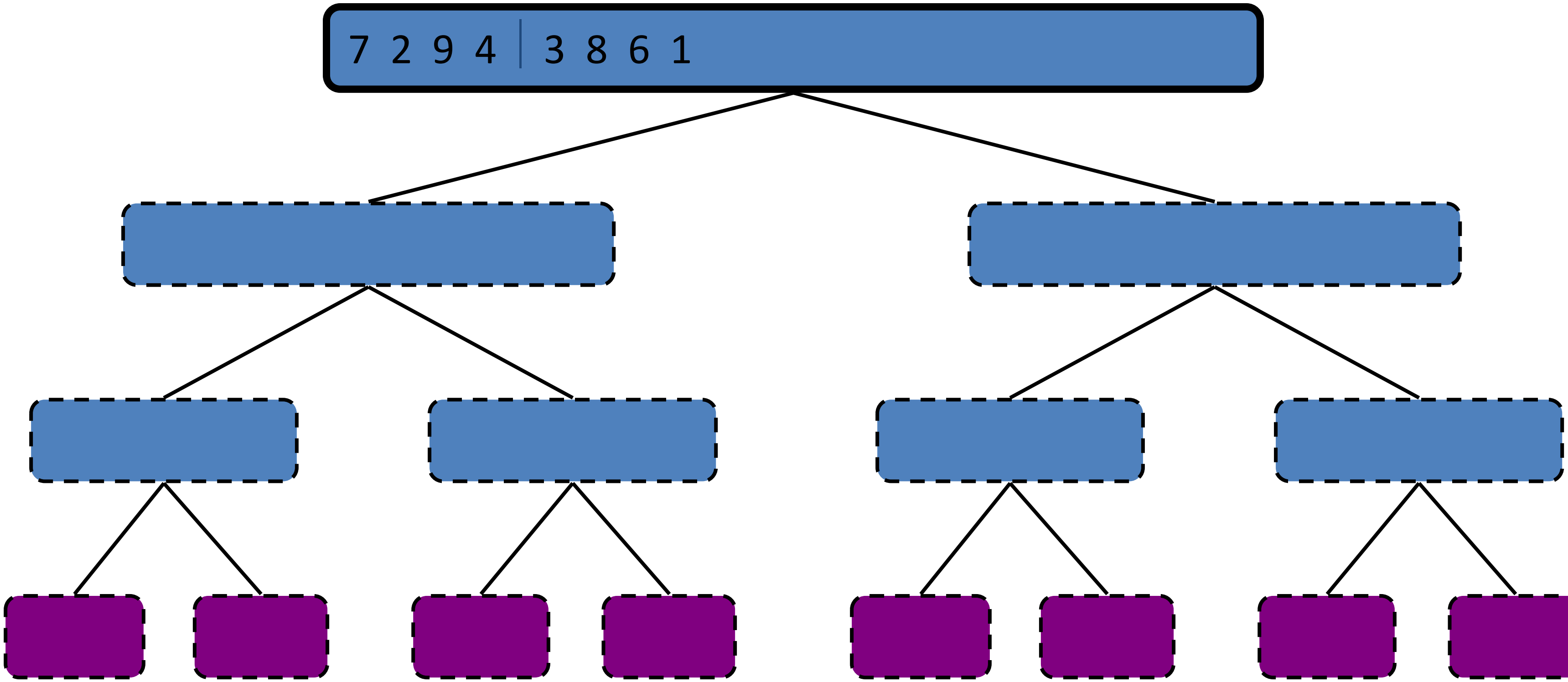
Mergesort

- Pseudocode:

$$
\begin{aligned}
&\text{Mergesort}(A[1:n])\\
&\quad \textbf{If } n == 1 \textbf{ then}\\
&\qquad \textbf{return } A\\
&\quad \text{Mergesort }(A[1:\tfrac{n}{2}])\\
&\quad \text{Mergesort }(A[\tfrac{n}{2}+1:n])\\
&\quad C \leftarrow \text{Merge}(A[1:\tfrac{n}{2}], A[\tfrac{n}{2}+1:n])\\
&\quad \textbf{return } C
\end{aligned}
$$

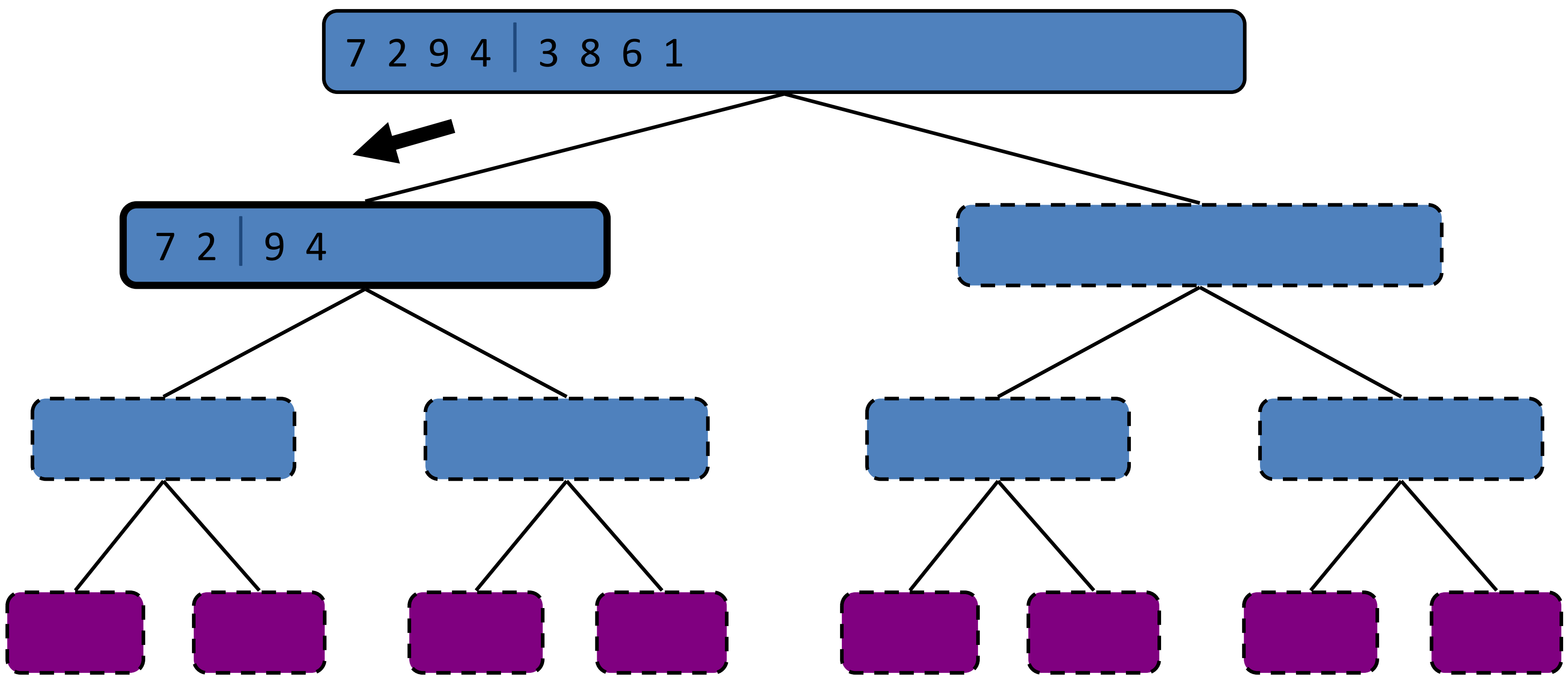
Mergesort (Example)

