L15 Positional scored based voting rules

CS 295 Introduction to Algorithmic Game Theory
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Based on works of Ariel Procaccia

Recap

Theorem (Gibbard-Satterthwaite)**.** *Let f be a monotone social choice function onto A with $|A| \geq 3$, then f is a dictatorship.*

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Choose a voter at random and ask him/her to vote. How to we "measure" the performance of the mechanism? What are the guarantees?

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Answer: Positional scoring-based rules.

Positional scoring-based rules

Definition (Positional score based rule)**.** *Let n be the number of voters and m the number of candidates. Each voter i has preference $>_i$. A positional scoring rule is defined by a vector of nonnegative real numbers $a = (a_1, \ldots, a_n)$ so that the score of candidate x is given by*

$$sc(x, >) = \sum_{i=1}^{n} a_{>_i(x)}.$$

Examples:

- Plurality: $a = (1,0,\ldots,0)$.
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Goal: Design positional scoring rules that are incentive compatible and close to deterministic score-based rules (winner is the candidate with maximum score).

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$$\Pr[\text{choose } x] = \frac{a_{>_i(x)}}{\sum_{j=1}^{m} a_j}.$$

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Theorem (General Guarantee). *Let f be a positional scoring rule with parameters a. Then the approximation ratio of Rule 1 with respect to f is* $\Omega\left(\frac{1}{\sqrt{m}}\right)$.

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Proof. Assume $>$ is a preference profile and a is the candidate with the maximum score, i.e.,

$$sc(a, >) = \text{OPT}.$$

Let SUM be the total score of all candidates, that is

$$\text{SUM} = \sum_{x \in A} sc(x, >) = n \sum_{j=1}^{m} a_j.$$

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Proof cont. $sc(a,>) = \mathrm{OPT}.$ $\mathrm{SUM} = \sum_{x \in A} sc(x,>) = n \sum_{j=1}^{m} a_j.$

Rule 1 chooses candidate x with probability

$$\sum_{i \in I} \Pr[\text{choose voter i}] \times \Pr[\text{i chooses x}] = \frac{1}{n} \sum_{i \in I} \frac{a_{>_i(x)}}{\sum_{j=1}^{n} a_j}$$

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Hence, the expected score of the winner is

$$\frac{sc(a,>)}{\text{SUM}} \cdot \text{OPT} + \sum_{x\neq a, x\in A} \frac{sc(x,>)}{\text{SUM}} \cdot sc(x,>).$$

We need to lower bound the above.

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Recall CS inequality that is

$$\left(\sum_j b_j^2\right)\left(\sum_j c_j^2\right) \geq \left(\sum_j b_j c_j\right)^2 \text{ hence}$$

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Finally observe that

$$\text{SUM} - \text{OPT} = \sum_{x \neq a, x \in A} sc(x.>).$$

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Therefore we conclude

$$\frac{\mathrm{OPT}}{\mathrm{SUM}}\cdot \mathrm{OPT} + \sum_{x\neq a, x\in A} \frac{sc(x,>)}{\mathrm{SUM}}\cdot sc(x,>) \geq \frac{\mathrm{OPT}}{\mathrm{SUM}}\cdot \mathrm{OPT} + \frac{1}{m-1}\frac{(\mathrm{SUM}\text{ - }\mathrm{OPT})^2}{\mathrm{SUM}}$$

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The function $f(x) = \frac{\frac{x}{\mathrm{SUM}}\cdot x + \frac{1}{m-1}\frac{(\mathrm{SUM\ -\ x})^2}{\mathrm{SUM}}}{x}$ is minimized for $x = \frac{\mathrm{SUM}}{\sqrt{m}}$ and this gives $\approx \frac{2}{\sqrt{m}}$ approximation ratio.

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The winner will get expected score $\Omega\left(\frac{1}{\sqrt{m}}\right) \times \text{OPT}$

Further Approximations

Borda: $a = (m-1, m-2, \ldots, 0)$.

Theorem (Borda Guarantee). *Rule 1 gives a* $1/2 + \Omega\left(\frac{1}{m}\right)$*-approximation with respect to Borda.*

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Proof. If we assume Borda voting rule, we can get better approximation. Recall

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The winner gets expected score **at least**

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The idea is that $\mathrm{SUM} = \frac{nm(m-1)}{2}$ and $\mathrm{OPT} \leq n(m-1)$.

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$\frac{OPT}{SUM} \leq \frac{2}{m}$ **so we can improve the previous analysis!**

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Veto: $a = (1,1,\ldots,1,0)$.

Theorem (Veto Guarantee). *Rule 1 gives a* $1 - O\left(\frac{1}{m}\right)$*-approximation with respect to Veto.*

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Remark:

- Rule 1 is tight for Plurality!
- The proof uses Yao's min-max principle:

Best randomized algorithm over worst deterministic input same guarantees as worst distribution input of best deterministic algorithm.

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Remark:

- Rule 1 is **not** tight for Plurality!
- The proof uses Yao's min-max principle.

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Rule 2: Choose a pair of alternatives uniformly at random. If one is preferred to the other by a majority of agents then it is the winner. Break ties at random.

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Duple := voting rule of range at most 2.

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