L17 More on Markets

CS 295 Introduction to Algorithmic Game Theory Ioannis Panageas

Recap

Definition (Market). *A market consists of:*

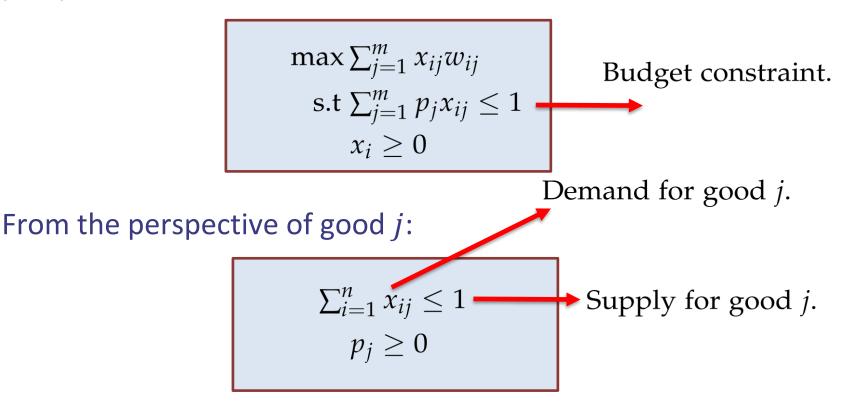
- A set B of n buyers/traders.
- A set \mathcal{G} of m goods.
- *Each buyer i has* 1 *amount of* \$.
- One unit for each good.
- w_{ij} denotes the utility derived by *i* on obtaining a unit amount of good of *j*.
- Each good j is associated with a price p_j .

Definition (Fisher Market). A market so that the utilities are linear: Let x_{ij} be the amount of units buyer i gets of good j then

$$u_i = \sum_{j \in \mathcal{G}} x_{ij} w_{ij}.$$

Recap

Given an arbitrary vector of prices $p \ge 0$, from each buyer's *i* perspective:



Can we find (*x*, *p*) s.t all are satisfied simultaneously?

Market dynamics:

Each time step the buyers face the same market parameters, (goods, budget constraint, utility function) while the buyers make their bidding decisions according to the previous market actions

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Notation:

• $b_{ij}^{(t)}$ the bid of buyer *i* for good *j* at time *t*.

•
$$p_j^{(t)} = \sum_{i \in \mathcal{B}} b_{ij}^{(t)}$$
 price for good *j*.

• Allocation
$$x_{ij}^{(t)} = \frac{b_{ij}^{(t)}}{p_j^{(t)}}$$
.

• Utility of agent *i* from good *j* is $u_{ij}^{(t)} = x_{ij}^{(t)} w_{ij}$.

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- Utility $u_i^{(t)} = \sum_{j \in \mathcal{G}} u_{ij}^{(t)}$. Bid $b_i^{(t)} = \sum_{j \in \mathcal{G}} b_{ij}^{(t)}$.

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Theorem (Convergence). The proportional response dynamics converges to a market equilibrium in the Fisher market with linear utility functions. For linear functions, it converges to an ϵ -market equilibrin in $O\left(\frac{1}{\epsilon^2}\right)$ iterations.

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Remark:

- The convergence result holds for CES utilities with a different rate.
- Similar rate to Multiplicative Weights Method (not a coincidence).

Proof. We will show only convergence (no rates). We need to come up with a potential function.

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Let (x^*, p^*) be a market equilibrium (optimum for EG program). We set

$$b_{ij}^* = x_{ij}^* \cdot p_j^*.$$

The potential function will be

$$\Phi^{(t)} = \sum_{i \in \mathcal{B}} \mathrm{KL}(b_i^* || b_i^{(t)}).$$

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Remark:

- KL divergence $KL(x||y) = \sum x_i \log \frac{x_i}{y_i}$ for distributions *x*, *y*.
- $KL(x||y) \ge 0$, pseudo-distance, symmetry not satisfied.

Proof cont.

The potential function will be

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Recall the KKT-conditions:
$$\frac{w_{ij}}{u_i^*} - p_j^* = 0 \text{ if } x_{ij}^* > 0.$$

Therefore
$$b_{ij}^* = p_j^* x_{ij}^* = \frac{w_{ij} x_{ij}^*}{u_i^*} = \frac{u_{ij}^*}{u_i^*}.$$

Proof cont.

