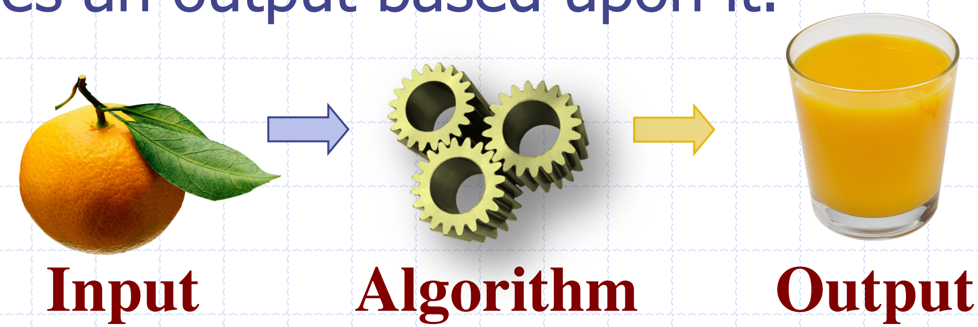
Lecture 2
Math overview

CS 161 Design and Analysis of Algorithms

Ioannis Panageas

Algorithms and Data Structures

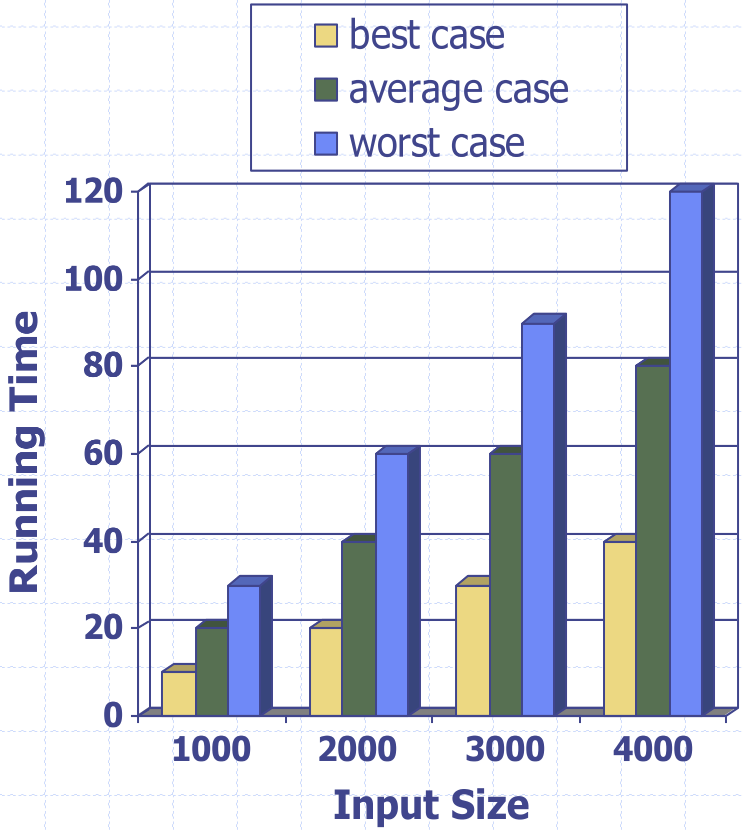
- An **algorithm** is a step-by-step procedure for performing some task in a finite amount of time.
  - Typically, an algorithm takes input data and produces an output based upon it.



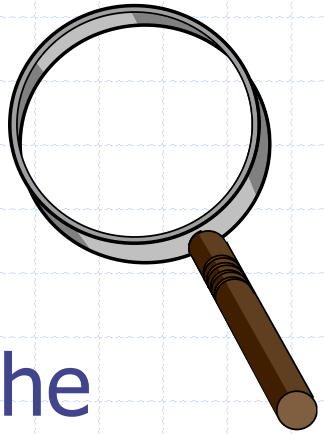
- A **data structure** is a systematic way of organizing and accessing data.

Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus primarily on the **worst case running time**.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



Theoretical Analysis



- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, n
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Pseudocode

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Pseudocode Details

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  Algorithm *method* (*arg* [, *arg*…])

  Input …

  Output …
- Method call

  method (*arg* [, *arg*…])
- Return value

  return *expression*
- Expressions:

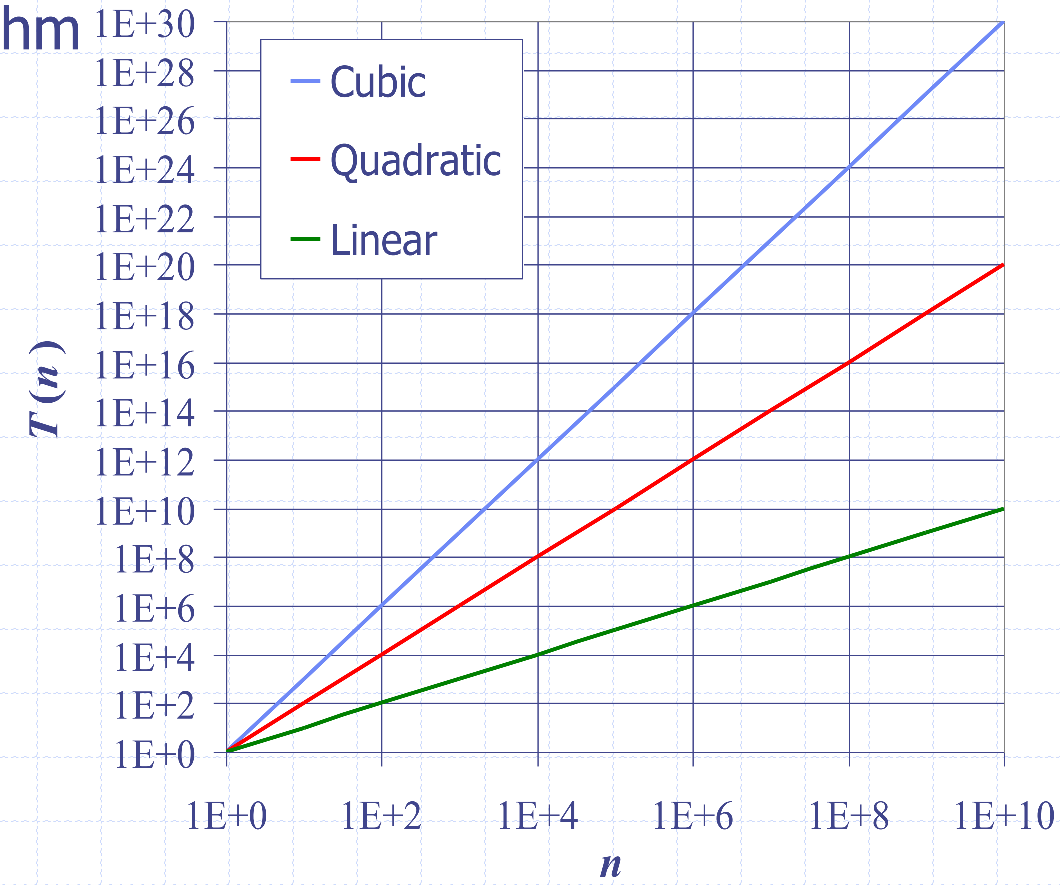
  $\leftarrow$ Assignment

  $=$ Equality testing

  n^2 Superscripts and other mathematical formatting allowed

Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant ≈ 1
  - Logarithmic ≈ $\log n$
  - Linear ≈ n
  - N-Log-N ≈ $n \log n$
  - Quadratic ≈ n^2
  - Cubic ≈ n^3
  - Exponential ≈ 2^n
- In a log-log chart, the slope of the line corresponds to the growth rate