Example: Sort **7 2 9 4 3 8 6 1**



Mergesort (Example)

Recursive call, left part



Mergesort (Example)

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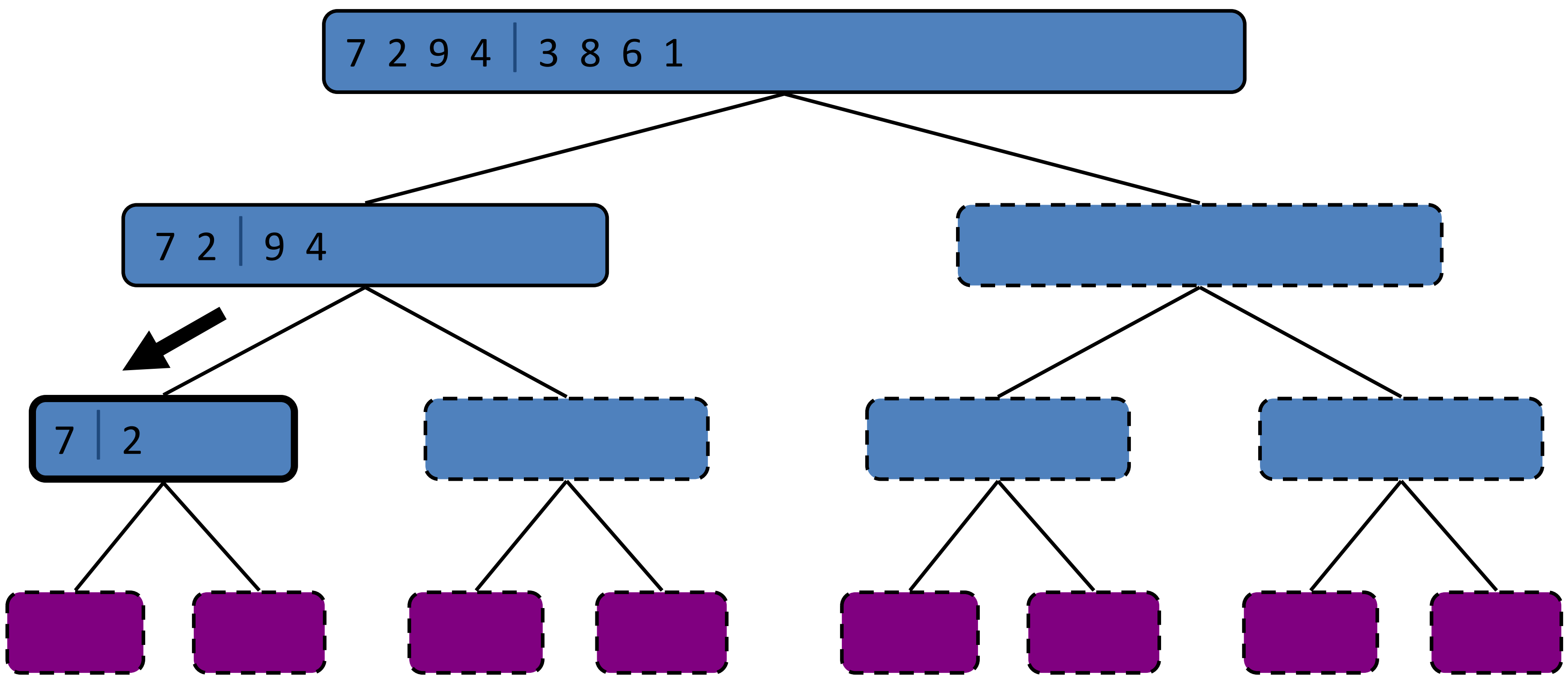
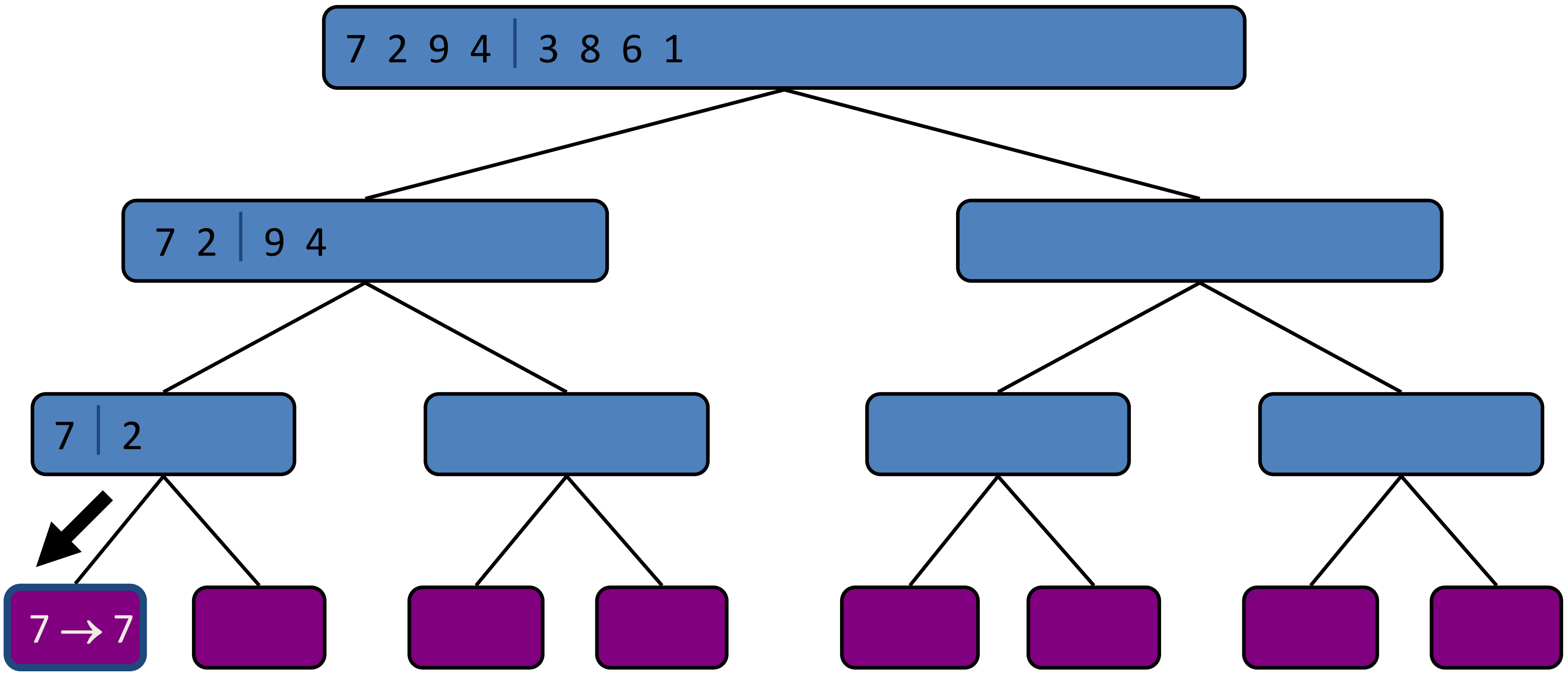


