L07 Price of Anarchy

CS 295 Introduction to Algorithmic Game Theory
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Price of Anarchy

Suppose 100 drivers commute from A to B. Drivers want to minimize the time.
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Price of Anarchy

Suppose 100 drivers commute from A to B. Drivers want to minimize the time.

Question: What if we add a new link?
Price of Anarchy

Suppose 100 drivers commute from A to B. Drivers want to minimize the time. Delay is now 2 hours for everybody at the unique Nash equilibrium.

Adding a fast link is not always a good idea!
Suppose 100 drivers commute from A to B. Drivers want to minimize the time.

Delay is now 2 hours for everybody at the unique Nash equilibrium.

Braess’s paradox

Adding a fast link is not always a good idea!

\[
\text{PoA} = \frac{\text{performance of worst case NE}}{\text{optimal performance if agents do not decide on their own}}
\]

Price of Anarchy (Koutsoupias, Papadimitriou 99’).
Non-atomic selfish routing

Example: Simpler example. **Pigou network.**

\[ c(x) = 1 \]

\[ c(x) = x \]
Non-atomic selfish routing

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\[ c(x) = 1 \]

\[ \frac{1}{2} \]

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\[ c(x) = x \]

Cost \( \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} \)
Non-atomic selfish routing

Example: Simpler example. **Pigou network.**

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Cost 1 and \( \text{PoA} = \frac{4}{3} \)
Non-atomic selfish routing

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A *non-atomic selfish routing* game is defined by:

- Graph \( G(V, E) \).
- Source destination pairs \( (s_1, t_1), \ldots, (s_k, t_k) \).
- \( r_i \) traffic from \( s_i \rightarrow t_i \).
- \( c_e(.) \geq 0 \) cost function of edge \( e \), continuous and non-decreasing.
- Flow is an equilibrium if all traffic is routed on *cheapest paths.*

*Intro to AGT*
Non-atomic selfish routing

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Cost of path: $c_p(f) = \sum_{e \in p} c_e(f)$

Social Cost := $\sum_p f_p c_p(f)$
Non-atomic selfish routing

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Cost of path: \( c_p(f) = \sum_{e \in p} c_e(f) \)

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Remark: Equilibrium flow exists and is unique!
Non-atomic selfish routing

A bad Example. **Pigou network with large degree $d$.**

\[ c(x) = 1 \]

$\epsilon$

\[ c(x) = x^d \]

\[ 1 - \epsilon \]

Cost $\approx \epsilon$ and PoA $= \frac{1}{\epsilon}$ for $d$ large.
Non-atomic selfish routing

A bad Example. Pigou network with large degree $d$.

Questions:
1. When is PoA small (bounded)?
2. Can we find bounds on PoA for specific classes of cost functions?
Price of Anarchy in Non-atomic selfish routing with \textit{Linear} costs

\textbf{Definition (Linear costs).} Linear costs are of the form \( c_e(x) = a_e \cdot x + b_e \).
Price of Anarchy in Non-atomic selfish routing with *Linear* costs

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**Theorem** (*Roughgarden-Tardos 00*, PoA for linear costs). For every network with *linear costs*:

$$\text{cost of Nash flow} \leq \frac{4}{3} \cdot \text{cost of optimal flow}.$$
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**Proof.** Let \( f^* \) be a Nash flow and \( f \) another flow. We first show (*Variational Inequality*)

\[
\sum_{e} f_e^* c_e(f_e^*) \leq \sum_{e} f_e c_e(f_e^*).
\]
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\sum_e f_e^* c_e(f_e^*) \leq \sum_e f_e c_e(f_e^*).
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Observe that $f^*$ equilibrium flow $\Rightarrow$ if $f_p^* > 0$ then $c_p(f^*) \leq c_{p'}(f^*)$ for all paths $p'$.
Price of Anarchy in Non-atomic selfish routing with *Linear* costs