Functions Graphed Using "Normal" Scale

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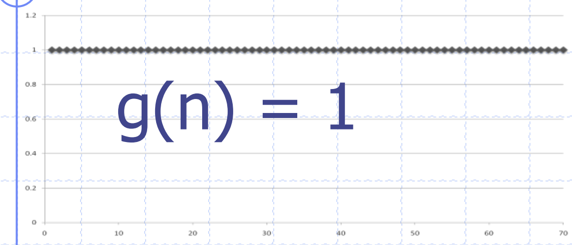
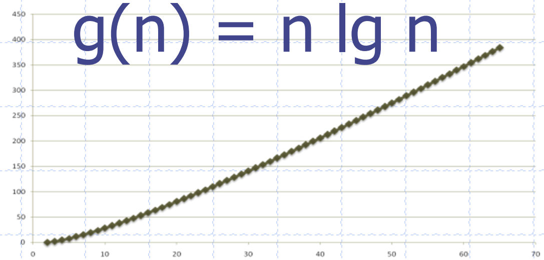
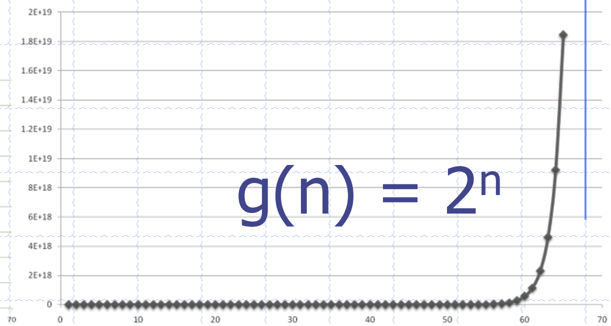
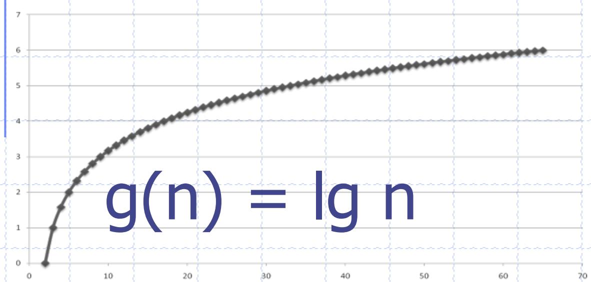
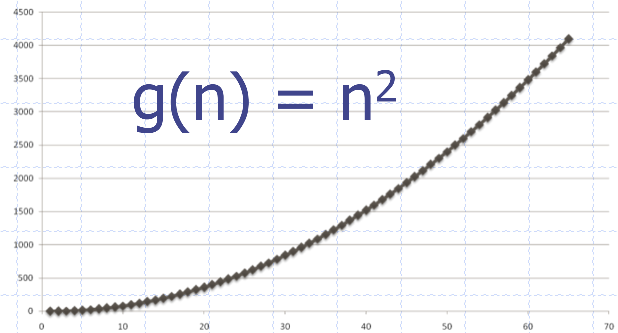
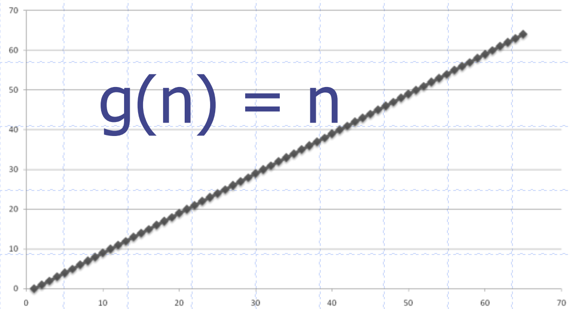


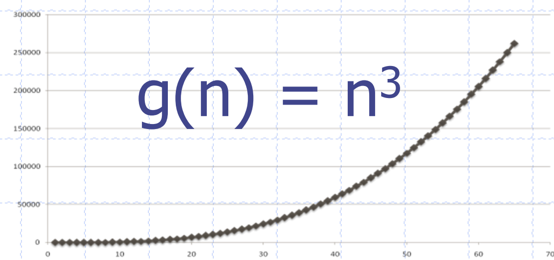












Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important

- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method

Counting Primitive Operations

- Example: By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

Algorithm arrayMax(A, n):

Input: An array A storing $n \geq 1$ integers.

Output: The maximum element in A.