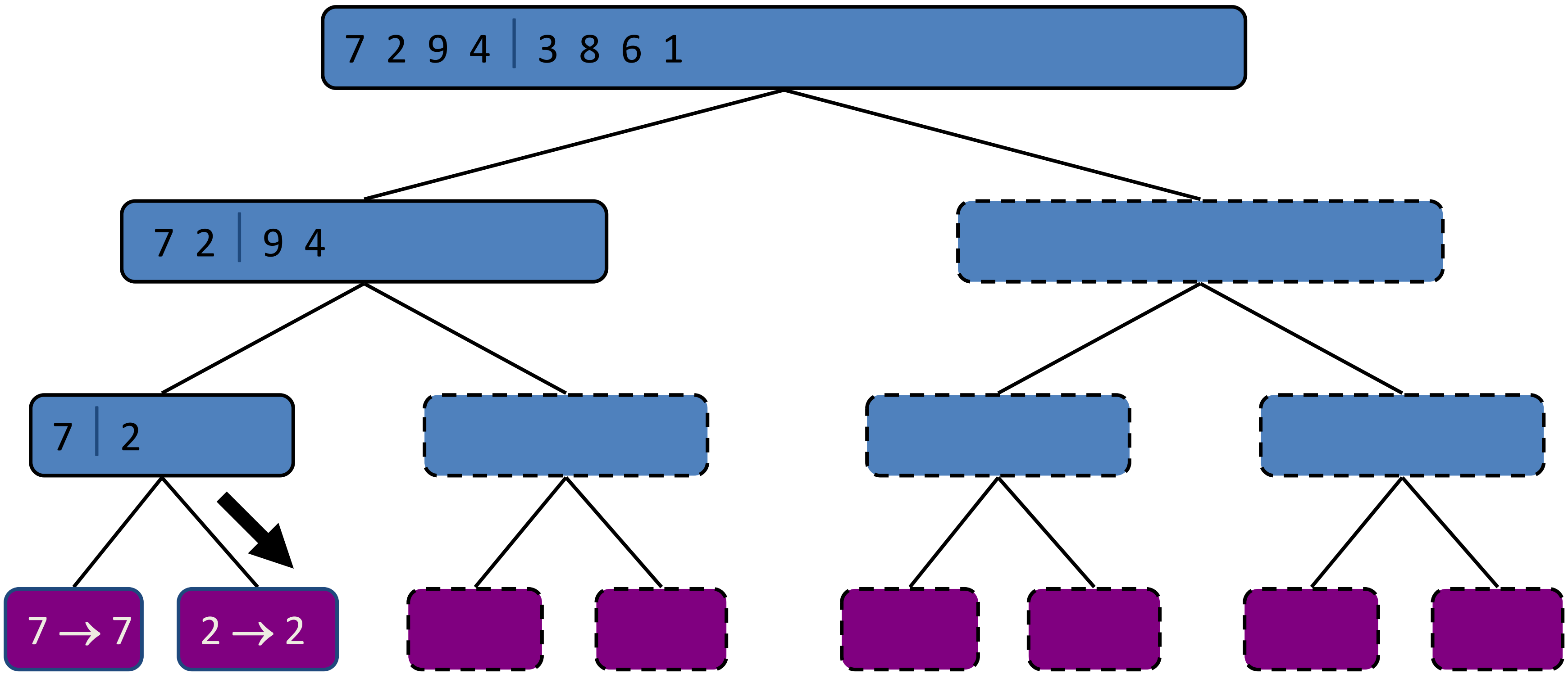
Mergesort (Example)

Recursive call, base case



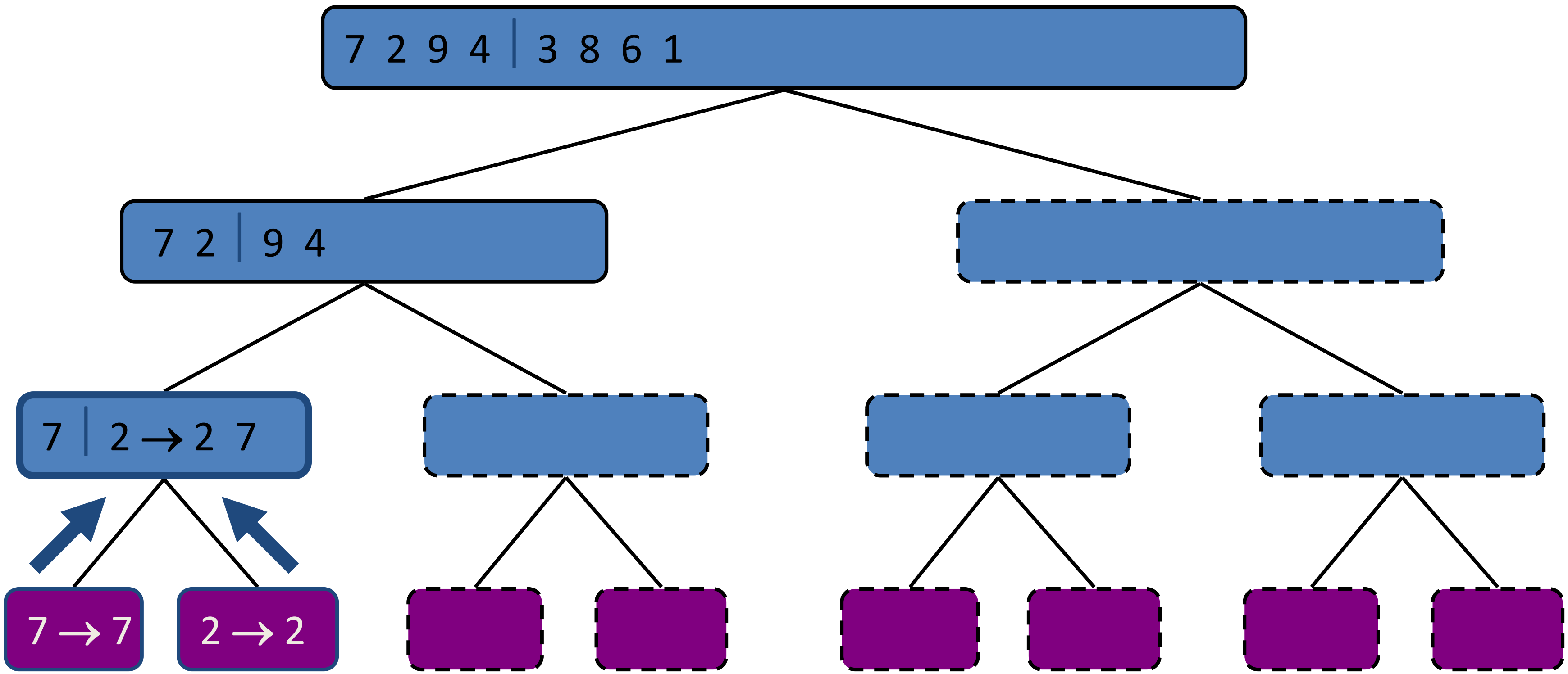
Mergesort (Example)