*Proof cont.* Therefore all paths $p$ so that $f_p^* > 0$ have same cost say $L$. Hence $\sum_p f_p^* c_p(f^*) = L \cdot F$ where $F = \sum_p f_p^*$ is the total flow.
Price of Anarchy in Non-atomic selfish routing with *Linear* costs

*Proof cont.* Therefore all paths \( p \) so that \( f_p^* > 0 \) have same cost say \( L \). Hence

\[
\sum_p f_p^* c_p(f^*) = L \cdot F \quad \text{where} \quad F = \sum_p f_p^* \quad \text{is the total flow.}
\]

Since \( c_p(f^*) \geq L \) we conclude

\[
\sum_p f_p c_p(f^*) \geq L \sum_p f_p = L \cdot F
\]
Price of Anarchy in Non-atomic selfish routing with Linear costs

Proof cont. Therefore all paths $p$ so that $f_p^* > 0$ have same cost say $L$. Hence $\sum_p f_p^* c_p(f^*) = L \cdot F$ where $F = \sum_p f_p$ is the total flow.

Since $c_p(f^*) \geq L$ we conclude

$$\sum_p f_p c_p(f^*) \geq L \sum_p f_p = L \cdot F$$

Combining the above

$$\sum_e f_e c_e(f^*) = \sum_p f_p c_p(f^*) \geq L \cdot F = \sum_p f_p^* c_p(f^*) = \sum_e f_e^* c_e(f^*)$$
Price of Anarchy in Non-atomic selfish routing with *Linear* costs

\[ \sum_{e} f_{e}c_{e}(f^{*}) \geq \sum_{e} f^{*}_{e}c_{e}(f^{*}). \]

*Proof cont.* We get that

\[ \sum_{e} f^{*}_{e}c_{e}(f^{*}) \leq \sum_{e} f_{e}c_{e}(f) + \sum_{e} f_{e}(c_{e}(f^{*}) - c_{e}(f)) \]
Price of Anarchy in Non-atomic selfish routing with *Linear* costs

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We also have that

\[ \sum_{e} f_{e}(c_{e}(f^{*}) - c_{e}(f)) \leq \frac{1}{4} \sum_{e} f_{e}^{*}c_{e}(f^{*}) \]
Price of Anarchy in Non-atomic selfish routing with *Linear* costs

\[
\sum_{e} f_e c_e(f^*) \geq \sum_{e} f^*_e c_e(f^*).
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Proof cont. We get that

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We also have that

\[
\sum_{e} f_e (c_e(f^*) - c_e(f)) \leq \frac{1}{4} \sum_{e} f^*_e c_e(f^*)
\]

- Case \(c_e(f^*) < c_e(f)\) trivially \(f_e (c_e(f^*) - c_e(f)) \leq \frac{1}{4} f^*_e c_e(f^*)\).
Price of Anarchy in Non-atomic selfish routing with *Linear* costs

\[ \sum_{e} f_e c_e(f^*) \geq \sum_{e} f_e^* c_e(f^*). \]

**Proof cont.** We get that

\[ \sum_{e} f_e^* c_e(f^*) \leq \sum_{e} f_e c_e(f) + \sum_{e} f_e (c_e(f^*) - c_e(f)) \]

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- Case \( c_e(f^*) < c_e(f) \) trivially \( f_e (c_e(f^*) - c_e(f)) \leq \frac{1}{4} f_e^* c_e(f^*) \).
- Case \( c_e(f^*) \geq c_e(f) \) \( \Rightarrow f_e^* \geq f_e \). Linear costs \( \Rightarrow \) LHS = \( a_e f_e (f_e^* - f_e) \) and RHS \( \geq \frac{1}{4} a_e f_e^* 2 \).
Price of Anarchy in Non-atomic selfish routing with \textit{Linear} costs

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\textit{Proof cont.} We get that

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We also have that

\[ \sum_{e} f_e (c_e(f^*) - c_e(f)) \leq \frac{1}{4} \sum_{e} f_e^* c_e(f^*) \]

