L12 Monotone Allocations and Myerson’s Lemma

CS 295 Introduction to Algorithmic Game Theory
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Inspired and some figures by Tim Roughgarden notes
Recap

Three desirable guarantees

1. **DSIC**: Truthful bidding is a dominant strategy. Easy to play for bidders, Predict outcome.

2. Social surplus maximization:
   \[ \sum_{i=1}^{n} x_i v_i \]
   where \( x_i \) is the amount allocated to \( i \).

3. The auction can be implemented in polynomial time.
An Example: Sponsored Search Auctions

Every time you type a query into a search engine, an auction is run to decide which advertisers’ links are shown, the order of the links, and how advertisers are charged.
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- Items for sale are k “slots”
- Bidders are the advertisers.
- Each slot j has CTR (click-through-rate) $a_j$.
- Each bidder $i$ has private valuation $v_i$ and gets value $a_j \cdot v_i$. Note $a_1 \geq \ldots \geq a_k$.
Definitions

Definition (**Single parameter environments**). A single parameter environment is defined:

- $n$ bidders with private $v_i$,
- **Feasible set** $\mathcal{X}$, each element of which is a $n$-dimensional vector $(x_1, ..., x_n)$ in which $x_i$ is the amount of "stuff" given to $i$.

Examples:

1. **Single-item auctions**: $\mathcal{X}$ is 0-1 vectors with at most one 1, i.e., $\sum x_i \leq 1$.
2. **k identical goods, each bidder gets at most one**: $\mathcal{X}$ is 0-1 vectors with $\sum x_i \leq k$.
3. **In sponsored search**: $\mathcal{X}$ is the set of $n$-vectors with $x_i$ being $a_j$ if slot $j$ is assigned to bidder $i$. 
More Definitions

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**Definition (Allocation and Payments)**. A sealed-bid auction is defined:

1. Bidders report bids $b = (b_1, \ldots, b_n)$,
2. Auctioneer chooses feasible allocation $x(b) \in \mathcal{X}$.
3. Auctioneer chooses payments $p(b) \in \mathbb{R}^n$.
4. Bidder $i$ gets utility $u_i = v_i \cdot x_i(b) - p_i(b)$.
Monotone Allocations and Myerson’s Lemma

**Definition (Monotone Allocations).** An allocation rule $x$ for a single-parameter environment is *monotone* if for every bidder $i$ and bids $b_{-i}$ by rest of bidders, the allocation

$$x_i(z, b_{-i})$$

is nondecreasing in $z$. 

Intro to AGT
Monotone Allocations and Myerson’s Lemma

Definition (Monotone Allocations). An allocation rule $x$ for a single-parameter environment is monotone if for every bidder $i$ and bids $b_{-i}$ by rest of bidders, the allocation

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is nondecreasing in $z$.

Theorem (Myerson’s Lemma). Let $(x, p)$ be a mechanism. We assume that $p_i(b) = 0$ whenever $b_i = 0$, for all bidders $i$.

1. It holds that if $(x, p)$ is DSIC mechanism then $x$ is monotone.

2. If $x$ is a monotone allocation, then there is a unique payment rule such that $(x, p)$ is DSIC.
Myerson’s Lemma: Monotone

Proof. Suppose \((x, p)\) is a DSIC and let \(0 \leq y \leq z\).

If bidder \(i\) has private valuation \(z\), to avoid reporting \(y\), DSIC demands

\[ z \cdot x_i(z) - p_i(z) \geq y \cdot x_i(y) - p_i(y) \text{ for all } i. \]
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Combining the two inequalities:

\[ z \cdot (x_i(y) - x_i(z)) \leq p(y) - p(z) \leq y \cdot (x_i(y) - x_i(z)) \]
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Proof cont. Assume \( x \) is monotone for the rest of the proof and \( x \) is piecewise constant (simple function). If there is a jump at \( z \) (say of magnitude \( h \)) then as \( y \to z \) from left we get

\[ z \cdot h \leq p(y) - p(z) \leq y \cdot h. \]
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We conclude that (given \( p_i(0) = 0 \))

\[ p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot \text{jump in } x_i(., b_{-i}) \text{ at } z_j, \]

where \( z_1, \ldots, z_l \) are the breakpoints of \( x_i(., b_{-i}) \) in \([0, b_i] \).
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If we divide both sides on the top inequality and let \( y \rightarrow z \) we get

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\[ p_i(b_i, b_{-i}) = \int_{0}^{b_i} z \cdot \frac{dx_i(z, b_{-i})}{dz} \, dz. \]
Myerson’s Lemma: DSIC

Proof cont. By picture.
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Answer: Fix $i, b_{-i}$ and set $B = \max_{j \neq i} b_j$. Then $x_i(z, b_{-i})$ is 0 for $0 \leq z < B$ for and 1 for $z \geq B$. Moreover, $p_i(z, b_{-i}) = B$ for $z \geq B$ and 0 for $0 \leq z < B$. 
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Approach:

- **Step 1:** Assume, without justification, that bidders bid truthfully. How should we assign bidders to slots so that we can maximize surplus?
- **Step 2:** Given our answer to Step 1, how should we set selling prices so that DSIC holds?
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Back to Sponsored Search Auctions

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$$p_i(b) = \sum_{j=i}^{k} b_{j+1}(a_j - a_{j+1})$$