$currentMax \leftarrow A[0]$

for $i \leftarrow 1$ **to** $n - 1$ **do**

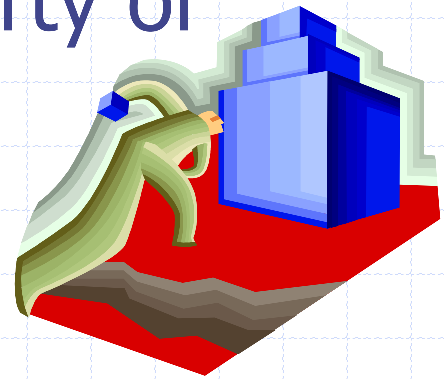
  if $currentMax < A[i]$ **then**

    $currentMax \leftarrow A[i]$

return $currentMax$

Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm arrayMax



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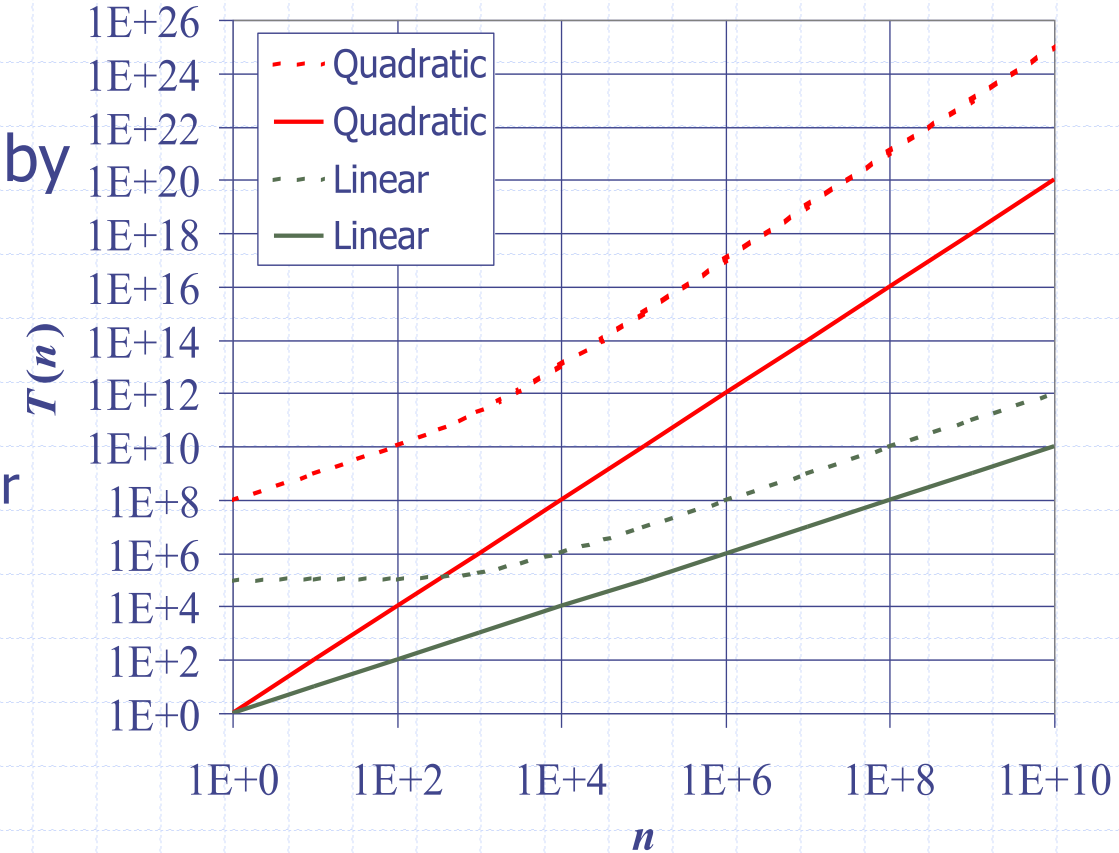
Why Growth Rate Matters

if runtime is...	time for n + 1	time for 2 n	time for 4 n
$c \lg n$	$c \lg (n + 1)$	$c (\lg n + 1)$	$c(\lg n + 2)$
$c\, n$	$c\, (n + 1)$	$2c\, n$	$4c\, n$
$c\, n \lg n$	$\sim c\, n \lg n + c\, n$	$2c\, n \lg n + 2cn$	$4c\, n \lg n + 4cn$
$c\, n^2$	$\sim c\, n^2 + 2c\, n$	**$4c\, n^2$**	$16c\, n^2$
$c\, n^3$	$\sim c\, n^3 + 3c\, n^2$	$8c\, n^3$	$64c\, n^3$
$c\, 2^n$	$c\, 2^{n+1}$	$c\, 2^{2n}$	$c\, 2^{4n}$

runtime quadruples when problem size doubles

Constant Factors

- The growth rate is minimally affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function



Asymptotic Algorithm Analysis

- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We say that algorithm arrayMax "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Big-Oh Rules

- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Analyzing Recursive Algorithms

- Use a function, T(*n*), to derive a **recurrence relation** that characterizes the running time of the algorithm in terms of smaller values of *n*.

Algorithm recursiveMax(A, n):

Input: An array A storing $n \geq 1$ integers.

Output: The maximum element in A.

