L05 Computing NE in two player games

CS 295 Introduction to Algorithmic Game Theory

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Any Nash equilibrium with support S, T (x, y) must satisfy:

1a) $x_i \geq 0$ for all $i \in [n]$.

2a) $x_i = 0$ for all $i \notin S$.

3a) $\sum_{i \in S} x_i = 1$.

1b) $y_i \geq 0$ for all $i \in [m]$.

2b) $y_i = 0$ for all $i \notin T$.

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4a) $(Ry)_i \geq (Ry)_j \; \forall i \in S, j \in [n]$.

1b) $y_i \geq 0$ for all $i \in [m]$.

2b) $y_i = 0$ for all $i \notin T$.

3b) $\sum_{i \in T} y_i = 1$.

4b) $(C^\top x)_i \geq (C^\top x)_j \; \forall i \in T, j \in [m]$.

A trivial algorithm

LP (S, T)

$$(C^\top x)_i \geq (C^\top x)_j \; \forall i \in T, j \in [m].$$
$$(Ry)_i \geq (Ry)_j \; \forall i \in S, j \in [n].$$
$$\textstyle\sum_{i \in S} x_i = 1.$$
$$\textstyle\sum_{i \in T} y_i = 1.$$
$$x_i = 0 \text{ for all } i \notin S.$$
$$y_i = 0 \text{ for all } i \notin T.$$
$$x_i \geq 0 \text{ for all } i \in [n].$$
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Algorithm: For all index sets S, T, check feasibility of $LP\ (S, T)$. If a feasible solution (x, y) is found, it is a Nash.

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LP (S, T)

$$(C^\top x)_i \geq (C^\top x)_j \ \forall i \in T, j \in [m].$$

$$(Ry)_i \geq (Ry)_j \ \forall i \in S, j \in [n].$$

$$\sum_{i \in S} x_i = 1.$$

Running time $2^{n+m} \cdot \text{poly}(n, m)$!

Slow, not polynomial!

$$y_i \geq 0 \text{ for all } i \in [m].$$

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Lemke-Howson Algorithm

Assumption: Matrices R, C have non-negative entries. No loss of generality, NE are invariant under shifting.

Basic idea: The Lemke-Howson algorithm maintains a single guess of the supports, and in each iteration we change the guess only a little bit.

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$$P_1 = \{x \in \mathbb{R}^n : \forall i \in [n] \;\; x_i \geq 0 \;\&\; \forall j \in [m] \;\; (x^\top C)_j \leq 1\}.$$

$$P_2 = \{y \in \mathbb{R}^m : \forall i \in [m] \;\; y_i \geq 0 \;\&\; \forall j \in [n] \;\; (Ry)_j \leq 1\}.$$

$$\mathrm{nrml}(x) = \left(\textstyle\sum_{i \in [n]} x_i\right)^{-1} x \qquad \mathrm{nrml}(y) = \left(\textstyle\sum_{i \in [m]} y_i\right)^{-1} y$$

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Lemma. *Let $x^* \in P_1$, $y^* \in P_2$, x^*, y^* have all labels and assume x^*, y^* are not zero vectors. It holds that $(nrml(x^*), nrml(y^*))$ is a Nash equilibrium.*

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Lemma. *Let $x^* \in P_1$, $y^* \in P_2$, x^*, y^* have all labels together and assume x^*, y^* are not zero vectors. It holds that $(nrml(x^*), nrml(y^*))$ is a Nash equilibrium.*

Proof.

- For each $i \in [n]$, either $x_i^* = 0$ or $(Ry^*)_i = 1$ (i is best response of row player to $\text{nrml}(y^*)$).
- For each $j \in [m]$, either $y_j^* = 0$ or $(x^{*\top} C)_j = 1$ (j is best response of column player to $\text{nrml}(x^*)$).

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We conclude that

$$\text{if } x_i^* > 0 \Rightarrow (Ry^*)_i \geq (Ry^*)_j \ \ \forall j \in [n]$$

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Proof.

- For eac[h …]nse of row player t[…]
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They satisfy $\mathbf{LP}(\mathbf{Supp}(x^*), \mathbf{Supp}(y^*))$!

Inverse is also true!

