L03 LP Duality and zero-sum games

CS 295 Introduction to Algorithmic Game Theory

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Linear Programming

Problem (Linear Program (Feasibility))**.** *Suppose we are given a linear program in the standard form*

$$Ax \leq b$$
$$x \geq 0.$$

where A is of size $n \times m$.
Goal: Find a feasible solution x^* *(if there is one).*

Remark: We have n constraints and m variables.

Linear Programming

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Lemma (Equivalence). *These two problems are polynomial time equivalent.*

Linear Programming

Problem (Primal Formulation)**.** *Suppose we are given a linear program in the standard form*

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$$s.t\ Ax \leq b$$
$$x \geq 0.$$

Goal: Find optimal or return infeasible.

We can also define the dual formulation.

Problem (Dual Formulation)**.**

$$\min b^\top y$$
$$s.t\ A^\top y \geq c$$
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Linear Programming

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Problem (Dual Formulation).

$$\min b^\top y$$
$$s.t\ A^\top y \ge c$$
$$y \ge 0.$$

Remark: We have m constraints and n variables!

Linear Programming

Facts (Four possible cases)**.**

1. Primal bounded and feasible $\Rightarrow$ Dual bounded and feasible.
2. Primal unbounded and feasible $\Rightarrow$ Dual infeasible.
3. Primal infeasible $\Rightarrow$ Dual unbounded and feasible.
4. Primal infeasible $\Rightarrow$ Dual infeasible.

Linear Programming

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Let's focus on case 1.

Theorem (Weak duality)**.** *Assume that primal is feasible and bounded. It holds that*

$$\max_{x \in P} c^\top x \leq \min_{y \in D} b^\top y$$

Linear Programming

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Proof. Let $x \in P$. We have that $x^\top A^\top y \geq x^\top c$.

Linear Programming

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Proof. Let $x \in P$. We have that $x^\top A^\top y \geq x^\top c$.

Moreover, let $x \in D$. We have that $y^\top Ax \leq y^\top b$.

Linear Programming

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Proof. Let $x \in P$. We have that $x^\top A^\top y \geq x^\top c$.

Moreover, let $x \in D$. We have that $y^\top A x \leq y^\top b$.

Therefore, $c^\top x \leq y^\top A x \leq y^\top b$.

Since x, y were arbitrary it follows $\max_{x \in P} c^\top x \leq \min_{y \in D} b^\top y$.

Linear Programming

Theorem (Strong duality). *Assume that primal is feasible and bounded. It actually holds that*

$$\max_{x \in P} c^\top x = \min_{y \in D} b^\top y$$

Remark: The proof is much harder, it uses Farkas' lemma.

Linear Programming

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Example.

Primal

$$\begin{aligned} &\max z \\ \text{s.t } & 3x_1 - 2x_2 - z \geq 0 \\ & -x_1 + x_2 - z \geq 0 \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Linear Programming

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Example.

Primal

$$\max 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot z$$

$$\text{s.t} \begin{pmatrix} -3 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix} \leq 0$$

$$x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

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Example.

Primal	Dual
$\max 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot z$ s.t $\begin{pmatrix} -3 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix} \leq 0$, $x_1 + x_2 = 1$, $x_1, x_2 \geq 0$	$\min 0 \cdot y_1 + 0 \cdot y_2 + 1 \cdot w$ s.t $\begin{pmatrix} -3 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ w \end{pmatrix} \geq 0$, $y_1 + y_2 = 1$, $y_1, y_2 \geq 0$

Linear Programming

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Example. Sol $x_1, x_2 = (\frac{3}{7}, \frac{4}{7}), y_1, y_2 = (\frac{2}{7}, \frac{5}{7}), w = z = \frac{1}{7}$

Primal

They match, optimality!!

$$\max 0 \cdot x_1 + 0 \cdot \ldots$$

$$\text{s.t} \begin{pmatrix} -3 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix} \leq 0$$

$$x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

$$\ldots \cdot y_2 + 1 \cdot w$$

$$\text{s.t} \begin{pmatrix} -3 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ w \end{pmatrix} \geq 0$$

$$y_1 + y_2 = 1$$

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Linear Programming

Facts (polynomial time)**.**

1. Solving Linear program is in P.
2. First polynomial time algorithm was ellipsoid method (proof by Khachiyan)
3. Most efficient methods nowadays are interior point methods.
4. Simplex runs in exponential time in worst case. On average runs faster than the other methods!

Back to zero-sum Games

Question: What do we care about LP? Recall the example was from last week's lecture (zero-sum game)!

