

# L13 Myerson's Lemma cont (Bayesian).

CS 295 Introduction to Algorithmic Game Theory

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Inspired and some figures by Tim Roughgarden notes

# Recap (Single parameter)

Three desirable **guarantees**

1. **DSIC**: Being truthful is a dominant strategy.
2. Social **surplus maximization**.
3. Implementation in **polynomial time**.

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Three desirable **guarantees**

1. **DSIC**: Being truthful is a dominant strategy.
2. Social **surplus maximization**.
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**Theorem (Myerson's Lemma)**. Let  $(x, p)$  be a mechanism. We assume that  $p_i(b) = 0$  whenever  $b_i = 0$ , for all bidders  $i$ .

1. It holds that if  $(x, p)$  is DSIC mechanism then  $x$  is **monotone**.
2. If  $x$  is a monotone allocation, then there is a unique payment rule such that  $(x, p)$  is DSIC.

# A (computationally) hard example: Knapsack auctions

- Each bidder  $i$  has a publicly known size  $w_i$  and a private valuation  $v_i$ .
- The seller has capacity  $W$ .
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Remark:

- $k$ -identical item auction is a special case (why)?

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## Approach:

- **Step 1:** **Assume**, without justification, that bidders **bid truthfully**. How should we design the allocation so that we **can maximize surplus**?
- **Step 2:** Given our answer to Step 1, how should we **set the payments** so that **DSIC** holds?

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- **Step 2:** Given our answer to Step 1, how should we **set the payments** so that **DSIC** holds? **Payment** rule from **Myerson's** Lemma.

Remark: Theory people **are not happy** with the solution above.

# Relaxing Knapsack auctions: Approximation

## Approach:

- Step 1 was computationally **intractable**. **Instead**, how should we design the allocation so that we can **approximately** maximize surplus (**monotone allocation**)? Let  $b_1, \dots, b_n$  the bids of the agents:

First **remove** all  $i$ :  $w_i > W$ .

**Sort** and re-index bidders:  $\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \dots \geq \frac{b_n}{w_n}$ .

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**What guarantees the auctioneer has?**

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$$\text{If } A + B \geq \text{OPT} \text{ then} \\ \max(A, B) \geq \frac{\text{OPT}}{2}$$

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(relaxation of IP) has optimal solution  $x_1 = \dots = x_S = 1$  and  $x_{S+1} = \frac{W - \sum_{i=1}^S w_i}{w_{S+1}}$

LP relaxation

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Also we have

**OPT of knapsack  $\leq$  OPT of LP relaxation**

# Definitions: Bayesian Setting

**Definition (Bayesian - Single parameter setting).** *Bayesian setting single parameter environment is defined:*

- *$n$  bidders with private  $v_i$ .*
- *Feasible set  $\mathcal{X}$ , each element of which is a  $n$ -dimensional vector  $(x_1, \dots, x_n)$  in which  $x_i$  is the amount of "stuff" given to  $i$ .*
- *The private valuation  $v_i$  of agent  $i$  is assumed to be drawn from a distribution  $F_i$  with density  $f_i$  and support  $[0, v_{\max}]$ .*
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Intuition:

- 1 item, 1 person. Suppose *post price is*  $r$ . Revenue is

$$r \cdot (1 - F(r))$$

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$$\max_{r \in [0,1]} r - r^2 \Rightarrow r = \frac{1}{2}, \text{ rev} = \frac{1}{4}$$

# More Definitions

**Definition (Payments).** Assume bidders are truthful ( $b = v$ ). Recall by Myerson's Lemma:

$$p_i(v_i, v_{-i}) = \int_0^{v_i} z \cdot \frac{dx_i(z, v_{-i})}{dz} dz.$$

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Valuations are **random variables**, hence we care about the **expectation**:

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Plugging in the above:

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# Payments reduce to welfare

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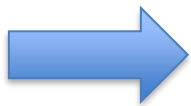
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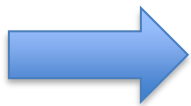
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$$\text{Rev} = \mathbb{E}_{v \sim F_1, \dots, F_n} \left[ \sum_i p_i(v) \right] = \mathbb{E}_{v \sim F_1, \dots, F_n} \left[ \sum_i x_i(v) \phi_i(v) \right]$$

# Monotone Allocations for regular $F$

Approach:

- **Step 1: Assume**, without justification, that bidders **bid truthfully**. How should we design the allocation so that we **can maximize virtual social welfare**,  $\sum x_i(v)\phi_i(v)$ ?
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Example (**Uniform is Regular**): Let  $F$  be the uniform in  $[0,1]$ . The valuation is  $2v - 1$  which is **strictly increasing**.

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- 2) Since virtual is strictly increasing, the winner is the **highest bidder**, thus the **allocation is monotone!**

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**Question:** What is the allocation rule and the payment?

- 1) Give the item to the bidder with **highest positive virtual** valuation.
- 2) Since virtual is strictly increasing, the winner is the **highest bidder**, thus the **allocation is monotone!**
- 3) The winner  $i$  pays  $\phi_i(v_i)$ .

Observe that this is a Vickrey auction with **reserve price**  $\phi^{-1}(0)$ . If valuations come from  $[0,1]$ , to maximize welfare, set  $r = \frac{1}{2}$ .