Homework 1
Solutions

CS 280
Algorithmic Game Theory

Ioannis Panageas

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Problem 1: the two LPs

The row player maximizes. The column player minimizes.

LP1: $\max\ z$ subject to $R^\top x \geq z\mathbf{1}_m$, $\mathbf{1}_n^\top x = 1$, $x \geq 0$

LP2: $\max\ z''$ subject to $-Ry \geq z''\mathbf{1}_n$, $\mathbf{1}_m^\top y = 1$, $y \geq 0$

The scalars z, z'' are unrestricted.

$z^* = (x^*)^\top R y^*$, (x^*, y^*) is a Nash equilibrium.

Goal: prove that (x^*, z^*) and $(y^*, -z^*)$ are optimal.

Problem 1: feasibility

Write $\Delta_d = \{p \geq 0 : \sum_{j=1}^{d} p_j = 1\}$ for the probability simplex.

$$\underbrace{x^\top R y^* \leq z^*}_{\text{row player}} \leq \underbrace{(x^*)^\top R y}_{\text{column player}} \qquad (x \in \Delta_n,\ y \in \Delta_m).$$

Choose each pure strategy in turn.

$$Ry^* \leq z^* \mathbf{1}_n, \qquad R^\top x^* \geq z^* \mathbf{1}_m.$$

(x^*, z^*) is feasible for LP1, $(y^*, -z^*)$ for LP2.

Problem 1: optimality

Take any feasible (x, z) for LP1: $R^\top x \geq z\mathbf{1}_m$.

$$z = z\mathbf{1}_m^\top y^* \leq (R^\top x)^\top y^*$$
$$= x^\top R y^* \leq z^*.$$

Take any feasible (y, z'') for LP2: $-Ry \geq z''\mathbf{1}_n$.

$$z'' = z''\mathbf{1}_n^\top x^* \leq -(x^*)^\top R y \leq -z^*.$$

The last inequalities use Nash equilibrium. Both feasible equilibrium solutions attain their bounds.

$$\boxed{\mathrm{OPT}(\mathrm{LP1}) = z^*, \quad \mathrm{OPT}(\mathrm{LP2}) = -z^*.}$$

Problem 2: a common strategy

Player 1 uses the same strategy x in both games.

$$u_1 = x^\top A y + x^\top B z,$$

$$u_2 = -x^\top A y, \qquad u_3 = -x^\top B z.$$

Guarantee α against player 2 and β against player 3.

$$\begin{aligned} \text{Primal:} \quad & \max \ \alpha + \beta \\ & A^\top x \geq \alpha \mathbf{1}_m \\ & B^\top x \geq \beta \mathbf{1}_k \\ & x \in \Delta_n. \end{aligned}$$

Problem 2: LP duality

$$\text{Dual:} \quad \min\ r$$

$$Ay + Bz \le r\mathbf{1}_n$$
$$y \in \Delta_m, \quad z \in \Delta_k.$$

Both LPs are feasible. Strong duality gives

$$\alpha^* + \beta^* = r^*.$$

$$\underbrace{\alpha^* + \beta^*}_{\text{primal bound}} \le (x^*)^\top A y^* + (x^*)^\top B z^* \le \underbrace{r^*}_{\text{dual bound}}.$$

Problem 2: best responses

$$(x^*)^\top A y^* = \alpha^*, \qquad (x^*)^\top B z^* = \beta^*.$$

For every unilateral deviation:

$$\text{Player 1:} \quad x^\top (A y^* + B z^*) \leq r^*.$$

$$\text{Player 2:} \quad -(x^*)^\top A y \leq -\alpha^*.$$

$$\text{Player 3:} \quad -(x^*)^\top B z \leq -\beta^*.$$

The bounds equal the players' current payoffs.