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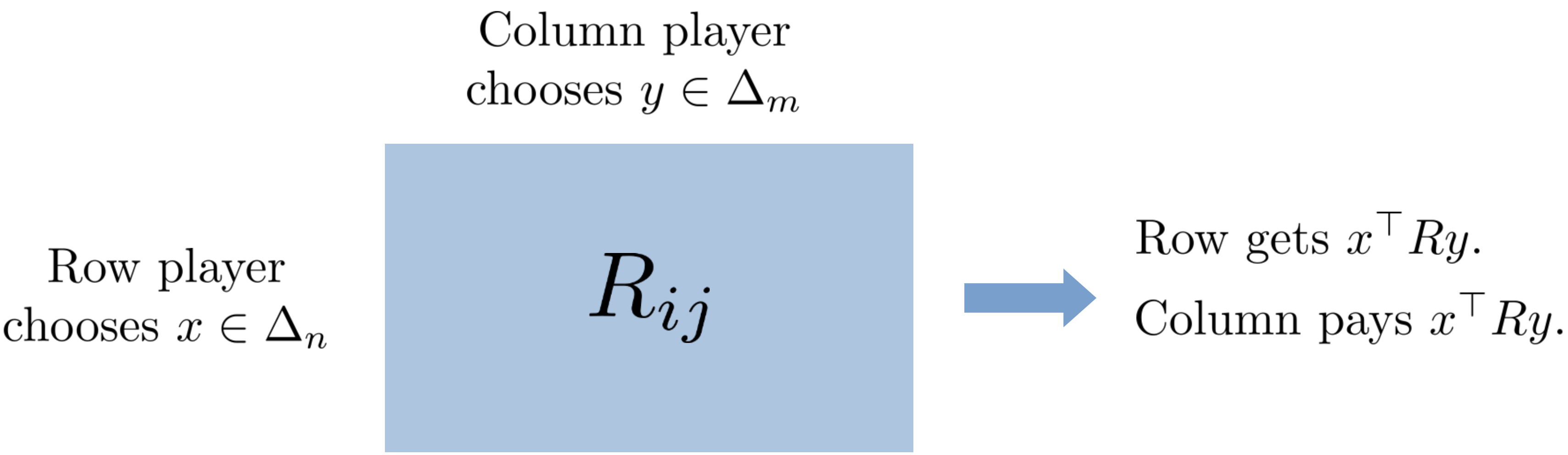
Answer: We can formulate the problem of computing Nash in zero-sum using LP!



Zero-sum Games as LPs

Assume player x plays first and wants to get at least z. For all pure strategies of y, x should get at least z. Formally:

$$x^\top R \geq z \cdot \mathbf{1}^\top$$

Zero-sum Games as LPs

Assume player x plays first and wants to get at least z. For all pure strategies of y, x should get at least z. Formally:

$$x^\top R \geq z \cdot \mathbf{1}^\top$$
$$\text{or } -x^\top R + z \cdot \mathbf{1}^\top \leq 0$$

Moreover, x should be a randomized strategy. Formally:

$$x^\top \mathbf{1} = 1$$
$$x \geq \mathbf{0}$$

Zero-sum Games as LPs

LP for player x:

$$\max z$$
$$x^\top R \geq z \cdot \mathbf{1}^\top$$
$$x^\top \mathbf{1} = 1$$
$$x \geq \mathbf{0}$$

Remark: Notice that the maximum above is the same as

$$\max_{x \in \Delta_n} \min_{y \in \Delta_m} x^\top R y$$

Zero-sum Games as LPs

Consider the dual of the previous LP:

$$\min z'$$

$$-y^\top R^\top + z' \cdot \mathbf{1}^\top \geq 0$$

$$y^\top \mathbf{1} = 1$$

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Set $z'' = -z'$ the above becomes

$$-\max z''$$
$$y^\top \cdot (-R)^\top \geq z'' \cdot \mathbf{1}^\top$$
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Zero-sum Games as LPs

Consider the dual of the previous LP:

$$\min z'$$

$$-y^\top R^\top + z' \cdot \mathbf{1}^\top \geq 0$$

This is the LP as if y player would play first with sign flipped!

Set $z'' = -z'$ the above becomes

$$-\max z''$$

$$y^\top \cdot (-R)^\top \geq z'' \cdot \mathbf{1}^\top$$

$$y^\top \mathbf{1} = 1$$

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Nash equilibrium and LP

$LP1$

$$\max z$$
$$x^\top R \geq z \cdot \mathbf{1}^\top$$
$$x^\top \mathbf{1} = 1$$
$$x \geq \mathbf{0}$$

$LP2$

$$\max z''$$
$$y^\top (-R)^\top \geq z'' \cdot \mathbf{1}^\top$$
$$y^\top \mathbf{1} = 1$$
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Theorem. *Let (x^*, z^*) be optimal for LP1, and (y^*, z''^*) be optimal for LP2, then (x^*, y^*) is a Nash equilibrium of the zero sum game with payoff matrix R. The payoff of the row player is z and of the column player is $z'' = -z$.*

Nash equilibrium and LP

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Proof.

Since (x^*, z) is feasible we have $x^{*\top} R y^* \geq z$.

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Since (y^*, z'') is feasible we have $-y^{*\top} R^\top x^* \geq z''$.

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Finally from strong duality we have $z'' = -z$!

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$$x^{*\top} R y^* = z!$$

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Proof.

No matter what y does if x chooses x^*, y pays at least z.
No matter what x does if y chooses y^*, x gets at most z.
Thus it is a Nash!

Nash equilibrium and LP

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Theorem. *Let (x^*, y^*) be a Nash equilibrium and set $z^* = x^{*\top} R y^*$. (x^*, z^*) is optimal solution for LP1, and $(y^*, -z^*)$ is optimal solution for LP2.*

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Theorem. *Let (x^*, y^*) be a Nash equilibrium and set $z^* = {x^*}^\top R y^*$. (x^*, z^*) is optimal solution for LP1, and $(y^*, -z^*)$ is optimal solution for LP2.*

Proof. Homework!

Corollaries

Theorem (Von Neuman minimax Theorem). *It holds that*

$$\max_{x \in \Delta_n} \min_{y \in \Delta_m} x^\top R y = \min_{y \in \Delta_m} \max_{x \in \Delta_n} x^\top R y$$

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Theorem (Convexity of Nash Equilibria). *The set of Nash equilibria in a zero-sum game is convex.*