- Case \( c_e(f^*) < c_e(f) \): Since \( xy - y^2 \leq \frac{x^2}{4} \) \( \Rightarrow \) LHS \( \leq \) RHS. \( c_e(f^*) \).
- Case \( c_e(f^*) \geq c_e(f) \) \( \Rightarrow \) LHS \( \geq \) RHS. Linear costs \( \Rightarrow \) LHS \( = \) \( a_e f_e(f_e^* - f_e) \) and RHS \( \geq \frac{1}{4} a_e f_e^* 2. \)
Price of Anarchy in Non-atomic selfish routing with *Linear* costs

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\sum_{e} f_{e} c_{e}(f^{*}) \geq \sum_{e} f^{*}_{e} c_{e}(f^{*}).
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*Proof cont.* We conclude that

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\sum_{e} f^{*}_{e} c_{e}(f^{*}) \leq \sum_{e} f_{e} c_{e}(f) + \frac{1}{4} \sum_{e} f^{*}_{e} c_{e}(f^{*})
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Or equivalently

\[
\sum_{e} f_{e}^{*} c_{e}(f^{*}) \leq \frac{4}{3} \sum_{e} f_{e} c_{e}(f).
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Price of Anarchy in Non-atomic selfish routing with \textit{Linear} costs

\[
\sum_{e} f_{e} c_{e}(f^*) \geq \sum_{e} f_{e}^{*} c_{e}(f^*).
\]

Proof cont. We have shown that the Pigou example is tight!

Or equivalently

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\]

\textbf{Theorem} (Roughgarden 02', \textit{PoA for polynomial costs}). For every network with \textit{polynomial costs} with degree \( d \):

\[
\text{cost of Nash flow} \leq \Theta \left( \frac{d}{\log d} \right) \cdot \text{cost of optimal flow}.
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Price of Anarchy in Non-atomic selfish routing with *Linear* costs

\[ \sum_{e} f_{e} c_{e}(f^*) \geq \sum_{e} f_{e}^* c_{e}(f^*). \]

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Price of Anarchy in Congestion Games

**Theorem** (Christodoulou-Koutsoupias, PoA for linear costs). For every congestion game with **linear costs**:

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\text{cost of worst Nash} \leq \frac{5}{2} \cdot \text{cost of optimal welfare}.
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Proof. Let \( l^* \) be a Nash equilibrium in which \( i \) uses path \( P_i \) and assume \( i \) deviates to path \( \tilde{P}_i \). It holds (Variational Inequality)

\[
\sum_{e \in P_i} c_e(l^*_e) \leq \sum_{e \in P_i \cap \tilde{P}_i} c_e(l^*_e) + \sum_{e \in \tilde{P}_i \setminus P_i} c_e(l^*_e + 1)
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\]

\[
= \sum_{e \in \tilde{P}_i} c_e(l^*_e + 1).
\]
Proof cont. Consider any configuration \( \tilde{l} \), where each agent \( j \) uses path \( \tilde{P}_j \). Summing for all agents \( i \)

\[
\sum_{i \in [n]} \sum_{e \in P_i} c_e(l_e^*) \leq \sum_{i \in [n]} \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1).
\]
Price of Anarchy in Congestion Games

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\[
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Price of Anarchy in Congestion Games

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\[
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\]

\[
= \sum_{e} a_e \tilde{l}_e (l_e^* + 1) + b_e \tilde{l}_e.
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Price of Anarchy in Congestion Games

\[ \sum_{e \in P_i} c_e(l_e^*) \leq \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1). \]

Proof cont. Consider any configuration \( \tilde{l} \), where each agent \( j \) uses path \( \tilde{P}_j \). Summing for all agents \( i \)

\[ \sum_{i \in [n]} \sum_{e \in P_i} c_e(l_e^*) \leq \sum_{e} \left( \frac{5}{3} \tilde{l}_e^2 + \frac{1}{3} l_e^* \right) + b_e \tilde{l}_e. \]

