Lecture 16
Dynamic Programming

CS 161 Design and Analysis of Algorithms

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- [CLRS] Chapter 15

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- [GT]: Chapter 12
- [CLRS] Chapter 15
- [Kleinberg and Tardos], Chapter 6

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  - This requires careful indexing of subproblems

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	D&C / Recursion	Memoized Recursion	Dynamic Programming
Basic approach	recursion	recursion	iteration
Use of recurrence	top-down	top-down	bottom-up
Store subproblem solutions	No	Yes	Yes
Space needed for stack	Yes	Yes	No

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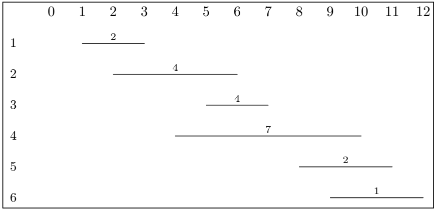
- **Problem:** Find a non-overlapping set of intervals that maximizes the total value.
- **Example:**

j	$s(j)$	$f(j)$	$v(j)$
1	1	3	2
2	2	6	4
3	5	7	4
4	4	10	7
5	8	11	2
6	9	12	1



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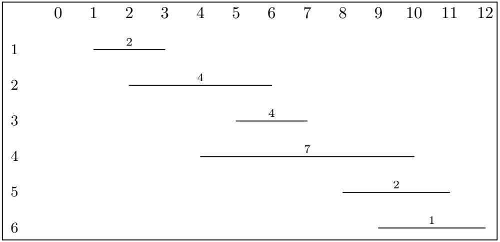

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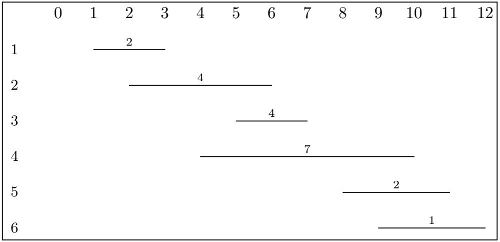

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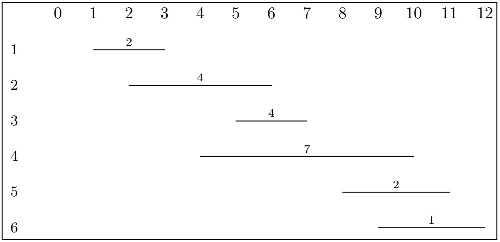

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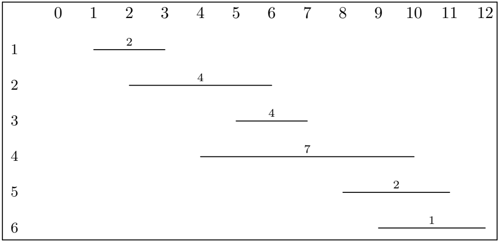

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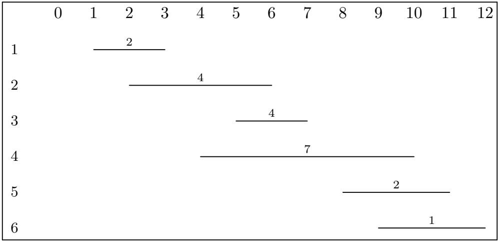

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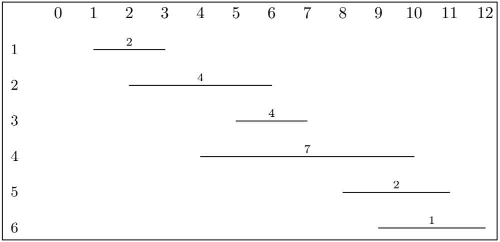

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def OPT(j):
    if j = 0:  return 0
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- Hence, $O(n)$ calls.

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    2. Once we know how to compute the optimum cost or value, we then consider how to compute a configuration that has the optimum cost or value.

