L17 More on Markets

CS 295 Introduction to Algorithmic Game Theory

Ioannis Panageas

Recap

Definition (Market). *A market consists of:*

- *A set $\mathcal{B}$ of n buyers/traders.*
- *A set $\mathcal{G}$ of m goods.*
- *Each buyer i has 1 amount of \$.*
- *One unit for each good.*
- *w_{ij} denotes the utility derived by i on obtaining a unit amount of good of j.*
- *Each good j is associated with a price p_j.*

Definition (Fisher Market). *A market so that the utilities are linear: Let x_{ij} be the amount of units buyer i gets of good j then*

$$u_i = \sum_{j \in \mathcal{G}} x_{ij} w_{ij}.$$

Recap

Given an arbitrary vector of prices $p \geq 0$, from each buyer's i perspective:

$$\max \sum_{j=1}^{m} x_{ij} w_{ij}$$
$$\text{s.t} \sum_{j=1}^{m} p_j x_{ij} \leq 1$$
$$x_i \geq 0$$

Budget constraint.

Demand for good j.

From the perspective of good j:

$$\sum_{i=1}^{n} x_{ij} \leq 1$$
$$p_j \geq 0$$

Supply for good j.

Can we find (x, p) s.t all are satisfied simultaneously?

Proportional Response Dynamics

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Each time step the buyers face the same market parameters, (goods, budget constraint, utility function) while the buyers make their bidding decisions according to the previous market actions

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Notation:

- $b_{ij}^{(t)}$ the bid of buyer i for good j at time t.
- $p_j^{(t)} = \sum_{i \in \mathcal{B}} b_{ij}^{(t)}$ price for good j.
- Allocation $x_{ij}^{(t)} = \frac{b_{ij}^{(t)}}{p_j^{(t)}}$.
- Utility of agent i from good j is $u_{ij}^{(t)} = x_{ij}^{(t)} w_{ij}$.
- Utility $u_i^{(t)} = \sum_{j \in \mathcal{G}} u_{ij}^{(t)}$. Bid $b_i^{(t)} = \sum_{j \in \mathcal{G}} b_{ij}^{(t)}$.

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For each agent i and good j set

$$b_{ij}^{(t+1)} = \frac{u_{ij}^{(t)}}{u_i^{(t)}}$$

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Theorem (Convergence). *The proportional response dynamics converges to a market equilibrium in the Fisher market with linear utility functions.*
For linear functions, it converges to an ϵ-market equilibrin in $O\left(\frac{1}{\epsilon^2}\right)$ iterations.

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Remark:

- The convergence result holds for CES utilities with a different rate.
- Similar rate to Multiplicative Weights Method (not a coincidence).

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Let (x^*, p^*) be a market equilibrium (optimum for EG program). We set

$$b^*_{ij} = x^*_{ij} \cdot p^*_j.$$

The potential function will be

$$\Phi^{(t)} = \sum_{i \in \mathcal{B}} \mathrm{KL}(b^*_i || b^{(t)}_i).$$

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Remark:

- KL divergence $\mathrm{KL}(x||y) = \sum x_i \log \frac{x_i}{y_i}$ for distributions x, y.
- $\mathrm{KL}(x||y) \geq 0$, pseudo-distance, symmetry not satisfied.

Proportional Response Dynamics: Proof of Convergence

Proof cont.

The potential function will be

$$\Phi^{(t)} = \sum_{i \in \mathcal{B}} \mathrm{KL}(b_i^* || b_i^{(t)}).$$

Recall the KKT-conditions: $\frac{w_{ij}}{u_i^*} - p_j^* = 0 \text{ if } x_{ij}^* > 0.$

Therefore $b_{ij}^* = p_j^* x_{ij}^* = \frac{w_{ij} x_{ij}^*}{u_i^*} = \frac{u_{ij}^*}{u_i^*}.$

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Observe now that

$$b_{ij}^* \ln \frac{b_{ij}^*}{b_{ij}^{(t+1)}} = b_{ij}^* \ln \frac{b_{ij}^* u_i^{(t)}}{u_{ij}^{(t)}}$$

$$= b_{ij}^* \ln \frac{u_{ij}^* u_i^{(t)}}{u_i^* u_{ij}^{(t)}} = b_{ij}^* \ln \frac{u_{ij}^*}{u_{ij}^{(t)}} - b_{ij}^* \ln \frac{u_i^{(t)}}{u_i^*}$$

Proportional Response Dynamics: Proof of Convergence

Proof cont.

Moreover $\frac{u_{ij}^*}{u_{ij}^{(t)}} = \frac{b_{ij}^* p_j^{(t)}}{b_{ij}^{(t)} p_j^*}$. Combining the above we get

$$b_{ij}^* \ln \frac{b_{ij}^*}{b_{ij}^{(t+1)}} = b_{ij}^* \ln \frac{b_{ij}^*}{b_{ij}^{(t)}} - b_{ij}^* \ln \frac{p_j^*}{p_j^{(t)}} - b_{ij}^* \ln \frac{u_i^*}{u_i^{(t)}}$$

Proportional Response Dynamics: Proof of Convergence

Proof cont.