Since \( y(z + 1) \leq \frac{5}{3} y^2 + \frac{1}{3} z^2 \) for naturals \( y, z \)

HW2

\[ \sum_{e} a_e \tilde{l}_e (l_e^* + 1) + b_e \tilde{l}_e. \]
Price of Anarchy in Congestion Games

\[\sum_{i\in[n]} \sum_{e\in P_i} c_e(l^*_e) \leq \sum_{e} a_e \left( \frac{5}{3} \tilde{l}_e^2 + \frac{1}{3} l^*_e \right) + b_e \tilde{l}_e\]

Proof cont. Observe that

\[\frac{5}{3} C(\tilde{l}) = \frac{5}{3} \sum_{i\in[n]} \sum_{e\in \tilde{P}_i} c_e(\tilde{l}_e) = \sum_{e} \frac{5}{3} a_e \tilde{l}_e^2 + \frac{5}{3} b_e \tilde{l}_e \geq \sum_{e} \frac{5}{3} a_e \tilde{l}_e^2 + b_e \tilde{l}_e\]
Price of Anarchy in Congestion Games

\[ \sum_{i \in [n]} \sum_{e \in P_i} c_e(l_e^*) \leq \sum_{e} a_e \left( \frac{5}{3} \tilde{l}_e^2 + \frac{1}{3} l_e^* \right) + b_e \tilde{l}_e \]

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Therefore

\[ C(l^*) \leq \frac{5}{3} C(\tilde{l}) + \frac{1}{3} \sum_{e} a_e l_e^* \]

Intro to AGT
Price of Anarchy in Congestion Games

\[
\sum_{i \in [n]} \sum_{e \in P_i} c_e(l_e^*) \leq \sum_{e} a_e \left( \frac{5}{3} \tilde{l}_e^2 + \frac{1}{3} l_e^* 2 \right) + b_e \tilde{l}_e
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Therefore

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C(l^*) \leq \frac{5}{3} C(\tilde{l}) + \frac{1}{3} \sum_{e} a_e l_e^* 2
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\[
\leq \frac{5}{3} C(\tilde{l}) + \frac{1}{3} C(l^*)
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Proof cont. Observe:

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Therefore

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C(l^*) \leq \frac{5}{3} C(\tilde{l}) + \frac{1}{3} \sum_{e} a_e l^*_e^2
\]

\[
\leq \frac{5}{3} C(\tilde{l}) + \frac{1}{3} C(l^*)
\]

Remark:
1. The above bound is tight!
2. For polynomial cost functions the PoA is exponential in \( d \).
Definition (Balls and Bins). Consider

- set of $n$ balls and $n$ bins $\{e_1, \ldots, e_n\}$.
- Each ball $i$ chooses a bin $j$ and pays the load of the bin $j$.
- Define social cost the maximum load.
- What is PoA? Is it $\frac{5}{2}$?
Price of Anarchy and Balls & Bins

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- set of \( n \) balls and \( n \) bins \( \{e_1, \ldots, e_n\} \).
- Each ball \( i \) chooses a bin \( j \) and pays the load of the bin \( j \).
- Define social cost the maximum load.
- What is PoA?

\[
\begin{align*}
    c_{e_1}(x) &= x \\
    c_{e_2}(x) &= x \\
    \vdots \\
    c_{e_{n-1}}(x) &= x \\
    c_{e_n}(x) &= x
\end{align*}
\]
Price of Anarchy and Balls & Bins

**Theorem** (Koutsoupias-Papadimitriou, PoA for balls & bins). *The PoA is*

\[ \Omega \left( \frac{\ln n}{\ln \ln n} \right). \]

*Proof.* We will use second moment method.