if $n = 1$ **then**

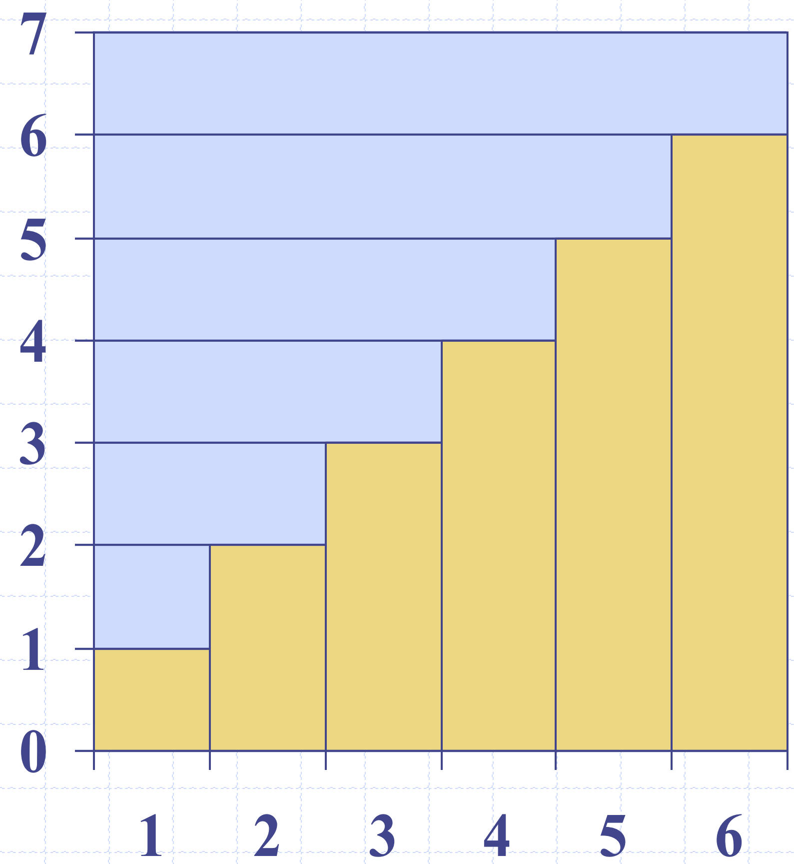
  return $A[0]$

return $\max\{\text{recursiveMax}(A, n-1),\ A[n-1]\}$

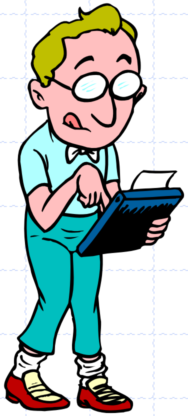
$$T(n) = \begin{cases} 3 & \text{if } n = 1 \\ T(n-1) + 7 & \text{otherwise,} \end{cases}$$

Arithmetic Progression

- Assume the running time of P is $O(1 + 2 + \ldots + n)$
- The sum of the first n integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm P runs in $O(n^2)$ time



Math you need to Review



- Summations
- Powers
- Logarithms
- Proof techniques
- Basic probability

- Properties of powers:
  $a^{(b+c)} = a^b a^c$
  $a^{bc} = (a^b)^c$
  $a^b / a^c = a^{(b-c)}$
  $b = a^{\log_a b}$
  $b^c = a^{c*\log_a b}$
- Properties of logarithms:
  $\log_b(xy) = \log_b x + \log_b y$
  $\log_b (x/y) = \log_b x - \log_b y$
  $\log_b xa = a\log_b x$
  $\log_b a = \log_x a/\log_x b$

O ("big oh")

Informally:

- $g \in O(f)$ if g is bounded above by a constant multiple of f (for sufficiently large values of n).
- $g \in O(f)$ if "g grows no faster than (a constant multiple of) f."
- $g \in O(f)$ if the ratio g/f is bounded above by a constant (for sufficiently values of n).

O ("big oh")

Formally:

- $g \in O(f)$ if and only if:

$$\exists_{C>0}\, \exists_{n_0>0}\, \forall_{n>n_0}\, g(n) \leq C \cdot f(n).$$

- Equivalently: $g \in O(f)$ if and only if:

$$\exists_{C>0}\, \exists_{n_0>0}\, \forall_{n>n_0}\, \frac{g(n)}{f(n)} \leq C.$$

- Sometimes we write: $g = O(f)$ rather than $g \in O(f)$

Examples of O-notation:

Example 1: $f(n) = n$, $g(n) = 1000n$: $g \in O(f)$.

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Proof: Let $C = 1000$. Then $g(n) \leq C \cdot f(n)$ for all n.

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Alternate Proof: If $n \geq 1$, $n^{1/2} \geq 1$, so $n^{3/2} \leq n^2$.
Hence we can choose $C = 1$ and $n_0 = 1$.

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Proof: $\lim_{n \to \infty} \frac{g(n)}{f(n)} = \infty$.
Hence there is no $C > 0$ such that $g(n) \leq C \cdot f(n)$ for sufficiently large n.

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Proof: If $n \geq 1$, then $n \leq n^2$ and $1 \leq n^2$. Hence:

$$\begin{aligned} g(n) &= 5n^2 + 23n + 2 \\ &\leq 5n^2 + 23n^2 + 2n^2 \end{aligned}$$

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$$\begin{aligned} g(n) &= 5n^2 + 23n + 2 \\ &\leq 5n^2 + 23n^2 + 2n^2 \\ &\leq 30n^2 \end{aligned}$$

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So we can take $C = 30$, $n_0 = 1$.

