L17 More on Markets

CS 295 Introduction to Algorithmic Game Theory
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Recap

**Definition (Market).** A market consists of:

- A set $\mathcal{B}$ of $n$ *buyers/traders*.
- A set $\mathcal{G}$ of $m$ *goods*.
- Each buyer $i$ has 1 *amount of $*$.
- One unit for each good.
- $w_{ij}$ denotes the utility derived by $i$ on obtaining a unit amount of good of $j$.
- Each good $j$ is associated with a *price* $p_j$.

**Definition (Fisher Market).** A market so that the utilities are linear: Let $x_{ij}$ be the amount of units buyer $i$ gets of good $j$ then

$$u_i = \sum_{j \in \mathcal{G}} x_{ij} w_{ij}.$$
Given an arbitrary vector of prices $p \geq 0$, from each buyer’s $i$ perspective:

\begin{align*}
\text{max} & \quad \sum_{j=1}^{m} x_{ij} w_{ij} \\
\text{s.t} & \quad \sum_{j=1}^{m} p_{j} x_{ij} \leq 1 \\
& \quad x_{i} \geq 0
\end{align*}

Demand for good $j$.

From the perspective of good $j$:

$$\sum_{i=1}^{n} x_{ij} \leq 1$$

Supply for good $j$.

Can we find $(x, p)$ s.t all are satisfied simultaneously?
Proportional Response Dynamics

Market dynamics:

Each time step the buyers face the same market parameters, (goods, budget constraint, utility function) while the buyers make their bidding decisions according to the previous market actions.
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Notation:

- $b_{ij}^{(t)}$ the bid of buyer $i$ for good $j$ at time $t$.
- $p_j^{(t)} = \sum_{i \in B} b_{ij}^{(t)}$ price for good $j$.
- Allocation $x_{ij}^{(t)} = \frac{b_{ij}^{(t)}}{p_j^{(t)}}$.
- Utility of agent $i$ from good $j$ is $u_{ij}^{(t)} = x_{ij}^{(t)} w_{ij}$.
- Utility $u_i^{(t)} = \sum_{j \in G} u_{ij}^{(t)}$. Bid $b_i^{(t)} = \sum_{j \in G} b_{ij}^{(t)}$.
Proportional Response Dynamics

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For each agent $i$ and good $j$ set

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b_{ij}^{(t+1)} &= \frac{u_{ij}^{(t)}}{u_i^{(t)}}
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**Theorem (Convergence).** The proportional response dynamics converges to a market equilibrium in the Fisher market with linear utility functions. For linear functions, it converges to an $\epsilon$-market equilibrin in $O\left(\frac{1}{\epsilon^2}\right)$ iterations.
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**Remark:**

- The convergence result holds for CES utilities with a different rate.
- Similar rate to Multiplicative Weights Method (not a coincidence).
Proof. We will show only convergence (no rates). We need to come up with a potential function.
Proportional Response Dynamics: Proof of Convergence

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Let \((x^*, p^*)\) be a market equilibrium (optimum for EG program). We set

\[ b_{ij}^* = x_{ij}^* \cdot p_j^*. \]

The potential function will be

\[
\Phi(t) = \sum_{i \in B} \text{KL}(b_i^* || b_i^{(t)}).
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Remark:

- KL divergence \(\text{KL}(x||y) = \sum x_i \log \frac{x_i}{y_i}\) for distributions \(x, y\).
- \(\text{KL}(x||y) \geq 0\), pseudo-distance, symmetry not satisfied.
Proportional Response Dynamics: Proof of Convergence

Proof cont.

The potential function will be

$$\Phi(t) = \sum_{i \in B} \text{KL}(b_i^* \| b_i^{(t)}).$$

Recall the KKT-conditions:

$$\frac{w_{ij}}{u_i^*} - p_j^* = 0 \text{ if } x_{ij}^* > 0.$$

Therefore

$$b_{i,j}^* = p_j^* x_{i,j}^* = \frac{w_{ij} x_{i,j}^*}{u_i^*} = \frac{u_{i,j}^*}{u_i^*}.$$
Proportional Response Dynamics: Proof of Convergence

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Observe now that

\[
\begin{align*}
    b_{ij}^* \ln \frac{b_{ij}^*}{b_{ij}^{(t+1)}} &= b_{ij}^* \ln \frac{b_{ij}^* u_i^{(t)}}{u_{ij}} \\
    &= b_{ij}^* \ln \frac{u_{ij}^* u_i^{(t)}}{u_i^* u_{ij}} = b_{ij}^* \ln \frac{u_{ij}^*}{u_{ij}} - b_{ij}^* \ln \frac{u_i^{(t)}}{u_i^*}
\end{align*}
\]

Intro to AGT
Proportional Response Dynamics: Proof of Convergence

Proof cont.