Recursive call, base case



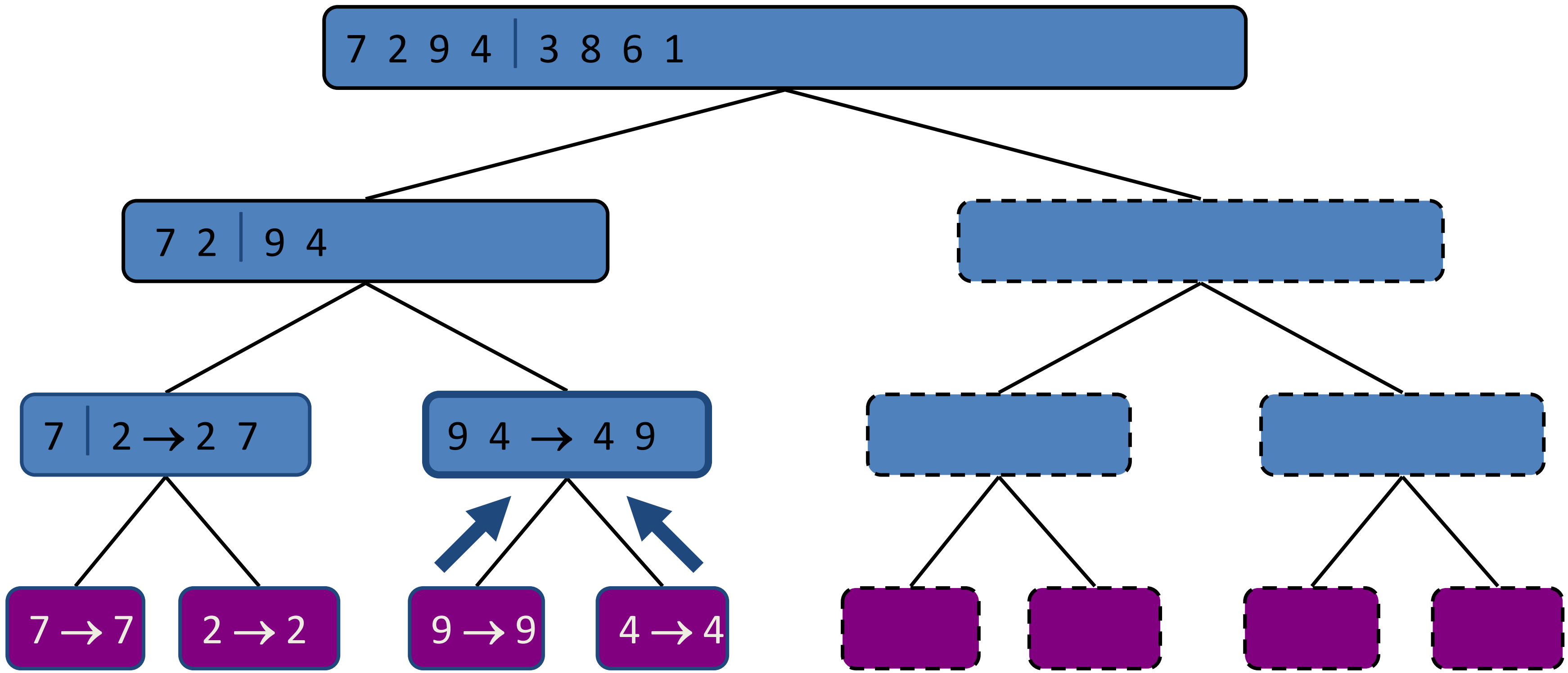
Mergesort (Example)

Merge



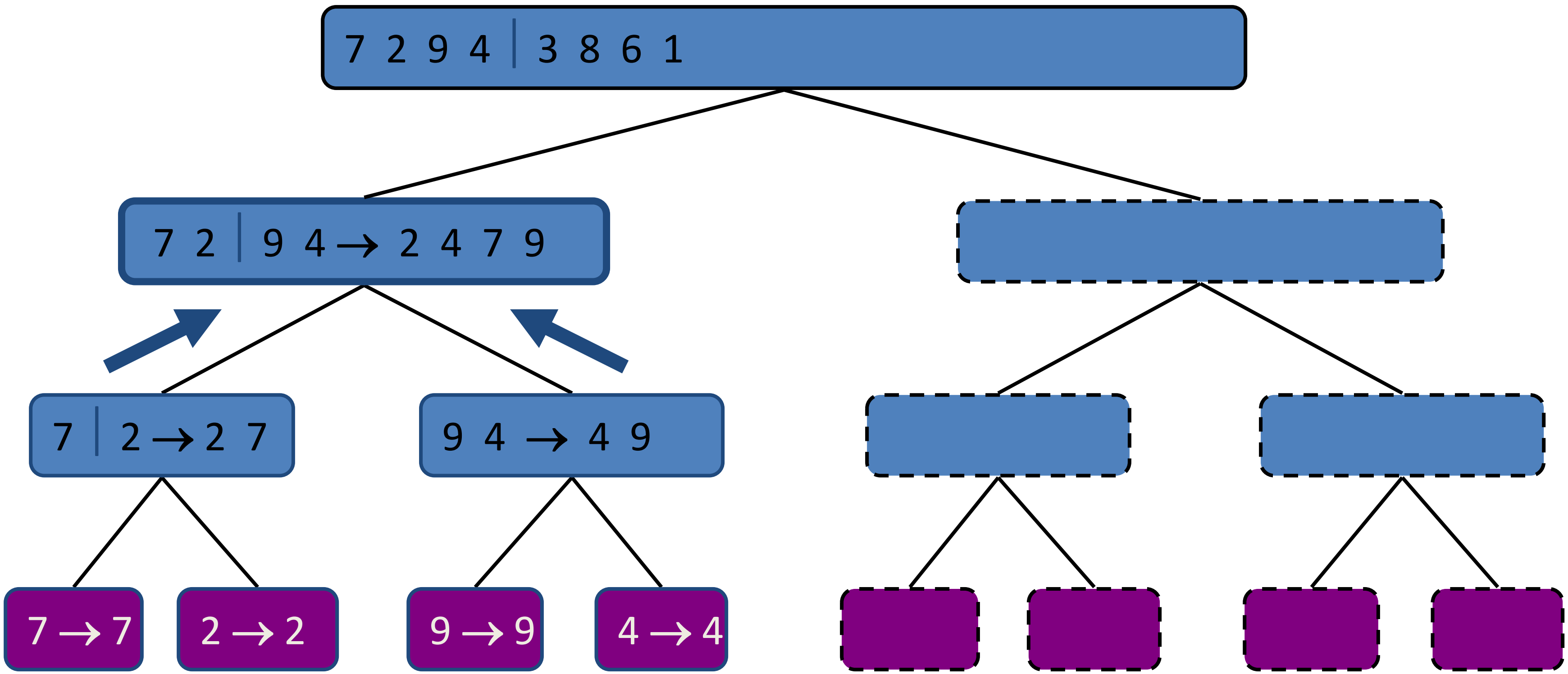
Mergesort (Example)