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Observe now that

$$b_{ij}^* \ln \frac{b_{ij}^*}{b_{ij}^{(t+1)}} = b_{ij}^* \ln \frac{b_{ij}^* u_i^{(t)}}{u_{ij}^{(t)}}$$
$$= b_{ij}^* \ln \frac{u_{ij}^* u_i^{(t)}}{u_i^* u_{ij}^{(t)}} = b_{ij}^* \ln \frac{u_{ij}^*}{u_{ij}^{(t)}} - b_{ij}^* \ln \frac{u_i^{(t)}}{u_i^*}$$

Proof cont.

Moreover $\frac{u_{ij}^*}{u_{ij}^{(t)}} = \frac{b_{ij}^* p_j^{(t)}}{b_{ij}^{(t)} p_j^*}$. Combining the above we get

$$b_{ij}^* \ln \frac{b_{ij}^*}{b_{ij}^{(t+1)}} = b_{ij}^* \ln \frac{b_{ij}^*}{b_{ij}^{(t)}} - b_{ij}^* \ln \frac{p_j^*}{p_j^{(t)}} - b_{ij}^* \ln \frac{u_i^*}{u_i^{(t)}}$$

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Proof cont.

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We finally get

$$= \Phi^{(t)} - \mathrm{KL}(p^* || p^{(t)}) - \sum_{i,j} b_{ij}^* \ln \frac{u_i^*}{u_i^{(t)}}$$

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Definitions

Definition (Exchange Market). *An exchange market consists of:*

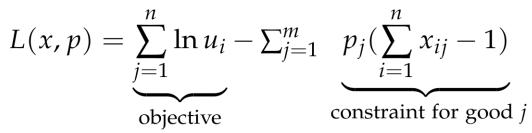
- A set \mathcal{A} of n agents.
- A set G of m divisble goods.
- Each agent *i* has an endowment $w_i = (w_{i1}, ..., w_{im})$, with w_{ij} capturing the amount of good *j* agent *i* has.
- u_{ij} denotes the utility derived by *i* on obtaining a unit amount of good of *j*.
- Each good j is associated with a price p_j .

Remark:

- Each agent first earns money by selling its endowment and then buys a utility maximizing (optimal) bundle of goods subject to budget constraints.
- Arrow-Debreu showed **existence** of a market equilibrium.
- PPAD-hard for $ho=-\infty$, for $0\leq
 ho\leq 1$ is in P (e.g., DPSV)

Eisenberg-Gale Convex Program

 x^* satisfies the KKT conditions.



Remark: Langrangian involves objective and constraints!

KKT conditions: x are primal variables, p are dual variables.**Primal feasibility**:**Dual feasibility**: $x_{ij} \ge 0$ for all $i \in \mathcal{B}, j \in \mathcal{G}$. $p_j \ge 0$ for all $j \in \mathcal{G}$.

$$\frac{\partial L(x,p)}{\partial x_{ij}} = \frac{w_{ij}}{u_i} - p_j = 0 \text{ if } x_{ij} > 0.$$

$$\frac{\partial L(x,p)}{\partial x_{ij}} = \frac{w_{ij}}{u_i} - p_j \le 0 \text{ if } x_{ij} = 0.$$

$$\frac{\partial L(x,p)}{\partial p_j} = 1 - \sum_{i=1}^n x_{ij} = 0 \text{ if } p_j > 0.$$

$$\frac{\partial L(x,p)}{\partial p_j} = 1 - \sum_{i=1}^n x_{ij} \ge 0 \text{ if } p_j = 0.$$
Intro to AGT

Other utility functions

CES (Constant elasticity of substitution) utility functions:

$$u_i(x) = \left(\sum_{j=1}^m u_{ij} x_{ij}^{\rho}\right)^{\frac{1}{\rho}}$$
, for $-\infty < \rho \le 1$.

Remark:

- $u_i(x)$ is concave function.
- If $u_{ij} = 0$, then the corresponding term in the utility function is always 0.
- If $u_{ij} > 0$, $x_{ij} = 0$, and $\rho < 0$ then $u_i(x) = 0$ no matter what the other x_{ij} 's are.

$$\rho = 1 \longrightarrow$$
 Linear utility form
 $\rho \rightarrow -\infty \longrightarrow$ Leontief utility form

 $\rho \rightarrow 0$ \longrightarrow Cobb-Douglas form

Elasticity of substitution $\sigma = \frac{1}{1-\rho}$.