L14 Social Choice Theory

CS 295 Introduction to Algorithmic Game Theory

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Definitions

Definition (Setting). *The social choice theory is defined by:*

- *Set I of n voters.*
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- *Each voter i has a set of preferences L (permutations of A).*

Definition (Social Choice function). *A social choice function is defined by*

$$f : L \times ... \times L \to A.$$

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Example: Three candidates a, b, c. Three voters with preferences

1) $'>_1' \coloneqq a > b > c$
2) $'>_2' \coloneqq b > c > a$
3) $'>_3' \coloneqq c > a > b$

Further Definitions

Two desirable properties

Definition (Unanimity). *A social welfare function F satisfies unanimity if*

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Definition (Independence of irrelevant alternatives). *A social welfare function F satisfies independence of irrelevant alternatives if for all* $a, b \in A$ *and* $>_1, \ldots, >_n, \overline{>}_1, \ldots, \overline{>}_n$ *with* $> = F(>_1, \ldots, >_n)$ *and* $\overline{>} = F(\overline{>}_1, \ldots, \overline{>}_n)$

$$a <_i b \Leftrightarrow a \overline{<}_i b \text{ for all } i \Rightarrow a < b \Leftrightarrow a \overline{<} b.$$

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Definition (Dictatorship)**.** *A voter i is a dictator if*

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Theorem (Arrow). *Every social welfare function over a set of more than 2 candidates ($|A| \geq 3$), that satisfies unanimity and indepedence of irrelevant alternatives is a dictatorship.*

Example: This is a negative result!

Proof of Arrow's Theorem

Proof. Let F be a social welfare function that satisfies unanimity and independence of irrelevant alternatives. Consider $>_1, ..., >_n$ and $\overline{>}_1, ..., \overline{>}_n$, $F(>_1, \ldots, >_n) = >$, $F(\overline{>}_1, \ldots, \overline{>}_n) = \overline{>}$ and $a, b, c, d \in A$ so that

for all voters i, $a <_i b \Leftrightarrow c \overline{<}_i d$.

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Idea: Assume $a < b$ and w.l.o.g $b \neq c$. Merge $<_i$ and $\overline{<}_i$ into $<_i$ by placing c just below a (unless $c = a$) and d above b (unless $c = a$) and preserve the internal order. Hence by unanimity we have

$$c < a \text{ and } b < d \text{ which implies } c < d.$$

Proof of Arrow's Theorem

Proof cont. Let $a \neq b$. Define a preference profile π^i for every $0 \leq i \leq n$ that the first i ranked players rank a above b, that is

$$a >_j b \Leftrightarrow j \leq i.$$

In π^0 we have $a < b$ and π^n we have $a > b$. Let i^* be the pivoting index. We need to show that i^* is a dictator.

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Let $c \neq d$. Assume $c >_{i^*} d$. We will show that $c > d$ where $> = F(>_1, \ldots, >_n)$, and the claim would follow. Consider $e \neq c, d$.

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- For $i < i^*$ move e to the top in $>_i$.
- For $i > i^*$ move e to the bottom in $>_i$.
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Using (IIA) we have not changed the social ranking between c, d. Note that players' preferences for c, e are identical to those for a, b in π^{i^*}, and players' preferences for e, d are identical to those for a, b in π^{i^*-1}.

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Using Claim we conclude that $c > e$ and $e > d$. Therefore by transitivity

$$c > d.$$

Social Choice: Gibbard-Satterthwaite

Definition (Motonone). *A social choice function* f *is monotone if* $a = f(<_1, \ldots, <_i, \ldots <_n)$ *and* $b = f(<_1, \ldots, <'_i, \ldots, <_n)$ *implies*

$$b <_i a \text{ and } a <'_i b.$$

Remark:

A social choice function is incentive compatible if it cannot strategically be manipulated. A function is incentive compatible if and only if it is monotone.

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Using Arrow's theorem, we can show another negative result:

Theorem (Gibbard-Satterthwaite). *Let f be a monotone social choice function onto A with $|A| \geq 3$, then f is a dictatorship.*

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Let S subset of the candidates and $<$ a preference. We denote

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Let S subset of the candidates and $<$ a preference. We denote

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the order by moving candidates in S to the top of $<$. Formally:

- If $a, b \in S$ then $a < b \Leftrightarrow a <^S b$.
- If $a, b \notin S$ then $a < b \Leftrightarrow a <^S b$.
- If $a \notin S$ and $b \in S$ then $a <^S b$.

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F is an extension of f. We need to show that F is a social welfare function so that if f is monotone and onto then F satisfies unanimity and IIA.

Claim: For any S and $<_1, \ldots, <_n$ we have $f(<_1^S, \ldots, <_n^S) \in S$.

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Since f is onto, there is a preference profile $<'_1, \ldots, <'_n$ such that

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Sequentially we change $<'_i$ to $<_i^S$ for $i = 1, \ldots, n$. Observe at no point, f will output a candidate $b \notin S$ since $b <^S a'$ for $a' \in S$ and a' being the previous outcome. Claim follows.

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We need to prove that F satisfies the following:

- Antisymmetry: By Claim we have $f(<_1^{\{a,b\}}, \ldots, <_n^{\{a,b\}}) \in \{a, b\}$.
- Transitivity: Take $S = \{a, b, c\}$ and assume $a < b$, $b < c$ and $c < a$. Use Claim to conclude that $a > b$ (contradiction).

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- Transitivity: Take $S = \{a, b, c\}$ and assume $a < b$, $b < c$ and $c < a$. Use Claim to conclude that $a > b$ (contradiction).
- Unanimity: If $b <_i a$ for all i then by Claim $f(<_1^{\{a,b\}}, \ldots, <_n^{\{a,b\}}) = a$.
- IIA: By monotonicity and Claim we have that if $b <_i a \Leftrightarrow b <'_i a$ then $f(<_1^{\{a,b\}}, \ldots, <_n^{\{a,b\}}) = f(<_1'^{\{a,b\}}, \ldots, <_n'^{\{a,b\}})$

Randomization and Positional scoring-based rules

Remark:

Due to negative result of Gibbard-Satterthwaite, we need to use randomization (toss coins).

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Example:

Choose a voter at random and ask him/her to vote. How to we "measure" the performance of the mechanism? What are the guarantees?

Answer: Positional scoring-based rules.

Positional scoring-based rules

Definition (Positional score based rule). *Let n be the number of voters and m the number of candidates. Each voter i has preference $>_i$. A positional scoring rule is defined by a vector of nonnegative real numbers $a = (a_1, \ldots, a_n)$ so that the score of candidate x is given by*

$$sc(x, >) = \sum_{i=1}^{n} a_{>_i(x)}.$$

Examples:

- Plurality: $a = (1,0,\ldots,0)$.
- Borda: $a = (m-1, m-2, \ldots, 0)$.
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Goal: Design positional scoring rules that are incentive compatible and close to deterministic score-based rules (winner is the candidate with maximum score).