Similarly

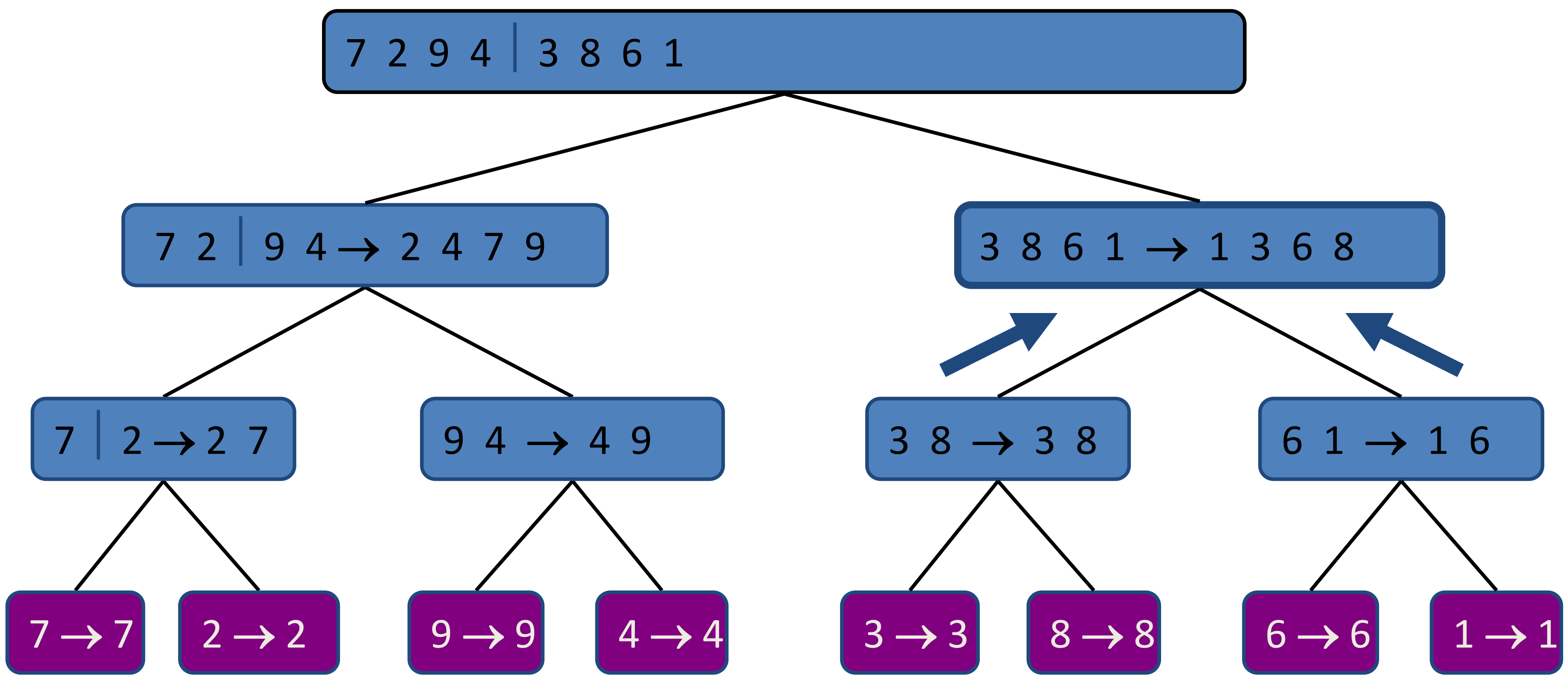


Mergesort (Example)

Merge



Mergesort (Example)



Mergesort (Example)

