L13 Myerson's Lemma cont (Bayesian).

CS 295 Introduction to Algorithmic Game Theory

Ioannis Panageas

Inspired and some figures by Tim Roughgarden notes

Recap (Single parameter)

Three desirable guarantees

1. DSIC: Being truthful is a dominant strategy.
2. Social surplus maximization.
3. Implementation in polynomial time.

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Theorem (Myerson's Lemma). *Let (x, p) be a mechanism. We assume that $p_i(b) = 0$ whenever $b_i = 0$, for all bidders i.*

1. *It holds that if (x, p) is DSIC mechanism then x is monotone.*

2. *If x is a monotone allocation, then there is a unique payment rule such that (x, p) is DSIC.*

A (computationally) hard example: Knapsack auctions

- Each bidder i has a publicly known size w_i and a private valuation v_i.
- The seller has capacity W.
- Feasibility set X is all 0-1 n-vectors $(x_1, \ldots, x_n)$ so that $\sum x_i w_i \leq W$.

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Remark:

- k-identical item auction is a special case (why)?

Knapsack auctions

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we design the allocation so that we can maximize surplus?
- Step 2: Given our answer to Step 1, how should we set the payments so that DSIC holds?

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$$\begin{aligned} &\max_x \textstyle\sum_{i=1}^n x_i b_i \\ &\text{s.t } \textstyle\sum_{i=1}^n x_i w_i \leq W, \\ &\qquad x_i \in \{0,1\} \text{ for all } i. \end{aligned}$$

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- Step 2: Given our answer to Step 1, how should we set the payments so that DSIC holds? Payment rule from Myerson's Lemma.

Remark: Theory people are not happy with the solution above.

Relaxing Knapsack auctions: Approximation

Approach:

- Step 1 was computationally **intractable**. Instead, how should we design the allocation so that we can **approximately** maximize surplus (**monotone allocation**)? Let $b_1, \dots, b_n$ the bids of the agents:

  First remove all i: $w_i > W$.

  Sort and re-index bidders: $\boxed{\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}.}$

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  Choose as many as possible (say S) so that $\sum_{i=1}^{S} w_i \leq W$ and $\sum_{i=1}^{S+1} w_i > W$.
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What guarantees the auctioneer has?

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If $A + B \geq$ OPT then $\max(A, B) \geq \frac{OPT}{2}$

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LP relaxation

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Also we have $\boxed{\textbf{OPT of knapsack} \leq \textbf{OPT of LP relaxation}}$

Definitions: Bayesian Setting

Definition (Bayesian - Single parameter setting). *Bayesian setting single parameter environment is defined:*

- *n bidders with private v_i.*
- *Feasible set $\mathcal{X}$, each element of which is a n-dimensional vector $(x_1, \ldots, x_n)$ in which x_i is the amount of "stuff" given to i.*
- *The private valuation v_i of agent i is assumed to be drawn from a distribution F_i with density f_i and support $[0, v_{\max}]$.*
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$$\max_{r \in [0,1]} r - r^2 \Rightarrow r = \frac{1}{2}, \text{rev} = \frac{1}{4}$$

More Definitions

Definition (Payments). *Assume bidders are truthful ($b = v$). Recall by Myerson's Lemma:*

$$p_i(v_i, v_{-i}) = \int_0^{v_i} z \cdot \frac{dx_i(z, v_{-i})}{dz} dz.$$

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Plugging in the above:

$$\boxed{\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x_i'(z, v_{-i}) dz\right] f(v_i) dv_i.}$$

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Set $\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$ (called **virtual** valuations) and we get

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$$\Rightarrow \quad \mathrm{Rev} = \mathbb{E}_{v \sim F_1, \ldots, F_n}\left[\sum_i p_i(v)\right] = \mathbb{E}_{v \sim F_1, \ldots, F_n}\left[\sum_i x_i(v) \phi_i(v)\right]$$

Monotone Allocations for regular F

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- Step 1: Assume, without justification, that bidders bid truthfully. How should we design the allocation so that we can maximize virtual social welfare, $\sum x_i(v)\phi_i(v)$?
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Example (Uniform is Regular): Let F be the uniform in $[0,1]$. The valuation is $2v - 1$ which is strictly increasing.

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2) Since virtual is strictly increasing, the winner is the highest bidder, thus the allocation is monotone!
3) The winner i pays $\phi_i(v_i)$.

Observe that this is a Vickrey auction with reserve price $\phi^{-1}(0)$. If valuations come from $[0,1]$, to maximize welfare, set $r = \frac{1}{2}$.