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- Run a post-processing step that uses this additional information

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```
def Iterative_OPT:
   M[0] = 0
   for j = 1 to n:
      if v(j)+M[p(j)] > M[j-1]:
         M[j] = v(j)+M[p(j)]
         keep[j] = True
      else:
         M[j] = M[j-1]
         keep[j] = False
```

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Once we have computed the two arrays `M[ ]` and `keep[ ]`:

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```
def PrintSolution(j):
   if j = 0:  return;
   if keep[j]:
      PrintSolution(p(j))
      print(j)
   else:
      PrintSolution(j-1)

PrintSolution(n)
```

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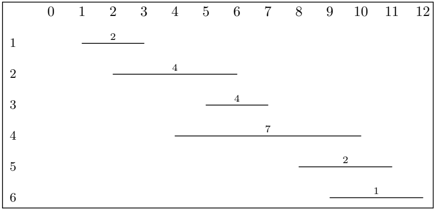

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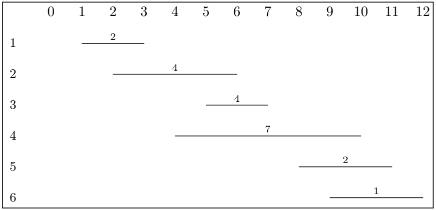

	0	1	2	3	4	5	6
`M:`	0	2	4	6	9	9	9
`keep:`		T	T	T	T	F	F

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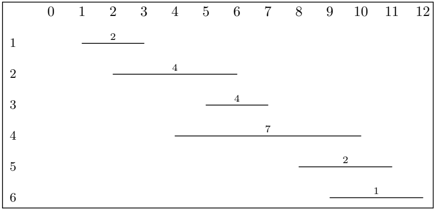

	0	1	2	3	4	5	6
`M:`	0	2	4	6	9	9	9
`keep:`		T	T	T	T	F	F

Selected intervals: $\{1, 4\}$.

Our example

j	$s(j)$	$f(j)$	$v(j)$	$p(j)$
1	1	3	2	0
2	2	6	4	0
3	5	7	4	1
4	4	10	7	1
5	8	11	2	3
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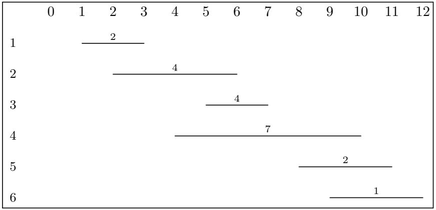

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The array M contains the solutions of the subproblems. We will refer to this as the memoization table

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We saw this in the case of the weighted interval scheduling problem.

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Here, "smaller" means "earlier in the ordering"

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$$p(j) = \begin{cases} \text{The highest-numbered interval } i < j \text{ that does not} \\ \text{overlap interval } j \text{ if such an interval exists} \\ 0 \text{ otherwise} \end{cases}$$

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We will express this more formally on the next slide.

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- Note that if $w_i > j$, we can't use box i, so only the second choice is available.
- This recurrence equation gives us the dynamic programming solution (specified on next slide)

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4. Initial values:

$$\begin{array}{rcll} \mathrm{OPT}(i, 0) & = & 0 & \text{for all } i \geq 0 \\ \mathrm{OPT}(0, j) & = & 0 & \text{for all } j \geq 0 \end{array}$$

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5. Recurrence:

$$\mathrm{OPT}(i,j) = \begin{cases} \max\left(w_i + \mathrm{OPT}(i-1, j-w_i), \mathrm{OPT}(i-1,j)\right) & \text{if } w_i \leq j \\ \mathrm{OPT}(i-1,j) & \text{if } w_i > j \end{cases}$$

Truck Loading Problem DP Pseudocode: compute OPT Matrix

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```
def compute_opt_matrix(w):
  for i = 0 to n:  OPT[i,0] = 0
  for j = 0 to W: OPT[0,j] = 0
  for i = 1 to n:
     for j = 1 to W:
        if w[i] > j:
           OPT[i,j] = OPT[i-1,j]
        else:
           OPT[i,j] = max(w[i] + OPT[i-1,j-w[i]], OPT[i-1,j])
  return OPT
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  return OPT
```

This tells us the maximum possible weight, but we need to also compute which boxes to load to achieve this maximum weight . . .

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```
def compute_opt_strategy(w):
  for i = 0 to n:  OPT[i,0] = 0
  for j = 0 to W: OPT[0,j] = 0
  for i = 1 to n:
     for j = 1 to W:
        if (w[i] > j) or (w[i] + OPT[i-1,j-w[i]] <= OPT[i-1,j])
           OPT[i,j] = OPT[i-1,j]
           keep[i,j] = False
        else:
           OPT[i,j] = w[i] + OPT[i-1,j-w[i]]
           keep[i,j] = True
     return (OPT,keep)
```

Truck Loading Problem DP Pseudocode: compute choice of boxes

Introduce an new array `keep[i,j]`, which tells us whether we keep box i when we solve the subproblem with i boxes and capacity j.