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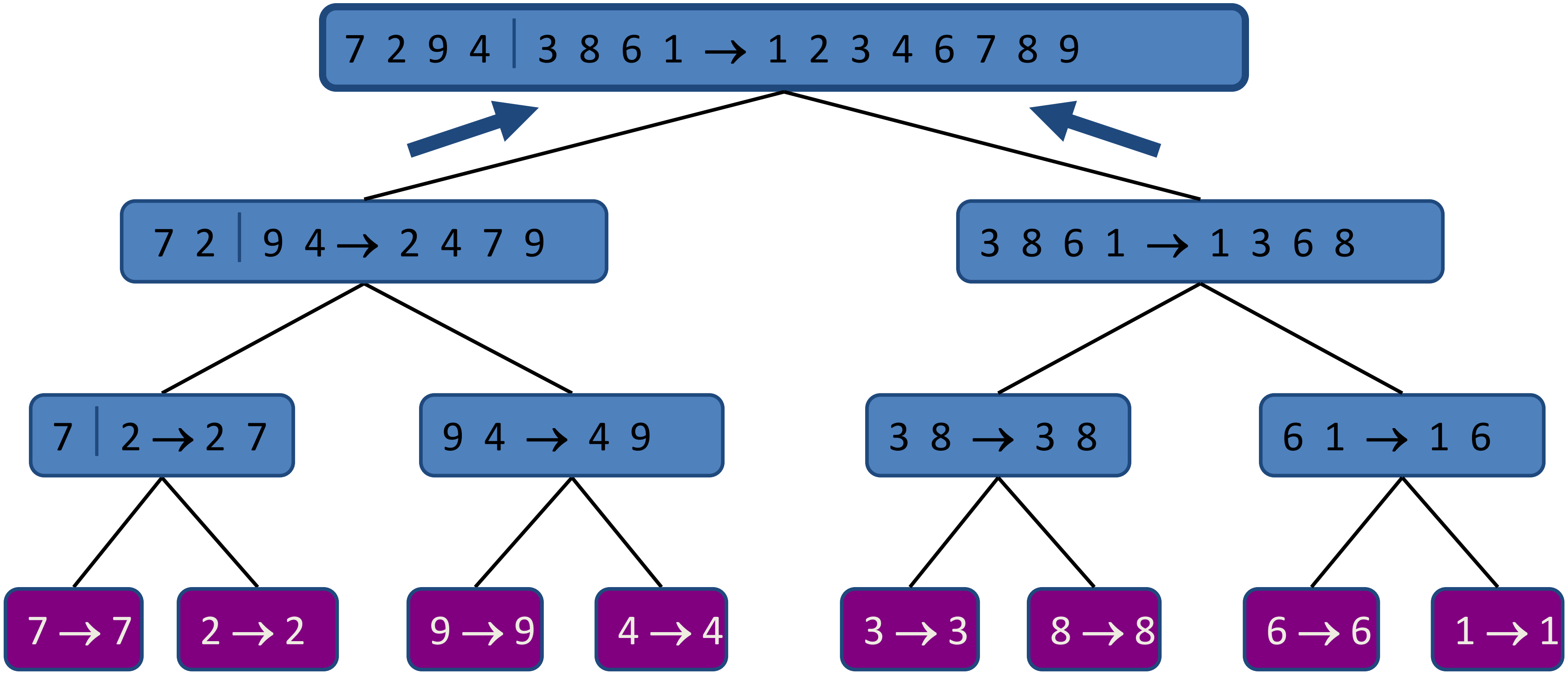

Mergesort

- Pseudocode:

$$
\begin{array}{l}
\text{Mergesort}(A[1:n]) \\
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\quad\quad \textbf{return } A \\
\quad \text{Mergesort }(A[1:\frac{n}{2}]) \\
\quad \text{Mergesort }(A[\frac{n}{2}+1:n]) \\
\quad C \leftarrow \text{Merge}(A[1:\frac{n}{2}], A[\frac{n}{2}+1:n]) \\
\quad \textbf{return } C
\end{array}
$$

- Running time:

$$T(n) = T(n/2) + T(n/2) + \Theta(n) + \Theta(1)$$

$$= 2T(n/2) + \Theta(n)$$

Mergesort

- Pseudocode:

$$
\begin{array}{l}
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\quad \textbf{If } n == 1 \textbf{ then} \\
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\quad \text{Mergesort}\,(A[1:\frac{n}{2}]) \\
\quad \text{Mergesort}\,(A[\frac{n}{2}+1:n]) \\
\quad C \leftarrow \text{Merge}(A[1:\frac{n}{2}], A[\frac{n}{2}+1:n]) \\
\quad \textbf{return } C
\end{array}
$$

- Running time:

$$
\begin{aligned}
T(n) &= T(n/2) + T(n/2) + \Theta(n) + \Theta(1) \\
&= 2T(n/2) + \Theta(n)
\end{aligned}
$$

How to analyze?

Master theorem

$$T(n) = \begin{cases} T(1) = \Theta(1) \\ aT(n/b) + f(n) \end{cases}$$

- The Master Theorem can find the order of $T(n)$ which is defined recursively.

Master theorem

$$T(n) = \begin{cases} T(1) = \Theta(1) \\ aT(n/b) + f(n) \end{cases}$$

- The **Master Theorem** can find the order of $T(n)$ which is defined recursively.

- Key idea: The answer depends on the comparison between $f(n)$ and $n^{\log_b a}$. So, there are **3** cases!

Master theorem

$$T(n) = \begin{cases} T(1) = \Theta(1) \\ aT(n/b) + f(n) \end{cases}$$

1. **If** $f(n)$ **is** $O(n^{\log_b a - \epsilon})$**, then** $T(n)$ **is** $\Theta(n^{\log_b a})$
2. If $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. If $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$,
   need to check $af(n/b) < f(n)$.

Case 1: $n^{\log_b a}$ **dominates** $f(n)$

Master theorem

$$T(n) = \begin{cases} T(1) = \Theta(1) \\ aT(n/b) + f(n) \end{cases}$$

1. If $f(n)$ is $O(n^{\log_b a - \epsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. **If $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$**
3. If $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$,
   need to check $af(n/b) < f(n)$.

Case 2: $n^{\log_b a}$ **have same order as** $f(n)$ (up to $\log^k n$)

Master theorem

$$T(n) = \begin{cases} T(1) = \Theta(1) \\ aT(n/b) + f(n) \end{cases}$$

$f(n) = \dfrac{n \log n}{n} \geq \dfrac{n^{1+\epsilon}}{n}$

$\log n \cancel{\geq} \quad n^{\epsilon}$ $\qquad \log n < n^{\epsilon}$

1. If $f(n)$ is $O(n^{\log_b a - \epsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. If $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. **If $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$, need to check $af(n/b) < f(n)$.**

Case 3: $n^{\log_b a}$ **is dominated by** $f(n)$ (+ another condition)

Master Theorem (Examples)

$$T(n) = 4T(n/2) + n$$

Master Theorem (Examples)

$$T(n) = 4T(n/2) + n$$

Solution:

We have $a = 4$, $b = 2$, hence $\log_b a = 2$.

Since $n^2 \gg n$, we are in case 1. Answer is $\Theta(n^2)$

Master Theorem (Examples)

$$T(n) = 2T(n/2) + \Theta(n)$$

Mergesort running time

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$$T(n) = 2T(n/2) + \Theta(n)$$

Mergesort running time

Solution:

We have $a = 2$, $b = 2$, hence $\log_b a = 1$.

Since n is $\Theta(n)$, we are in case 2 with $k = 0$.
Answer is $\Theta(n \log n)$

Master Theorem (Examples)

$$T(n) = 2T(n/2) + n\log n$$

Master Theorem (Examples)

$$T(n) = 2T(n/2) + n\log n$$

Solution:

We have $a = 2$, $b = 2$, hence $\log_b a = 1$.

Since n is $\Theta(n)$, we are in case 2 with $k = 1$.
Answer is $\Theta(n\log^2 n)$

Master Theorem (Examples)

$$T(n) = 9T(n/3) + n^3$$

Master Theorem (Examples)

$$T(n) = 9T(n/3) + n^3$$

Solution:

We have $a = 9$, $b = 3$, hence $\log_b a = 2$.