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Definition (Vertex)**.** *A vertex of polytope P_1 is given by n linearly independent equalities (the rest constraints of P_1 are strict inequalties). A vertex for P_2 is given by m linearly independent equalities (the rest constraints of P_1 are strict inequalties). For $P_1 \cup P_2$ is $n + m$. This is the non-degenerate case.*

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Algorithm (Lemke-Howson). *We define the following algorithm:*

1. Initialize $x = \mathbf{0}$ and $y = \mathbf{0}$.
2. $k = k_0 = 1$.
3. **Loop**
4. In P_1 find the neighbor vertex x' of x with label k' instead of k. Remove label k and add label k'.
5. **Set** $x = x'$.
6. **If** $k' = 1$ **STOP**.
7. In P_2 find the neighbor vertex y' of y with label k'' instead of k'. Remove label k' and add label k''.
8. **Set** $y = y'$.
9. **If** $k'' = 1$ **STOP**.
10. **Set** $k = k''$.

Analysis of Lemke-Howson

Theorem. *The Lemke-Howson algorithm outputs a Nash equilibrium.*

Proof. Define a graph with vertices in $P_1 \cup P_2$. Each vertex (x, y) has:

- One **duplicate** label. This vertex is adjacent to exactly two other vertices, since we can remove the duplicate label from x and pivot in P_1, or remove the duplicate label from y and pivot in P_2.

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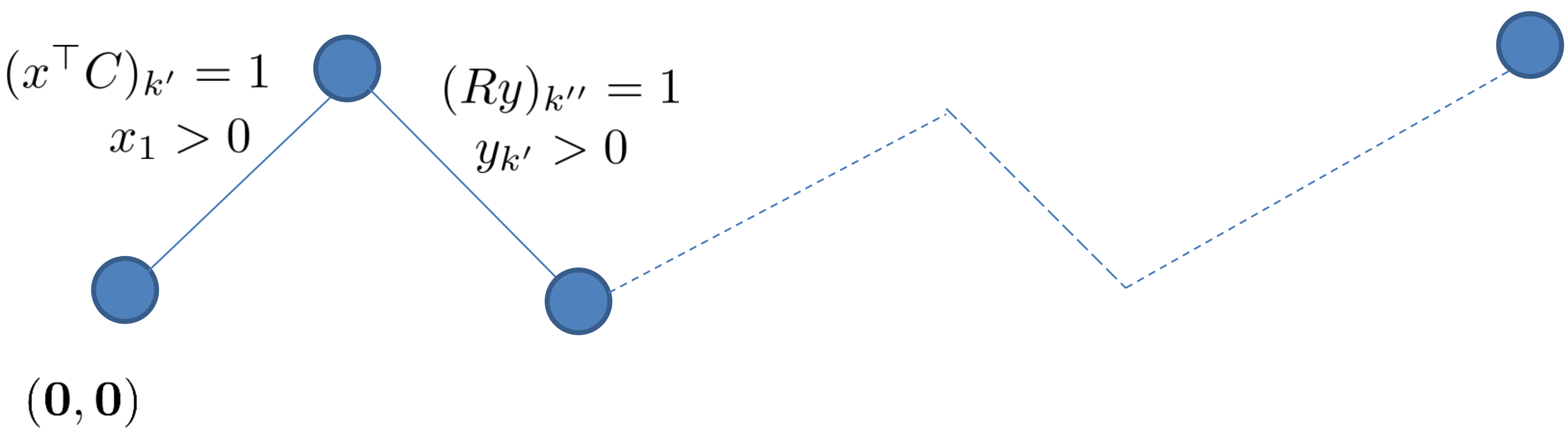

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1. Lemke-Howson algorithm begins at the configuration $(0,0)$.
2. $(0,0)$ has all labels and is therefore an endpoint of a path component.
3. The algorithm will terminate at the other endpoint of the path.
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From previous lemma, it must be a Nash equilibrium!

Other facts

Corollary (Odd Number). *For non-degenerate games, the number of Nash equilibria is odd!*

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Theorem (Savani, von Stengel'04). *The Lemke-Howson algorithm runs in exponential time in worst-case*

Approximating a Nash eq.

Definition (*k*-uniform). *A strategy x is called k-uniform when every coordinate x_i is a multiple of $1/k$.*

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Theorem (Approximate Nash with small support)**.** *Let $\epsilon > 0$. For any two player game, there always exists a k-uniform ϵ-approximate Nash equilibrium for $k = \frac{12 \log n}{\epsilon^2}$.*

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Remarks:

This was shown by Lipton, Markakis and Mehta using probabilistic method.

It gives a $n^{O\left(\frac{\log n}{\epsilon^2}\right)}$ algorithm. It was shown by Rubinstein that this is tight!