L12 Monotone Allocations and Myerson's Lemma

CS 295 Introduction to Algorithmic Game Theory

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Inspired and some figures by Tim Roughgarden notes

Recap

Three desirable guarantees

1. DSIC: Truthful bidding is a dominant strategy.

   Easy to play for bidders, Predict outcome.

2. Social surplus maximization:

   $$\sum_{i=1}^{n} x_i v_i$$

   where x_i is the amount allocated to i.

3. The auction can be implemented in polynomial time.

An Example: Sponsored Search Auctions

Every time you type a query into a search engine, an auction is run to decide which advertisers' links are shown, the order of the links, and how advertisers are charged.

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Probability to get a click

- Items for sale are k "slots"
- Bidders are the advertisers.
- Each slot j has CTR (click-through-rate) a_j.
- Each bidder i has private valuation v_i and gets value $a_j \cdot v_i$. Note $a_1 \geq \ \ldots \geq \ a_k$

Definitions

Definition (Single parameter environments). *A single parameter environment is defined:*

- *n bidders with private v_i,*
- *Feasible set $\mathcal{X}$, each element of which is a n-dimensional vector $(x_1, \ldots, x_n)$ in which x_i is the amount of "stuff" given to i.*

Examples:

1. Single-item auctions: $\mathcal{X}$ is 0-1 vectors with at most one 1, i.e., $\sum x_i \leq 1$.
2. k identical goods, each bidder gets at most one: $\mathcal{X}$ is 0-1 vectors with $\sum x_i \leq k$.
3. In sponsored search, $\mathcal{X}$ is the set of n-vectors with x_i being a_j if slot j is assigned to bidder i.

More Definitions

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Definition (Allocation and Payments). *A sealed-bid auction is defined:*

1. *Bidders report bids $b = (b_1, .., b_n)$,*
2. *Auctioneer chooses feasible allocation $x(b) \in \mathcal{X}$.*
3. *Auctioneer chooses payments $p(b) \in \mathbb{R}^n$.*
4. *Bidder i gets utility $u_i = v_i \cdot x_i(b) - p_i(b)$.*

Monotone Allocations and Myerson's Lemma

Definition (Monotone Allocations). *An allocation rule x for a single-parameter environment is monotone if for every bidder i and bids b_{-i} by rest of bidders, the allocation*

$$x_i(z, b_{-i}) \text{ is nondecreasing in } z.$$

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Theorem (Myerson's Lemma). *Let (x, p) be a mechanism. We assume that $p_i(b) = 0$ whenever $b_i = 0$, for all bidders i.*

1. *It holds that if (x, p) is DSIC mechanism then x is monotone.*
2. *If x is a monotone allocation, then there is a unique payment rule such that (x, p) is DSIC.*

Myerson's Lemma: Monotone

Proof. Suppose (x, p) is a DSIC and let $0 \leq y \leq z$.

If bidder i has **private valuation** z, to avoid reporting y, DSIC demands

$$z \cdot x_i(z) - p_i(z) \geq y \cdot x_i(y) - p_i(y) \text{ for all } i.$$

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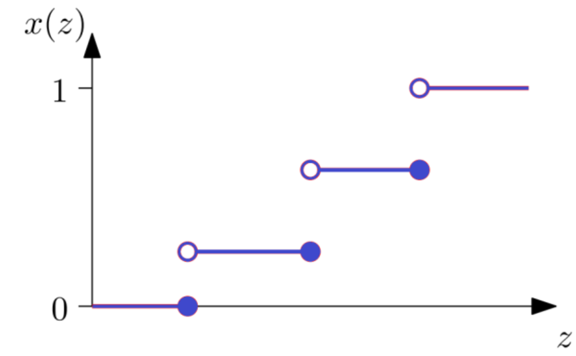
Proof cont. Assume x is monotone for the rest of the proof and x is piecewise constant (simple function). if there is a jump at z (say of magnitude h) then as $y \to z$ from left we get

$$z \cdot h \leq p(y) - p(z) \leq y \cdot h.$$



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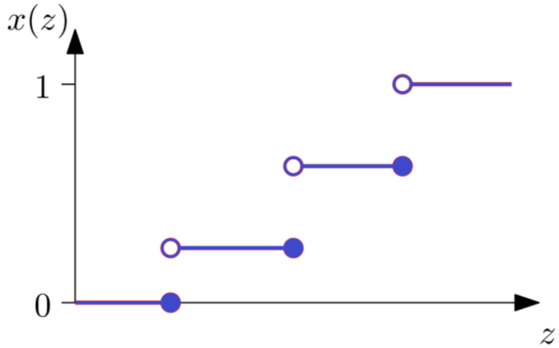

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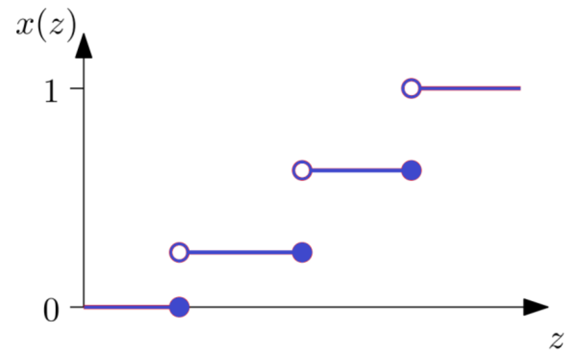

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We conclude that (given $p_i(0) = 0$)

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot \text{jump in } x_i(., b_{-i}) \text{ at } z_j,$$

where $z_1, ..., z_l$ are the breakpoints of $x_i(., b_{-i})$ in $[0, b_i]$.

