L11 Introduction to Mechanism Design

CS 295 Introduction to Algorithmic Game Theory
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Inspired by
by J. Hartline and T. Roughgarden notes

Warm-up: Single-item allocation

Definition (Single-item). *The single-item allocation problem is given by*

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- *n agents competing for the item,*
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- Select the agent i^* with highest report ($i^* = \text{argmax}_i\, b_i$).
- Allocate item to i^*.

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If your $v_i = 1\$$, how would you play?
You should always bid the highest number you can think of!
Outcome of mechanism is unpredictable, hard to reason about performance

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Theorem (lottery n-approximation). *The lottery mechanism has n-approximation ratio (i.e., $n = \frac{OPT}{alg}$).*

Proof. Consider $v_1 = 1$ and $v_i = \epsilon$ for $i \geq 2$. Expected surplus is

$$\frac{1}{n}\left(1 + (n-1)\epsilon\right).$$

Thus approximation ratio is $\frac{n}{1+(n-1)\epsilon} \to n$ as $\epsilon \to 0$.

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Charge payments, proportionally to the agents' bids => **Discourage low-valued** agents from making high bids.

Single-item auctions

Approach 2 (First-price auction)**.** *The first-price auction is defined:*

- *Agents report their bids* b_i.
- *Select agent* $i^* = \arg\max_i b_i$ *(highest bid).*
- i^* *gets the item and pays the amount of* b_{i^*}.

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First-price auctions are **hard to reason** about.
As a participant, it's hard to figure out **how to bid.**
As an auction designer, it's **hard to predict** what will happen.

Single-item auctions

Approach 3 (Second-price auction). *The second-price or Vickrey auction is defined:*

- *Agents report their bids* b_i.
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Theorem (Vickrey is truthful). *In second price auctions, every bidder i has a dominant strategy: Set her bid b_i equal to her private v_i (report truthfully). Dominant means the utility of bidder i is maximized no matter what other bidders do.*

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Remark: Utility of bidder i is $u_i := v_i - p_i$ **if he gets the item** and $u_i := 0$ **otherwise.**

Second price auctions

Proof. Fix an agent i and set $B = \max_{j \neq i} b_j$.

Consider the cases:

- If $b_i < B$ then i gets utility 0.
- If $b_i \geq B$ then i wins the item and $u_i = v_i - B$.

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Assume that $v_i < B$.

- If i reports truthfully, she gets utility 0 (did not win the item).
- Assume not and $b_i < v_i$ then the utility of i will still be 0.
- Assume not and $B > b_i > v_i$ then the utility will still be 0.
- Assume not and $b_i \geq B > v_i$ then the utility will be negative.

Second price auctions

Proof cont. Fix an agent i and set $B = \max_{j \neq i} b_j$.

Assume that $v_i \geq B$.

- If i reports truthfully, she gets utility $v_i - B \geq 0$ (won the item).
- Assume not and $b_i > u_i$ then the utility of i will still be $v_i - B \geq 0$.
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No matter what other bidders do, truthtelling is best strategy (**Dominant strategy**).

A general approach

An auction should satisfy following properties:

- Dominant strategy incentive compatible (DSIC), i.e., truthtelling is dominant strategy.

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- (Auctioneer incentives) If bidders are truthful, auction miximizes surplus

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  where x_i is 1 if i wins the item and 0 otherwise.
- (Computational Efficiency) The auction can be implemented in polynomial time.

An Example

Problem: Consider a society of n citizens and public good G.

- Each agent has (private) valuation v_i for the good.
- Cost of building G is (publicly known) C.

- G should be built if $\sum_{i=1}^{n} v_i > C$.

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Answer: For citizen i, if $v_i > \frac{C}{n}$, i should report $C + \epsilon$ so G will be built!

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Answer: **No DSIC!** $+ \epsilon$ so G will be built!

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Solution: Charge citizen i the amount $p_i := \max(0, C - \sum_{j \neq i} v_i)$.
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Definition (Quasi-linear environments). *Also known as Vickrey-Groves-Clark (VCG) environments:*

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- *Set of outcomes (finite) $\mathcal{X}$,*
- *Each agent i has a valuation $v_i : \mathcal{X} \to \mathbb{R}^+$,*
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Remark: This framework is called mechanism design with money.

VCG mechanisms

Definition (VCG mechanism)**.** *The family of mechanisms is defined as follows:*

- *Agents have valuations v_i and report their bids b_i.*
- *Set $x^* = \arg\max_{x \in \mathcal{X}} \sum_{i=1}^{n} b_i(x)$.*
- *Each agent pays* $$p_i := \underbrace{h_i(b_{-i})}_{\text{without } i} - \underbrace{\sum_{j \neq i} b_i(x^*)}_{\text{with } i}$$
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Theorem (VCG is DSIC). *Every VCG mechanism is DSIC.*

Proof. Fix an agent i and let $x^* = \arg\max \sum_{i=1}^{n} v_i(x)$. Assume that i reports $b_i \neq v_i$ and x' be the maximum if i reports b_i.

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By definition of x^* $u_i \geq u'_i$

- *Each agent has utility* $u_i = v_i(x^*) - p_i(x^*)$.

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Proof. Fix an agent i and let $x^* = \arg\max \sum_{i=1}^{n} v_i(x)$. Assume that i reports $b_i \neq v_i$ and x' be the maximum if i reports b_i. Observe that
$u_i = v_i(x^*) + \sum_{j \neq i} v_i(x^*) - h_i(v_{-i})$ if i reports v_i and
$u'_i = v_i(x') + \sum_{j \neq i} v_i(x') - h_i(v_{-i})$ if i reports b_i.

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Remark: For single-item, this is the second-price auction!
VCG might not be efficiently computable (e.g., combinatorial auctions)