Since $n^2 \ll n^3$, we are in case 3. Need to check that $\boldsymbol{9\left(\frac{n}{3}\right)^3 < n^3}$ which is equivalent to $\boldsymbol{\frac{n^3}{3} < n^3}$ (holds)

Answer is $\Theta(n^3)$

Master Theorem (Examples)

$$T(n) = T(n/2) + \Theta(1)$$

Binary search running time

Master Theorem (Examples)

$$T(n) = T(n/2) + \Theta(1)$$

Binary search running time

Solution:

We have $a = 1$, $b = 2$, hence $\log_b a = 0$.

Since n^0=1 is $\Theta(1)$, we are in case 2 with $k = 0$.
Answer is $\Theta(\log n)$

Divide and conquer method

- Steps of method:
  - **Divide** input into parts (smaller problems)
  - **Conquer** (solve) each part recursively
  - **Combine** results to obtain solution of original

$$
\begin{aligned}
T(n) = &\ \text{divide time} \\
&+ T(n_1) + T(n_2) + \ldots + T(n_k) \\
&+ \text{combine time}
\end{aligned}
$$

Case study I: Counting inversions

Given numbers $A_1, \ldots, A_n$ in an array A, compute the number of inversions.

(i, j) is an inversion: $A_i > A_j$ and $i < j$.

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Example [18, 29, 12, 15, 32, 10] has 9 inversions:

(18,12), (18,15), (18,10), (29,12), (29,15), (29,10), (12,10), (15,10), (32,10)

Case study I: Counting inversions

Given numbers $A_1, \ldots, A_n$ in an array A, compute the number of inversions.

(i, j) is an inversion: $A_i > A_j$ and $i < j$.

- Minimum number of inversions is zero (when sorted in increasing order)
- Maximum number of inversions is $\binom{n}{2}$ (when sorted in decreasing order)

Case study I: Counting inversions

- For all i, j with $i < j$, compare A_i with A_j and increase counter if $A_i > A_j$. Total number of comparisons $\frac{n(n-1)}{2}$. Running time $\Theta(n^2)$.

- Use Divide and conquer. Tricky part the combine step.

Case study I: Counting inversions

- For all i, j with $i < j$, compare A_i with A_j and increase counter if $A_i > A_j$. Total number of comparisons $\frac{n(n-1)}{2}$. Running time $\Theta(n^2)$.
- Use Divide and conquer. Tricky part the combine step.
- Question: Assume that $B_1, \ldots, B_k$ and $C_1, \ldots, C_l$ are both sorted. Can you compute the number of inversions of the sequence $B_1, \ldots, B_k, C_1, \ldots, C_l$?

Case study I: Counting inversions

- Question: Assume that $B_1, \dots, B_k$ and $C_1, \dots, C_l$ are both sorted. Can you compute the number of inversions of the sequence $B_1, \dots, B_k, C_1, \dots, C_l$?

If $B_i > C_j \geq B_{i-1}$ there are

including C_j

$B_1, \dots, B_i, \dots, B_k$	$C_1, \dots, C_j, \dots, C_l$
i	j

$C_j > C_1, C_2, \dots, C_{j-1}$

$C_j < \underbrace{B_i, B_{i+1}, B_{i+2}, \dots, B_k}$ $\quad k-i+1$

Case study I: Counting inversions

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If $B_i > C_j \geq B_{i-1}$ there are

$k - i + 1$ including C_j

$B_1, \ldots, B_i, \ldots, B_k$ $\qquad$ $C_1, \ldots, C_j, \ldots, C_l$

i $\qquad$ j

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

Solution: Index i for B, index j for C, index k for A, counter.

$$\textbf{While } k < \mathrm{len(B)} + \mathrm{len(C)} \textbf{ do}$$

$$\quad \textbf{If } B[i] <= C[j] \textbf{ then}$$

$$\quad\quad A[k] \leftarrow B[i]$$

$$\quad\quad i = i + 1, k = k + 1$$

$$\quad \textbf{else}$$

$$\quad\quad A[k] \leftarrow C[j]$$

$$\quad\quad \mathrm{counter} = \mathrm{counter} + \mathrm{len}(B) - i + 1$$

$$\quad\quad j = j + 1, k = k + 1$$

$B :$ 1 3 4 9 220 $\quad$ $C :$ 2 3 5 7 8 10

i $\quad$ j

$A :$

$\mathrm{counter} = 0$

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

Solution: Index i for B, index j for C, index k for A, counter.

While $k < \text{len(B)} + \text{len(C)}$ **do**

  If $B[i] <= C[j]$ **then**

    $A[k] \leftarrow B[i]$

    $i = i + 1, k = k + 1$

  else

    $A[k] \leftarrow C[j]$

    $\text{counter} = \text{counter} + \text{len}(B) - i + 1$

    $j = j + 1, k = k + 1$

B : 1 3 4 9 220 C : 2 3 5 7 8 10

i j

$A : 1$

$\text{counter} = 0$

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

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$\textbf{While } k < \text{len(B)} + \text{len(C)} \textbf{ do}$

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$\quad\quad A[k] \leftarrow C[j]$

$\quad\quad \text{counter} = \text{counter} + \text{len}(B) - i + 1$

$\quad\quad j = j + 1, k = k + 1$

$B: 1\ 3\ 4\ 9\ 220$ $\qquad$ $C: 2\ 3\ 5\ 7\ 8\ 10$

i $\qquad$ j

$A: 1\ 2$

$\text{counter} = 4$

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

Solution: Index i for B, index j for C, index k for A, counter.

$$
\begin{array}{l}
\textbf{While } k < \mathrm{len(B)} + \mathrm{len(C)} \textbf{ do} \\
\quad \textbf{If } B[i] <= C[j] \textbf{ then} \\
\quad\quad A[k] \leftarrow B[i] \\
\quad\quad i = i + 1, k = k + 1 \\
\quad \textbf{else} \\
\quad\quad A[k] \leftarrow C[j] \\
\quad\quad \mathrm{counter} = \mathrm{counter} + \mathrm{len}(B) - i + 1 \\
\quad\quad j = j + 1, k = k + 1
\end{array}
$$

$B: 1\ 3\ 4\ 9\ 220$ $\qquad C: 2\ 3\ 5\ 7\ 8\ 10$

$\qquad\quad i \qquad\qquad\qquad\qquad\qquad j$

$A: 1\ 2\ 3$

$\mathrm{counter} = 4$

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

Solution: Index i for B, index j for C, index k for A, counter.

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\begin{array}{l}
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\quad\quad A[k] \leftarrow C[j] \\
\quad\quad \mathrm{counter} = \mathrm{counter} + \mathrm{len}(B) - i + 1 \\
\quad\quad j = j + 1, k = k + 1
\end{array}
$$

$B: 1\ 3\ 4\ 9\ 220$ $\quad$ $C: 2\ 3\ 5\ 7\ 8\ 10$

i $\quad$ j

$A: 1\ 2\ 3\ 3$

$\mathrm{counter} = 7$

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

Solution: Index i for B, index j for C, index k for A, counter.