Moreover \( \frac{u_{i,j}^*}{u_{i,j}^{(t)}} = \frac{b_{i,j}^* p_{j}^{(t)}}{b_{i,j}^{(t)} p_{j}^*} \). Combining the above we get

\[
\frac{b_{i,j}^*}{b_{i,j}^{(t+1)}} \ln \frac{b_{i,j}^*}{b_{i,j}^{(t)}} = \frac{b_{i,j}^*}{b_{i,j}^{(t)}} \ln \frac{b_{i,j}^*}{b_{i,j}^{(t)}} - \frac{b_{i,j}^*}{b_{i,j}^{(t)}} \ln \frac{p_{j}^*}{p_{j}^{(t)}} - b_{i,j}^* \ln \frac{u_{i}^*}{u_{i}^{(t)}}
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Proportional Response Dynamics: Proof of Convergence

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The potential function becomes

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\Phi^{(t+1)} = \sum_{i \in \mathcal{B}} \text{KL}(b_i^* \| b_i^{(t)}) = \sum_{i,j} b_{i,j}^* \ln \frac{b_{i,j}^*}{b_{i,j}^{(t)}} - b_{i,j}^* \ln \frac{p_j^*}{p_j^{(t)}} - b_{i,j}^* \ln \frac{u_i^*}{u_i^{(t)}}.
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The potential function becomes

\[ \Phi^{(t+1)} = \sum_{i \in B} \text{KL}(b_i^* || b_i^{(t)}) = \sum_{i,j} b_{i,j}^* \ln \frac{b_{i,j}^*}{b_{i,j}^{(t)}} - b_{i,j}^* \ln \frac{p_j^*}{p_j^{(t)}} - b_{i,j}^* \ln \frac{u_i^*}{u_i^{(t)}}. \]

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Proportional Response Dynamics: Proof of Convergence

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Moreover \( \frac{u_{i,j}^*}{u_{i,j}^{(t)}} = \frac{b_{i,j}^* p_j^{(t)}}{b_{i,j}^{(t)} p_j^*} \). Combining the above we get

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\]

We finally get

\[
= \Phi^{(t)} - \text{KL}(p^* || p^{(t)}) - \sum_{i,j} b_{i,j}^* \ln \frac{u_i^*}{u_i^{(t)}}.
\]
Definitions

Definition (Exchange Market). An exchange market consists of:

- A set $\mathcal{A}$ of $n$ agents.
- A set $\mathcal{G}$ of $m$ divisible goods.
- Each agent $i$ has an endowment $w_i = (w_{i1}, ..., w_{im})$, with $w_{ij}$ capturing the amount of good $j$ agent $i$ has.
- $u_{ij}$ denotes the utility derived by $i$ on obtaining a unit amount of good of $j$.
- Each good $j$ is associated with a price $p_j$.

Remark:

- Each agent first earns money by selling its endowment and then buys a utility maximizing (optimal) bundle of goods subject to budget constraints.
- Arrow-Debreu showed existence of a market equilibrium.
- PPAD-hard for $\rho = -\infty$, for $0 \leq \rho \leq 1$ is in P (e.g., DPSV)
Eisenberg-Gale Convex Program

$x^*$ satisfies the KKT conditions.

\[
L(x, p) = \sum_{j=1}^{n} \ln u_i - \sum_{j=1}^{n} p_j \left( \sum_{i=1}^{n} x_{ij} - 1 \right)
\]

Remark: Langrangian involves objective and constraints!

**KKT conditions:** $x$ are primal variables, $p$ are dual variables.

Primal feasibility: $x_{ij} \geq 0$ for all $i \in \mathcal{B}, j \in \mathcal{G}$.

Dual feasibility: $p_j \geq 0$ for all $j \in \mathcal{G}$.

\[
\begin{align*}
\frac{\partial L(x,p)}{\partial x_{ij}} &= \frac{w_{ij}}{u_i} - p_j = 0 \text{ if } x_{ij} > 0. \\
\frac{\partial L(x,p)}{\partial x_{ij}} &= \frac{w_{ij}}{u_i} - p_j \leq 0 \text{ if } x_{ij} = 0.
\end{align*}
\]

\[
\begin{align*}
\frac{\partial L(x,p)}{\partial p_j} &= 1 - \sum_{i=1}^{n} x_{ij} = 0 \text{ if } p_j > 0. \\
\frac{\partial L(x,p)}{\partial p_j} &= 1 - \sum_{i=1}^{n} x_{ij} \geq 0 \text{ if } p_j = 0.
\end{align*}
\]

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Other utility functions

CES (Constant elasticity of substitution) utility functions:

\[ u_i(x) = \left( \sum_{j=1}^{m} u_{ij} x_{ij}^{\rho} \right)^{\frac{1}{\rho}}, \text{ for } -\infty < \rho \leq 1. \]

Remark:
• \( u_i(x) \) is concave function.
• If \( u_{ij} = 0 \), then the corresponding term in the utility function is always 0.
• If \( u_{ij} > 0, x_{ij} = 0, \) and \( \rho < 0 \) then \( u_i(x) = 0 \) no matter what the other \( x_{ij} \)'s are.

\[ \rho = 1 \quad \rightarrow \quad \text{Linear utility form} \]

\[ \rho \rightarrow -\infty \quad \rightarrow \quad \text{Leontief utility form} \]

\[ \rho \rightarrow 0 \quad \rightarrow \quad \text{Cobb-Douglas form} \]

Elasticity of substitution \( \sigma = \frac{1}{1-\rho} \).