More asymptotic notation: o ("little oh"), Ω ("big Omega")

- o ('little oh"):

$$g \in o(f) \quad \text{if and only if} \lim_{n \to \infty} \frac{g(n)}{f(n)} = 0.$$

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- Ω ("big Omega") (or just "Omega")

$$g \in \Omega(f) \quad \text{if and only if} \quad \exists_{C>0}\, \exists_{n_0>0}\, \forall_{n>n_0}\, g(n) \geq C \cdot f(n).$$

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Equivalently:

$$g \in \Omega(f) \quad \text{if and only if} \quad \exists_{C>0}\, \exists_{n_0>0}\, \forall_{n>n_0}\, \frac{g(n)}{f(n)} \geq C.$$

One more definition: Θ ("Theta")

- $g \in \Theta(f)$ if and only if:

$$g \in O(f) \text{ and } g \in \Omega(f).$$

- Equivalently, $g \in \Theta(f)$ if and only if:

$$\exists_{C_1>0}\, \exists_{C_2>0}\, \exists_{n_0>0}\, \forall_{n>n_0}\; C_1 \le \frac{g(n)}{f(n)} \le C_2.$$

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Then $g(n) = 1000 \cdot n > 1 \cdot n = C \cdot f(n)$.

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Then $g(n) = 1000 \cdot n > 1 \cdot n = C \cdot f(n)$.

To see that $g \in \Theta(f)$, we could argue that $g \in O(f)$ (shown earlier) and $g \in \Omega(f)$ (shown above).

Or we can take $C_1 = 1$, $C_2 = 1000$. Then

$$C_1 \leq \frac{g(n)}{f(n} \leq C_2.$$

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Because $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$.

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$g \in \Omega(f)$

Because $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$, so we can choose any C we want.

Examples of Asymptotic notation

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$$\begin{aligned} g(n) &= 5n^2 - 23n + 2 \\ &\geq 5n^2 - n^2 \\ &\geq 4n^2 \\ &= 4f(n) \end{aligned}$$

So we can take $C = 4$, $n_0 = 23$.

Another Example

Example 5: $\ln n = o(n)$

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Proof:

Examine the ratio $\frac{\ln n}{n}$ as $n \to \infty$.

If we try to evaluate the limit directly, we obtain the "indeterminate form" $\frac{\infty}{\infty}$.

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We need to apply L'Hôpital's rule (from calculus).

Example 5, continued: $\ln n = o(n)$

L'Hôpital's rule: If the ratio of limits

$$\frac{\lim_{n\to\infty} g(n)}{\lim_{n\to\infty} f(n)}$$

is an indeterminate form (i.e., ∞/∞ or $0/0$), then

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{g'(n)}{f'(n)}$$

where f' and g' are, respectively, the derivatives of f and g.

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By L'Hôpital's rule:

$$\begin{aligned} \lim_{n\to\infty} \frac{g(n)}{f(n)} &= \lim_{n\to\infty} \frac{g'(n)}{f'(n)} \\ &= \lim_{n\to\infty} \frac{1/n}{1} \end{aligned}$$

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By L'Hôpital's rule:

$$\begin{aligned}
\lim_{n\to\infty} \frac{g(n)}{f(n)} &= \lim_{n\to\infty} \frac{g'(n)}{f'(n)} \\
&= \lim_{n\to\infty} \frac{1/n}{1} \\
&= \lim_{n\to\infty} \frac{1}{n} \\
&= 0.
\end{aligned}$$

Hence $g(n) = o\left(f(n)\right)$.

Math background

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- Sums, Summations
- Logarithms, Exponents Floors, Ceilings, Harmonic Numbers
- Proof Techniques
- Basic Probability

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- If $S = \{s_1, \ldots, s_n\}$ is a finite set:

Sums, Summations

- Summation notation:

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Geometric sum

- Geometric sum:

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a},$$

provided that $a \neq 1$.

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- Previous formula holds for $a = 0$ because $a^0 = 1$ even when $a = 0$.
- Special case of geometric sum:

$$\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1.$$

Infinite Geometric sum

- From the previous slide:

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a},$$

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- Special case of infinite geometric sum:

$$\sum_{i=0}^{\infty} \frac{1}{2^i} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

Other Summations

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$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \Theta\left(n^2\right)$$

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- In general, for any fixed positive integer k:

$$\sum_{i=1}^{n} i^k = 1 + 2^k + 3^k + \cdots + n^k = \Theta\left(n^{k+1}\right)$$

Logarithms

Definition: $\log_b x = y$ if and only if $b^y = x$.