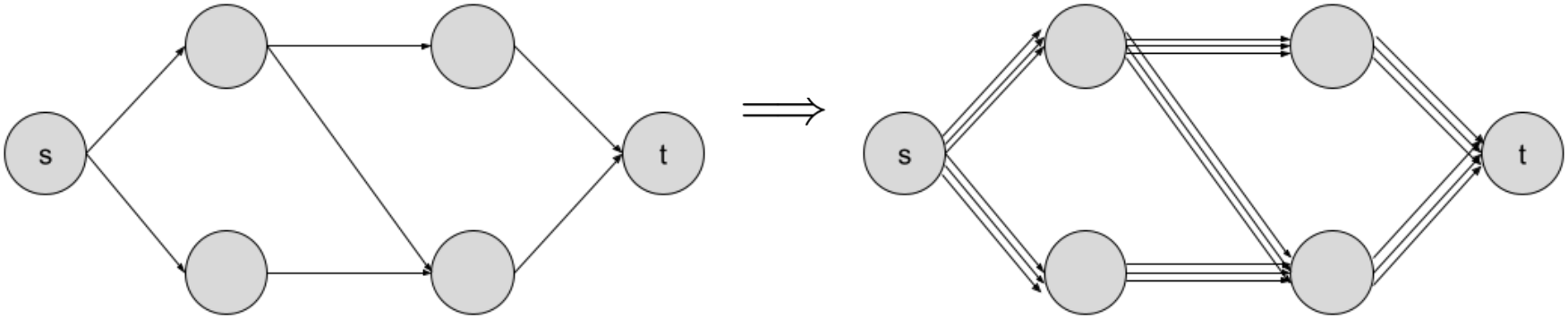
(x^*, y^*, z^*) is a Nash equilibrium.

Problem 3: the construction

Replace each edge e by n parallel edges.

Original network

Expanded network



Copy j has capacity 1 and unit cost $c_e(j)$.

$$0 \leq c_e(1) \leq c_e(2) \leq \cdots \leq c_e(n).$$

Problem 3.1: a profile gives a flow

Route one unit along each player's path.

On edge e, assign distinct copies $1, \ldots, l_e(P)$.

Unit capacities and flow conservation hold.

$$\text{value}(f^P) = n, \qquad f^P \text{ is integral.}$$

$$C(f^P) = \sum_{e \in E} \sum_{j=1}^{l_e(P)} c_e(j) = \Phi(P).$$

Problem 3.2: a flow gives a profile

Let f^* be an integral minimum-cost flow of value n.

1. Remove directed cycles. Costs are nonnegative.

2. Use copies $1, \ldots, l_e$ on each edge e.

3. Decompose into n simple paths, forming $\widehat{P}$.

Cleanup cannot increase cost, so optimality gives

$$\text{value}(\widehat{f}) = n, \qquad C(\widehat{f}) = C(f^*).$$

Problem 3.2: minimum potential

Let $\widehat{P}$ be the profile extracted from $\widehat{f}$.

Compare with the feasible flow f^P of any profile P.

$$\Phi(\widehat{P}) = C(\widehat{f}) = C(f^*)$$

$$\leq C(f^P) = \Phi(P).$$

$$\boxed{\widehat{P} \in \arg\min_P \Phi(P)}.$$

Problem 3.3: equilibrium and time

For every player i and alternative path P_i':

$$\text{cost}_i(P_i', \widehat{P}_{-i}) - \text{cost}_i(\widehat{P})$$

$$= \Phi(P_i', \widehat{P}_{-i}) - \Phi(\widehat{P}) \geq 0.$$

No player can lower their cost: $\widehat{P}$ is a pure Nash equilibrium.

The expanded graph has $n|E|$ edges.

Minimum-cost flow, cleanup, and decomposition take polynomial time.

Problem 4: exact supports

Given (A, B) and nonempty sets S, T, find a Nash equilibrium with

$$\text{supp}(x) = S, \qquad \text{supp}(y) = T.$$

Every specified action must have positive probability.

$$x_i \geq \delta \quad (i \in S), \qquad y_j \geq \delta \quad (j \in T).$$

Maximize the common margin δ.