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If we devide both sides on the top inequality and let $y \to z$ we get

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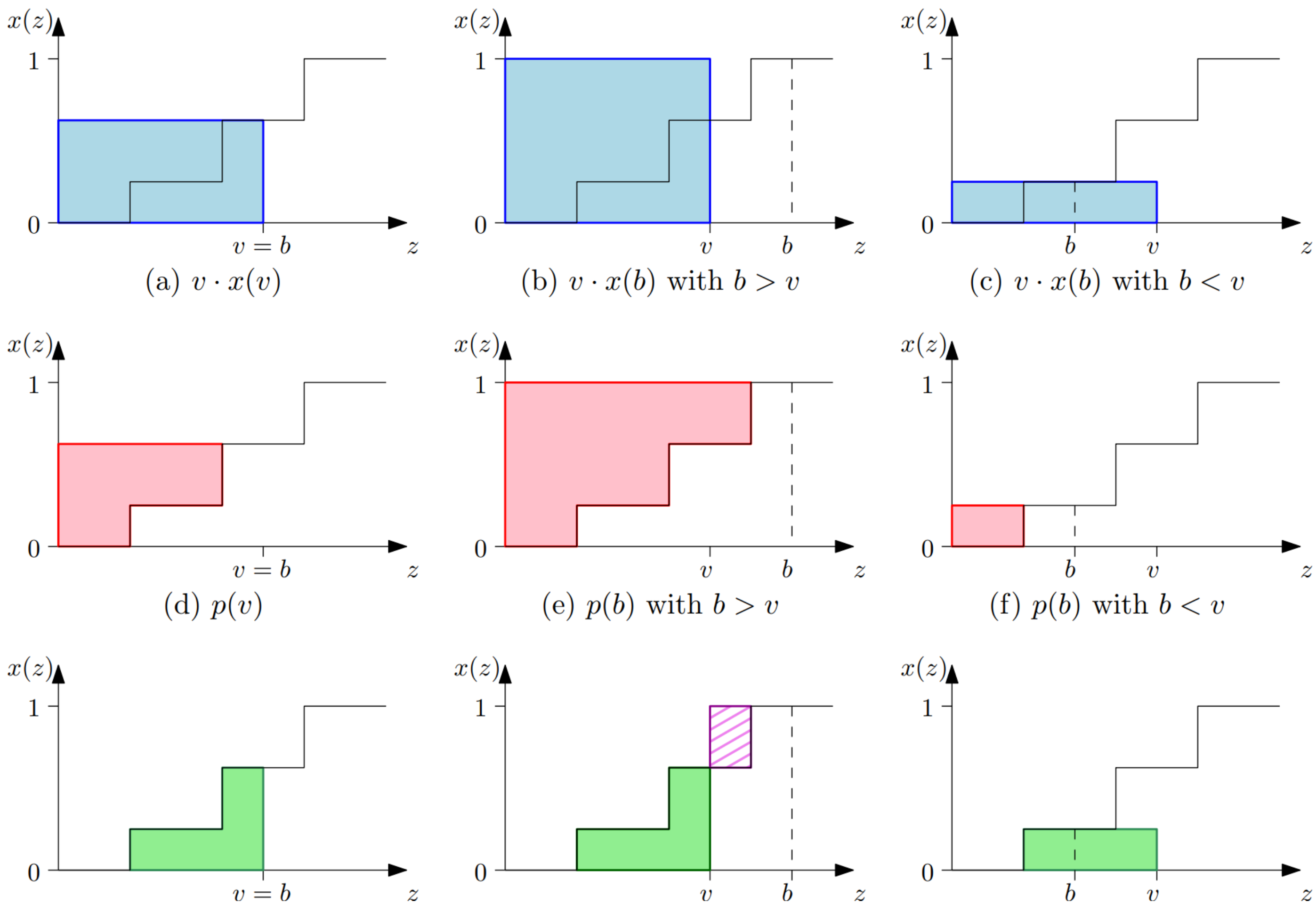
$$p_i'(z) \leq z \cdot x_i'(z)$$

$$\Rightarrow \quad p_i(b_i, b_{-i}) = \int_0^{b_i} z \cdot \frac{dx_i(z, b_{-i})}{dz} dz.$$

Myerson's Lemma: DSIC

Proof cont. By picture.



(a) $v \cdot x(v)$

(b) $v \cdot x(b)$ with $b > v$

(c) $v \cdot x(b)$ with $b < v$

(d) $p(v)$

(e) $p(b)$ with $b > v$

(f) $p(b)$ with $b < v$

(g) utility with $b = v$

(h) utility with $b > v$

(i) utility with $b < v$

Back to Sponsored Search Auctions

Remark: Myerson's Lemma regenerates the Vickrey auction as a special case. (why?)

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Answer: Fix i, b_{-i} and set $B = \max_{j \neq i} b_j$. Then $x_i(z, b_{-i})$ is 0 for $0 \leq z < B$ for and 1 for $z \geq B$. Moreover, $p_i(z, b_{-i}) = B$ for $z \geq B$ and 0 for $0 \leq z < B$.

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Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we assign bidders to slots so that we can maximize surplus?
- Step 2: Given our answer to Step 1, how should we set selling prices so that DSIC holds?

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$$\boxed{p_i(b) = \sum_{j=i}^{k} b_{j+1}(a_j - a_{j+1})}$$