Some useful properties:

1. $\log_b 1 = 0$.
2. $\log_b b^a = a$.
3. $\log_b(xy) = \log_b x + \log_b y$.
4. $\log_b(x^a) = a \log_b x$.
5. $x^{\log_b y} = y^{\log_b x}$.
6. $\log_x b = \frac{1}{\log_b x}$.
7. $\log_a x = \frac{\log_b x}{\log_b a}$.
8. $\log_a x = (\log_b x)(\log_a b)$.

Floors and ceilings

- $\lfloor x \rfloor$ = largest integer $\leq x$. (Read as Floor of x)
- $\lceil x \rceil$= smallest integer $\geq x$ (Read as Ceiling of x)

Factorials

- $n! = 1 \cdot 2 \cdots n$
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1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
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$\binom{n}{k}$ = The number of different ways of choosing k objects from a collection of n objects. (Pronounced "n choose k".)

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Special cases: $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = \frac{n(n-1)}{2}$

Harmonic Numbers

The *n*th Harmonic number is the sum:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

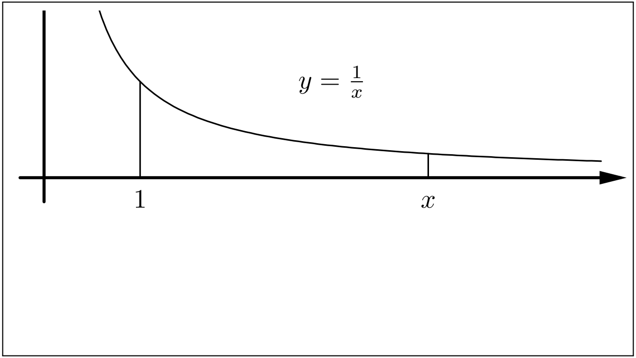
These numbers go to infinity:

$$\lim_{n \to \infty} H_n = \sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

Harmonic Numbers

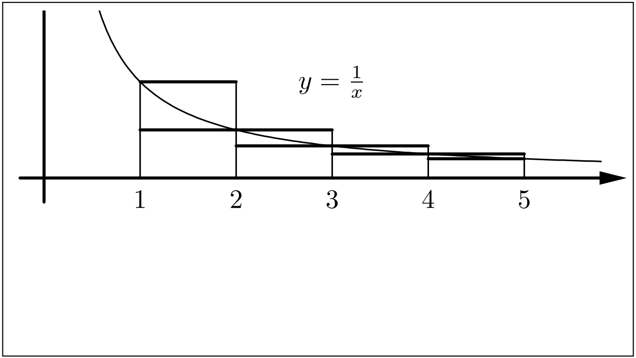
The harmonic numbers are closely related to logs. Recall:

$$\ln x = \int_1^x \frac{1}{t}\, dt$$



We will show that $H_n = \Theta(\log n)$.

Harmonic Numbers



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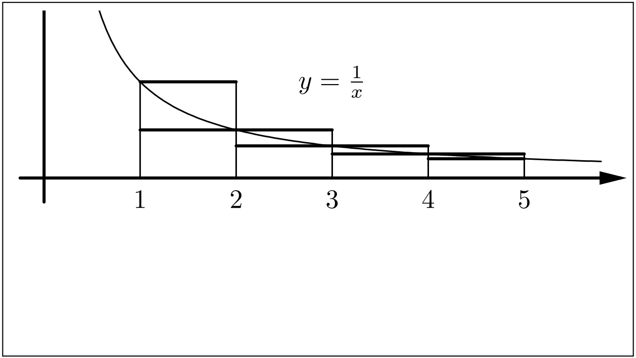

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \quad < \quad \ln n \quad < \quad 1 + \frac{1}{2} + \ldots + \frac{1}{n-1}$$

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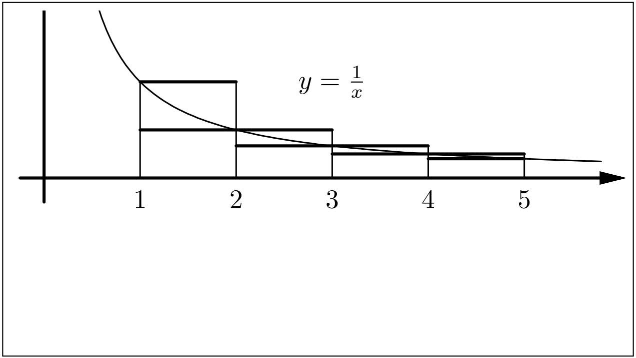