Problem 4: the linear program

$$\max \ \delta$$

Probabilities

$$x \in \Delta_n, \quad y \in \Delta_m$$
$$x_i = 0 \quad (i \notin S)$$
$$y_j = 0 \quad (j \notin T)$$
$$x_i \geq \delta \quad (i \in S)$$
$$y_j \geq \delta \quad (j \in T)$$
$$\delta \geq 0$$

Best responses

$$(Ay)_i \geq (Ay)_j$$
$$\forall i \in S, \ j \in [n]$$

$$(B^\top x)_i \geq (B^\top x)_j$$
$$\forall i \in T, \ j \in [m]$$

Each action in S and T must be a best response.

Problem 4: correctness

$\delta^* > 0$: exact supports, and every played action is a best response.

Conversely, an exact-support equilibrium gives

$$\delta = \min\left\{ \min_{i \in S} x_i,\ \min_{j \in T} y_j \right\} > 0.$$

Solve the rational LP in polynomial time.

Infeasible or $\delta^* = 0$: no exact-support equilibrium exists.

Problem 5: equal strategies

Players 1 and 2 share their action set, with

$$u_1(s_1, s_2, s_3, \ldots, s_n) = u_2(s_2, s_1, s_3, \ldots, s_n).$$

Restrict the mixed-strategy space to

$$D = \left\{ x \in \prod_{i=1}^{n} \Delta(S_i) : \; x_1 = x_2 \right\}.$$

D is nonempty, compact, and convex.

Problem 5: the gain map

For player i and pure action $s \in S_i$:

$$g_{i,s}(x) = \max\{u_i(s, x_{-i}) - u_i(x), 0\}.$$

$$\underbrace{G_i(x) = \sum_{r \in S_i} g_{i,r}(x)}_{\text{sum of player } i\text{'s positive deviation gains}} \geq 0.$$

$$F_{i,s}(x) = \frac{x_{i,s} + g_{i,s}(x)}{1 + G_i(x)}.$$

$$F_{i,s}(x) \geq 0, \qquad \sum_{s \in S_i} F_{i,s}(x) = 1.$$

F is continuous.

Problem 5: preserving symmetry

At any profile $x \in D$,

$$u_1(x) = u_2(x),$$

$$u_1(s, x_{-1}) = u_2(s, x_{-2}) \quad (s \in S_1).$$

$$g_{1,s} = g_{2,s}, \quad G_1 = G_2 \quad \Longrightarrow \quad F_1(x) = F_2(x).$$

Brouwer applies to the continuous map $F : D \to D$.

$$\boxed{\exists x^* \in D : \quad F(x^*) = x^*, \quad x_1^* = x_2^*.}$$

Problem 5: the fixed-point equation

At the fixed point x^*, omit stars and fix player i.

$$G_i = \sum_{r \in S_i} g_{i,r} \qquad \text{(sum of positive deviation gains).}$$

$$x_{i,s} = F_{i,s}(x) = \frac{x_{i,s} + g_{i,s}}{1 + G_i}.$$

Multiply by $1 + G_i$ and cancel $x_{i,s}$:

$$x_{i,s} + x_{i,s} G_i = x_{i,s} + g_{i,s}$$

$$\Longrightarrow \qquad \boxed{g_{i,s} = x_{i,s} G_i} \qquad (s \in S_i).$$

Problem 5: no profitable deviation

$$G_i = \sum_{r \in S_i} g_{i,r}, \qquad g_{i,s} = x_{i,s} G_i.$$

If $G_i > 0$, every played action has positive gain.

$$0 = \sum_{s \in S_i} x_{i,s} \big(u_i(s, x_{-i}) - u_i(x) \big)$$

$$= G_i \sum_{s \in S_i} x_{i,s}^2 > 0.$$

Thus every $G_i = 0$, so x^* is a Nash equilibrium.