Moreover $\dfrac{u^*_{ij}}{u^{(t)}_{ij}} = \dfrac{b^*_{ij} p^{(t)}_j}{b^{(t)}_{ij} p^*_j}$. Combining the above we get

$$b^*_{ij} \ln \frac{b^*_{ij}}{b^{(t+1)}_{ij}} = b^*_{ij} \ln \frac{b^*_{ij}}{b^{(t)}_{ij}} - b^*_{ij} \ln \frac{p^*_j}{p^{(t)}_j} - b^*_{ij} \ln \frac{u^*_i}{u^{(t)}_i}$$

The potential function becomes

$$\Phi^{(t+1)} = \sum_{i \in \mathcal{B}} \mathrm{KL}(b^*_i || b^{(t)}_i) = \sum_{i,j} b^*_{ij} \ln \frac{b^*_{ij}}{b^{(t)}_{ij}} - b^*_{ij} \ln \frac{p^*_j}{p^{(t)}_j} - b^*_{ij} \ln \frac{u^*_i}{u^{(t)}_i}.$$

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Moreover $\frac{u^*_{ij}}{u^{(t)}_{ij}} = \frac{b^*_{ij} p^{(t)}_j}{b^{(t)}_{ij} p^*_j}$. Combining the above we get

$$b^*_{ij} \ln \frac{b^*_{ij}}{b^{(t+1)}_{ij}} = b^*_{ij} \ln \frac{b^*_{ij}}{b^{(t)}_{ij}} - b^*_{ij} \ln \frac{p^*_j}{p^{(t)}_j} - b^*_{ij} \ln \frac{u^*_i}{u^{(t)}_i}$$

The potential function becomes

$$\Phi^{(t+1)} = \sum_{i \in \mathcal{B}} \mathrm{KL}(b^*_i || b^{(t)}_i) = \sum_{i,j} \underbrace{b^*_{ij} \ln \frac{b^*_{ij}}{b^{(t)}_{ij}}}_{\Phi^{(t)}} - b^*_{ij} \ln \frac{p^*_j}{p^{(t)}_j} - b^*_{ij} \ln \frac{u^*_i}{u^{(t)}_i}.$$

We finally get

$$= \Phi^{(t)} - \mathrm{KL}(p^* || p^{(t)}) - \sum_{i,j} b^*_{ij} \ln \frac{u^*_i}{u^{(t)}_i}.$$

Definitions

Definition (Exchange Market). *An exchange market consists of:*

- *A set $\mathcal{A}$ of n agents.*
- *A set $\mathcal{G}$ of m divisible goods.*
- *Each agent i has an endowment $w_i = (w_{i1}, ..., w_{im})$, with w_{ij} capturing the amount of good j agent i has.*
- *u_{ij} denotes the utility derived by i on obtaining a unit amount of good of j.*
- *Each good j is associated with a price p_j.*

Remark:

- Each agent first earns money by selling its endowment and then buys a utility maximizing (optimal) bundle of goods subject to budget constraints.
- Arrow-Debreu showed **existence** of a market equilibrium.
- PPAD-hard for $\rho = -\infty$, for $0 \leq \rho \leq 1$ is in P (e.g., DPSV)

Eisenberg-Gale **Convex** Program

x^* satisfies the KKT conditions.

$$L(x,p) = \underbrace{\sum_{j=1}^{n} \ln u_i}_{\text{objective}} - \Sigma_{j=1}^{m} \underbrace{p_j(\sum_{i=1}^{n} x_{ij} - 1)}_{\text{constraint for good } j}$$

Remark: Langrangian involves objective and constraints!

KKT conditions: x are primal variables, p are dual variables.

Primal feasibility:

$x_{ij} \geq 0$ for all $i \in \mathcal{B},\ j \in \mathcal{G}$.

Dual feasibility:

$p_j \geq 0$ for all $j \in \mathcal{G}$.

$$\left.\begin{array}{l}
\frac{\partial L(x,p)}{\partial x_{ij}} = \frac{w_{ij}}{u_i} - p_j = 0 \text{ if } x_{ij} > 0. \\
\frac{\partial L(x,p)}{\partial x_{ij}} = \frac{w_{ij}}{u_i} - p_j \leq 0 \text{ if } x_{ij} = 0. \\
\frac{\partial L(x,p)}{\partial p_j} = 1 - \Sigma_{i=1}^{n} x_{ij} = 0 \text{ if } p_j > 0. \\
\frac{\partial L(x,p)}{\partial p_j} = 1 - \Sigma_{i=1}^{n} x_{ij} \geq 0 \text{ if } p_j = 0.
\end{array}\right\} \text{Complementary Slackness}$$

Other utility functions

CES (Constant elasticity of substitution) utility functions:

$$u_i(x) = \left(\sum_{j=1}^{m} u_{ij} x_{ij}^{\rho} \right)^{\frac{1}{\rho}}, \text{ for } -\infty < \rho \leq 1.$$

Remark:

- $u_i(x)$ is concave function.
- If $u_{ij} = 0$, then the corresponding term in the utility function is always 0.
- If $u_{ij} > 0$, $x_{ij} = 0$, and $\rho < 0$ then $u_i(x) = 0$ no matter what the other x_{ij}'s are.

$\rho = 1$ ⟶ Linear utility form

$\rho \to -\infty$ ⟶ Leontief utility form

$\rho \to 0$ ⟶ Cobb-Douglas form

Elasticity of substitution $\sigma = \frac{1}{1-\rho}$.