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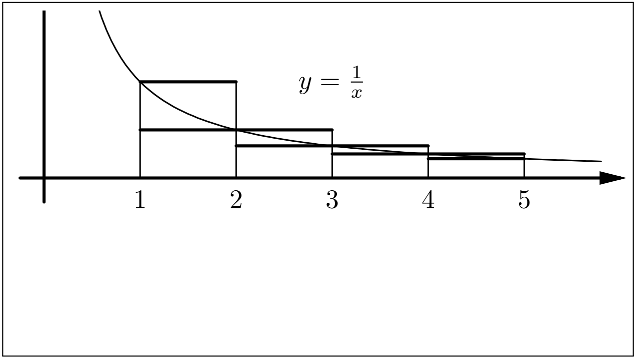

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Hence $\ln n + \frac{1}{n} < H_n < \ln n + 1$, so $H_n = \Theta(\log n)$.

Proof/Justification Techniques

- Proof by Example Can be used to prove
  - A statement of the form "There exists..." is true.
  - A statement of the form "For all..." is false.
  - A statement of the form "If P then Q" is false.

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- Illustration: Consider the statement:

  All numbers of the form $2^k - 1$ are prime.

  This statement is False: $2^4 - 1 = 15 = 3 \cdot 5$
- Note: The statement can be rewritten as:

  If n is an integer of the form $2^k - 1$, then n is prime.

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  See [GT] Section 1.3.3 for examples.

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  2. Inductive step: If $P(k)$ is true, then $P(k+1)$ is true.

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$$\sum_{i=1}^{n} i \cdot 2^i = (n-1) \cdot 2^{(n+1)} + 2$$

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Induction Example, continued

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Show $P(k+1)$ is true:

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2.$$

Induction Example, continued

$$\text{Assume: } \sum_{i=1}^{k} i \cdot 2^i = (k-1) \cdot 2^{(k+1)} + 2.$$

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&= (k-1) \cdot 2^{(k+1)} + 2 + (k+1) \cdot 2^{(k+1)}
\end{aligned}$$

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- Sample space $S = \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}$.
- The event "first coin is heads" is the subset $\{\mathtt{HH}, \mathtt{HT}\}$.

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- A probability function is a function $P(\cdot)$ that maps events (subsets of the sample space S) to real numbers such that:
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- Note: Property 4 implies that if $A \subseteq B$ then $P(A) \leq P(B)$.

Probability function (continued)

For finite sample spaces, this can be simplified:

- Sample space $S = \{s_1, \ldots, s_k\}$,
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Example: (2-coin example, continued). Define

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- Special case: An Indicator variable is a random variable that is always either 0 or 1.

Expectation

- The expected value, or expectation, of a random variable X represents its "average value".
- Formally: Let X be a random variable with a finite set of possible values $\mathrm{V} = \{x_1, \ldots, x_k\}$. Then

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Example: (2-coin example, continued). Let X be the number of heads when two coins are thrown. Then

$$\begin{aligned} E(X) &= 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) \\ &= 0 \cdot \left(\frac{1}{4}\right) + 1 \cdot \left(\frac{1}{2}\right) + 2 \cdot \left(\frac{1}{4}\right) \\ &= 1 \end{aligned}$$

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The expected value of the throw is:

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

Linearity of Expectation

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Example 2: Throw 100 six-sided dice. Let Y be the sum of the values. Then

$$E(Y) = 100 \cdot 3.5 = 350.$$

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Then $P(A_1) = \frac{1}{2}$, $P(A_2) = \frac{1}{2}$, and

$$P(A_1 \cap A_2) = P(\mathtt{HT}) = \frac{1}{4} = P(A_1) \cdot P(A_2).$$

So A_1 and A_2 are independent.

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A collection of n events $C = \{A_1, A_2, \ldots, A_n\}$ is mutually independent (or simply independent) if:

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Example: Suppose we flip 10 coins. Suppose the flips are fair ($P(\mathtt{H}) = P(\mathtt{T}) = 1/2$) and independent. Then the probability of any particular sequence of flips (e.g., `HHTTTHTHTH`) is $1/(2^{10})$.

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- The number of successful outcomes is $\binom{10}{7}$.
- Hence the probability of getting exactly 7 heads is:

$$\frac{\binom{10}{7}}{2^{10}} = \frac{120}{1024} = 0.117.$$

An average-case result about finding the maximum

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If there are 3,000,000,000 elements in the list, the expected update count is about 22.4