- Set every ball in a different bin. Hence optimal social cost is 1.
- Uniform \( \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \) is a Nash Equilibrium (symmetry).
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- With high probability, we show that uniform gives max load $\Omega \left( \frac{\ln n}{\ln \ln n} \right)$, which implies the expected max load is $\Omega \left( \frac{\ln n}{\ln \ln n} \right)$. 

Intro to AGT
Price of Anarchy and Balls & Bins

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- Uniform \( \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \) is a Nash Equilibrium (symmetry).
- With high probability, we show that uniform gives max load \( \Omega \left( \frac{\ln n}{\ln \ln n} \right) \), which implies the expected max load is \( \Omega \left( \frac{\ln n}{\ln \ln n} \right) \).

**Claim 1:** Bin \( i \) has at least \( k \ll n \) balls with probability at least:

$$\binom{n}{k} \frac{1}{n^k} \left( 1 - \frac{1}{n} \right)^{n-k} \geq \frac{1}{n^k} \left( \frac{n}{k} \right)^k \frac{1}{e} = \frac{1}{ek^k}.$$
Price of Anarchy and Balls & Bins

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\[ \Omega \left( \frac{\ln n}{\ln \ln n} \right). \]

**Proof.** We will use second moment method.

- Set every ball in a different bin. Hence optimal social cost is 1.
- Uniform (\(k\) is chosen with probability \(1/k\)).
- With high probability, there are at least \(k \ll n\) balls in bin even, which implies

\[ \left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{en}{k} \right)^k. \]

**Claim 1:** Bin \(i\) has at least \(k \ll n\) balls with probability at least:

\[ \left( \frac{n}{k} \right) \frac{1}{n^k} \left(1 - \frac{1}{n}\right)^{n-k} \geq \frac{1}{n^k} \left( \frac{n}{k} \right)^k \frac{1}{e} = \frac{1}{ek^k}. \]
Proof cont. Choosing \( k = \frac{\ln n}{3 \ln \ln n} \) we have \( k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln n}} = n^{1/3} \).

**Claim 1:** Thus bin \( i \) has at least \( \frac{\ln n}{3 \ln \ln n} \) balls with probability at least \( \frac{1}{en^{1/3}} \).
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*Proof cont.* Choosing \( k = \frac{\ln n}{3 \ln \ln n} \) we have \( k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln n}} = n^{1/3} \).

**Claim 1:** Thus bin \( i \) has at least \( \frac{\ln n}{3 \ln \ln n} \) balls with probability at least \( \frac{1}{en^{1/3}} \).

Let \( X_i \) be the indicator that bin \( i \) has at least \( \frac{\ln n}{3 \ln \ln n} \) balls and \( X \) be the expected number of all bins with at least \( \frac{\ln n}{3 \ln \ln n} \) balls.
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Proof cont. Choosing \( k = \frac{\ln n}{3 \ln \ln n} \) we have \( k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln n}} = n^{1/3} \).

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\[
X = \sum_i X_i \Rightarrow E[X] = \sum_i E[X_i].
\]
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Proof cont. Choosing \( k = \frac{\ln n}{3 \ln \ln n} \) we have \( k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln n}} = n^{1/3} \).

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\[
X = \sum_i X_i \Rightarrow E[X] = \sum_i E[X_i].
\]

Observe that \( E[X] \geq \frac{n^{2/3}}{e} \gg 1 \) but this does not imply \( X \geq 1 \) with high probability. We need to argue about the variance (second moment).
Proof cont. Choosing $k = \frac{\ln n}{3 \ln \ln n}$ we have $k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln n}} = n^{1/3}$.

Claim 1: Thus bin $i$ has at least $\frac{\ln n}{3 \ln \ln n}$ balls with probability at least $\frac{1}{en^{1/3}}$.

Let $X_i$ be the indicator that bin $i$ has at least $\frac{\ln n}{3 \ln \ln n}$ balls and $X$ be the expected number of all bins with at least $\frac{\ln n}{3 \ln \ln n}$ balls.

$$X = \sum_i X_i \Rightarrow E[X] = \sum_i E[X_i].$$

Observe that $E[X] \geq \frac{n^{2/3}}{e} \gg 1$ but this does not imply $X \geq 1$ with high probability. We need to argue about the variance (second moment).