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  for j = 0 to W: OPT[0,j] = 0
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     return (OPT,keep)
```

Running time: $O(n \cdot W)$

Truck Loading Problem DP Pseudocode: compute choice of boxes [continued]

Truck Loading Problem DP Pseudocode: compute choice of boxes [continued]

```
def print_solution(OPT,keep,i,j):
   if i == 0:  return
   if keep[i,j]:
      print_solution(OPT,keep,i-1,j-w[i])
      print (i)
   else:
      print_solution(OPT,keep,i-1,j)

// Main program starts here
(OPT,keep) = compute_opt_strategy(w)
print_solution(OPT,keep,n,W)
```

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  - Example:
    - $W = 100$
    - Item 1: $w_1 = 20$, $v_1 = 80$
    - Item 2: $w_2 = 90$, $v_2 = 90$.

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5. Recurrence:

$$\mathrm{OPT}(i,j) = \begin{cases} \max\left(v_i + \mathrm{OPT}(i-1, j-w_i), \mathrm{OPT}(i-1,j)\right) & \text{if } w_i \leq j \\ \mathrm{OPT}(i-1,j) & \text{if } w_i > j \end{cases}$$

Pseudocode for DP Solution to 0/1 Knapsack Problem

```
def compute_opt_strategy(w,v):
  for i = 0 to n:  OPT[i,0] = 0
  for j = 0 to W: OPT[0,j] = 0
  for i = 1 to n:
     for j = 1 to W:
        if (w[i] > j) or (v[i] + OPT[i-1,j-w[i]] <= OPT[i-1,j])
           OPT[i,j] = OPT[i-1,j]
           keep[i,j] = False
        else:
           OPT[i,j] = v[i] + OPT[i-1,j-w[i]]
           keep[i,j] = True
     return (OPT,keep)
```

Pseudocode for DP Solution to 0/1 Knapsack Problem [continued]

```
def print_solution(OPT,keep,i,j):
   if i == 0:  return
   if keep[i,j]:
      print_solution(OPT,keep,i-1,j-w[i])
      print (i)
   else:
      print_solution(OPT,keep,i-1,j)

// Main program starts here
(OPT,keep) = compute_opt_strategy(w,v)
print_solution(OPT,keep,n,W)
```

Optimal Matrix Chain Multiplication

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Parenthesization Matters

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$A_1 : 10 \times 15$
$A_2 : 15 \times 5$
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$$(A_1 \times A_2) \times ((((A_3 \times A_4) \times A_5) \times A_6) \times A_7)$$

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  - Total cost is $M(i,k) + M(k+1,j) + d_{i-1} d_k d_j$.

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  - Total cost is $M(i,k) + M(k+1,j) + d_{i-1}d_kd_j$.
- Choose the best index k:

$$M(i,j) = \min_{i \le k \le j-1} \left(M(i,k) + M(k+1,j) + d_{i-1}d_kd_j\right)$$

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Note:

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2. Function / Memoization table definition: $M(i, j)$ is the minimum number of multiplications required to compute the product $A_i \times \cdots \times A_j$ (using the best possible grouping).
3. Goal: $M(1, n)$
4. Initial values: $M(i, i) = 0$
5. Recurrence:

$$M(i, j) = \min_{i \leq k \leq j-1} \left(M(i, k) + M(k + 1, j) + d_{i-1} d_k d_j \right)$$

Note:

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```
def optMatrixChain(d):
    for i = 1 to n:
        M[i,i] = 0
    for len = 2 to n:
        for i = 1 to n - len + 1:
            j = i + len - 1
            M[i,j] = +∞
            for k = i to j-1:
                x = M[i,k] + M[k+1,j] + d[i-1]*d[k]*d[j]
                if x < M[i,j]:
                    M[i,j] = x
    return M
```

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    return M,S
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Solution to our example

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$A_1 : 10 \times 15$
$A_2 : 15 \times 5$
$A_3 : 5 \times 60$
$A_4 : 60 \times 100$
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j

1	2	3	4	5	6	7	i
0 —	750 1	3750 2	35750 2	41750 2	46750 2	56500 2	1
	0 —	4500 2	37500 2	41500 2	47000 2	56925 2	2
		0 —	30000 3	40000 4	44000 5	53400 6	3
			0 —	120000 4	168000 5	214000 5	4
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