While $k < \text{len(B)} + \text{len(C)}$ **do**

  If $B[i] <= C[j]$ **then**

    $A[k] \leftarrow B[i]$

    $i = i + 1, k = k + 1$

  else

    $A[k] \leftarrow C[j]$

    $\text{counter} = \text{counter} + \text{len}(B) - i + 1$

    $j = j + 1, k = k + 1$

$B : 1\ 3\ 4\ 9\ 220$ $C : 2\ 3\ 5\ 7\ 8\ 10$

i j

$A : 1\ 2\ 3\ 3\ 4$

$\text{counter} = 7$

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

Solution: Index i for B, index j for C, index k for A, counter.

$$
\begin{aligned}
&\textbf{While } k < \text{len(B)} + \text{len(C)} \textbf{ do}\\
&\quad \textbf{If } B[i] <= C[j] \textbf{ then}\\
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&\quad \textbf{else}\\
&\quad\quad A[k] \leftarrow C[j]\\
&\quad\quad \text{counter} = \text{counter} + \text{len}(B) - i + 1\\
&\quad\quad j = j + 1, k = k + 1
\end{aligned}
$$

B : 1 3 4 9 220 $\qquad$ C : 2 3 5 7 8 10

i $\qquad$ j

$A : 1\ 2\ 3\ 3\ 4\ 5$

$\text{counter} = 9$

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

Solution: Index i for B, index j for C, index k for A, counter.

$$
\begin{aligned}
&\textbf{While } k < \text{len(B)} + \text{len(C)} \textbf{ do}\\
&\quad \textbf{If } B[i] <= C[j] \textbf{ then}\\
&\qquad A[k] \leftarrow B[i]\\
&\qquad i = i + 1, k = k + 1\\
&\quad \textbf{else}\\
&\qquad A[k] \leftarrow C[j]\\
&\qquad \text{counter} = \text{counter} + \text{len}(B) - i + 1\\
&\qquad j = j + 1, k = k + 1
\end{aligned}
$$

$B: 1\ 3\ 4\ 9\ 220$ $\quad C: 2\ 3\ 5\ 7\ 8\ 10$

i $\quad j$

$A: 1\ 2\ 3\ 3\ 4\ 5\ 7$

$\text{counter} = 11$

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

Solution: Index i for B, index j for C, index k for A, counter.

$$\textbf{While } k < \text{len(B)} + \text{len(C)} \textbf{ do}$$

$\quad \textbf{If } B[i] <= C[j] \textbf{ then}$

$\quad\quad A[k] \leftarrow B[i]$

$\quad\quad i = i + 1, k = k + 1$

$\quad \textbf{else}$

$\quad\quad A[k] \leftarrow C[j]$

$\quad\quad \text{counter} = \text{counter} + \text{len}(B) - i + 1$

$\quad\quad j = j + 1, k = k + 1$

$B : 1\ 3\ 4\ 9\ 220$ $\qquad C : 2\ 3\ 5\ 7\ 8\ 10$

i $\qquad j$

$A : 1\ 2\ 3\ 3\ 4\ 5\ 7\ 8$

$\text{counter} = 13$

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

Solution: Index i for B, index j for C, index k for A, counter.

$$
\begin{array}{l}
\textbf{While } k < \text{len(B)} + \text{len(C)} \textbf{ do} \\
\quad \textbf{If } B[i] <= C[j] \textbf{ then} \\
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\quad \textbf{else} \\
\quad\quad A[k] \leftarrow C[j] \\
\quad\quad \text{counter} = \text{counter} + \text{len}(B) - i + 1 \\
\quad\quad j = j + 1, k = k + 1
\end{array}
$$

$B: 1\ 3\ 4\ 9\ 220$ $\qquad$ $C: 2\ 3\ 5\ 7\ 8\ 10$

i $\qquad$ j

$A: 1\ 2\ 3\ 3\ 4\ 5\ 7\ 8\ 9$

$\text{counter} = 13$

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

Solution: Index i for B, index j for C, index k for A, counter.

While $k <$ len(B) + len(C) **do**

    If $B[i] <= C[j]$ **then**

        $A[k] \leftarrow B[i]$

        $i = i + 1, k = k + 1$

    else

        $A[k] \leftarrow C[j]$

        $\text{counter} = \text{counter} + \text{len}(B) - i + 1$

        $j = j + 1, k = k + 1$

B : 1 3 4 9 220 C : 2 3 5 7 8 10

i j

A : 1 2 3 3 4 5 7 8 9 10

$\text{counter} = 14$

Case study I: Counting inversions

Problem: Given two sorted arrays B, C, merge them to a sorted array and count number of inversions simultaneously.

Solution: Index i for B, index j for C, index k for A, counter.

$$
\begin{array}{l}
\textbf{While } k < \mathrm{len(B)} + \mathrm{len(C)} \textbf{ do} \\
\quad \textbf{If } B[i] <= C[j] \textbf{ then} \\
\qquad A[k] \leftarrow B[i] \\
\qquad i = i + 1, k = k + 1 \\
\quad \textbf{else} \\
\qquad A[k] \leftarrow C[j] \\
\qquad \text{counter} = \text{counter} + \mathrm{len}(B) - i + 1 \\
\qquad j = j + 1, k = k + 1
\end{array}
$$

$B: 1\ 3\ 4\ 9\ 220$ $\quad$ $C: 2\ 3\ 5\ 7\ 8\ 10$

i $\quad$ j

$A: 1\ 2\ 3\ 3\ 4\ 5\ 7\ 8\ 9\ 10\ 220$

$\text{counter} = 14$

Case study I: Counting inversions

- For all i, j with $i < j$, compare A_i with A_j and increase counter if $A_i > A_j$. Total number of comparisons $\frac{n(n-1)}{2}$. Running time $\Theta(n^2)$.

- Use Divide and conquer. Tricky part the combine step.

- Solution: Run a modification of Mergesort that has a counter that counts inversions during merge steps.

Case study I: Counting inversions

- Pseudocode:

$$\text{Mergesort}(A[1:n])$$
$$\quad \textbf{If } n == 1 \textbf{ then}$$
$$\qquad \textbf{return } A, 0$$
$$\quad B, \textbf{countL} \;=\; \text{Mergesort }(A[1:\tfrac{n}{2}])$$
$$\quad C, \textbf{countR} \;=\; \text{Mergesort }(A[\tfrac{n}{2}+1:n])$$
$$\quad A \leftarrow \text{Merge}(B, C)$$
$$\quad \text{Get } \textbf{counter} \text{ from merging}$$
$$\quad \textbf{return } A, \text{countL} + \text{countR} + \text{counter}$$