Chebyshev's inequality gives $Pr[|X - E[X]| \geq tE[X]] \leq \frac{Var[X]}{t^2E^2[X]}$, 

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Proof cont. Choosing \( k = \frac{\ln n}{3 \ln \ln n} \) we have \( k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln n}} = n^{1/3} \).

Claim 1: Thus bin \( i \) has at least \( \frac{\ln n}{3 \ln \ln n} \) balls with probability at least \( \frac{1}{en^{1/3}} \).

Let \( X_i \) be the indicator that bin \( i \) has at least \( \frac{\ln n}{3 \ln \ln n} \) balls and \( X \) be the expected number of all bins with at least \( \frac{\ln n}{3 \ln \ln n} \) balls.

\[
X = \sum_{i} X_i \Rightarrow E[X] = \sum_{i} E[X_i].
\]

Observe that \( E[X] \geq \frac{n^{2/3}}{e} \gg 1 \) but this does not imply \( X \geq 1 \) with high probability. We need to argue about the variance (second moment).

Chebyshev’s inequality gives

\[
Pr[|X - E[X]| \geq tE[X]] \leq \frac{Var[X]}{t^2E^2[X]},
\]

thus

\[
Pr[X = 0] \leq Pr[|X - E[X]| \geq E[X]] \leq \frac{Var[X]}{E^2[X]}.
\]
Proof cont. \[ \Pr[X = 0] \leq \frac{Var[X]}{E^2[X]} \].

From negative correlation we have that \( Var[X] \leq \sum_i Var[X_i] \).

Moreover \( Var[X_i] = E[X_i^2] - E^2[X_i] \leq E[X_i^2] = E[X_i] \leq 1 \).
Proof cont. \[ Pr[X = 0] \leq \frac{\text{Var}[X]}{E^2[X]} \].

From negative correlation we have that \( \text{Var}[X] \leq \sum_i \text{Var}[X_i] \).

Moreover \( \text{Var}[X_i] = E[X_i^2] - E^2[X_i] \leq E[X_i^2] = E[X_i] \leq 1 \).

We conclude that

\[ Pr[X = 0] \leq \frac{n}{e^2 n^{4/3}} = \frac{n^{-1/3}}{e^2}. \]
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Proof cont. \( Pr[X = 0] \leq \frac{Var[X]}{E^2[X]} \).

From negative correlation we have that \( Var[X] \leq \sum_i Var[X_i] \).

Moreover \( Var[X_i] = E[X_i^2] - E^2[X_i] \leq E[X_i^2] = E[X_i] \leq 1 \).

We conclude that

\[
Pr[X = 0] \leq \frac{n}{e^2 n^{4/3}} = \frac{n^{-1/3}}{e^2}.
\]

Therefore

\[
Pr[X \geq 1] = 1 - Pr[X = 0] \geq 1 - \frac{n^{-1/3}}{e^2} \to 1.
\]
Congestion Games

A congestion game is defined by:

- \( n \) set of players.
- \( E \) set of edges/facilities/bins.
- \( S_i \subset 2^E \) the set of strategies of player \( i \).
- \( c_e : \{1, \ldots, n\} \rightarrow \mathbb{R}^+ \) cost function of edge \( e \).

For any \( s = (s_1, \ldots, s_n) \):

- \( l_e(s) \) number of players (load) that use edge \( e \).
- \( c_i(s) = \sum_{e \in s_i} c_e(l_e) \) the cost function of player \( i \).
Congestion Games

For this game:

\[ n = \{1, 2\} \text{ (red, green)} \]
\[ E \text{ are the edges of the network.} \]
\[ S_i \text{ is all } s - t \text{ paths.} \]
\[ c_e \text{ on edges.} \]