L16 Introduction to Markets

CS 295 Introduction to Algorithmic Game Theory
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Food Markets

Stock Markets

Matching Markets
Driven by a rule: Supply meets demand!

Food Markets

Stock Markets

Matching Markets
Definitions

Definition (Market). A market consists of:

- A set $\mathcal{B}$ of $n$ buyers/traders.
- A set $\mathcal{G}$ of $m$ goods.
- Each buyer $i$ has $e_i$ amount of $. W.l.o.g$ assume $e_i = 1$.
- $b_j$ denotes the amount of each good. W.l.o.g $b_j = 1$.
- $u_{ij}$ denotes the utility derived by $i$ on obtaining a unit amount of good of $j$.
- Each good $j$ is associated with a price $p_j$. 

Intro to AGT
Definitions

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- Each good $j$ is associated with a price $p_j$.

**Definition (Fisher Market).** A market so that the utilities are linear: Let $x_{ij}$ be the amount of units buyer $i$ gets of good $j$ then

$$u_i = \sum_{j \in \mathcal{G}} x_{ij} u_{ij}.$$
Definitions

Definition (Market clearance). A vector of price \((x^*, p^*)\) is called market equilibrium if for given prices \(p^*\), each buyer is assigned an optimal basket of goods relative the prices and buyer’s budget and there is no surplus or deficiency of any of the goods.

Goal: Compute allocations and prices in polynomial time!
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**Goal:** Compute allocations and prices in polynomial time!

Given an arbitrary vector of prices \(p \geq 0\), from each buyer’s \(i\) perspective:

\[
\max \sum_{j=1}^{m} x_{ij} u_{ij}
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Goal: Compute allocations and prices in polynomial time!

Given an arbitrary vector of prices \(p \geq 0\), from each buyer’s \(i\) perspective:

\[
\begin{align*}
\max_{x_i} & \sum_{j=1}^{m} x_{ij}u_{ij} \\
\text{subject to} & \sum_{j=1}^{m} p_jx_{ij} \leq 1 \\
& x_i \geq 0
\end{align*}
\]

Budget constraint.
Eisenberg-Gale Convex Program

Given an arbitrary vector of prices $p \geq 0$, from each buyer’s $i$ perspective:

From the perspective of good $j$:

$$\max \sum_{j=1}^{m} x_{ij} u_{ij}$$

subject to:

$$\sum_{j=1}^{m} p_{j} x_{ij} \leq 1$$
$$x_{i} \geq 0$$

Budget constraint.

Demand for good $j$.

$$\sum_{i=1}^{n} x_{ij} \leq 1$$
$$p_{j} \geq 0$$

Supply for good $j$. 

Intro to AGT
Eisenberg-Gale Convex Program

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From the perspective of good \( j \):

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\text{s.t } \sum_{j=1}^{m} p_j x_{ij} \leq 1
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x_i \geq 0
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Budget constraint.

Demand for good \( j \).

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\sum_{i=1}^{n} x_{ij} \leq 1
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Supply for good \( j \).

Can we find \((x, p)\) s.t all are satisfied simultaneously?
Eisenberg-Gale **Convex Program**

Consider the following **convex** program:

\[
\begin{align*}
\text{max } & \sum_{j=1}^{n} \ln u_i \\
\text{s.t } & u_i = \sum_{j=1}^{m} u_{ij} x_{ij} \text{ for all buyers } i \in \mathcal{B}, \\
& \sum_{i=1}^{n} x_{ij} \leq 1 \text{ for all goods } j \in \mathcal{G}, \\
& x_{ij} \geq 0 \text{ for all } i \in \mathcal{B}, j \in \mathcal{G}.
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**Remark:**

- The domain above is **compact** hence there is an optimal solution \( x^* \).
Eisenberg-Gale **Convex** Program

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\end{align*}$$

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- Maximizing a concave function is a convex program and can be solved in poly-time for affine constraints!
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\]

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- The domain above is **compact** hence there is an optimal solution \( x^* \).
- Note that there are no budget constraints!
- **Maximizing a concave** function is a convex program and can be solved in **poly-time** for **affine** constraints!

**Is \( x^* \) an equilibrium? What are the prices?**
Eisenberg-Gale **Convex** Program

$x^*$ satisfies the **KKT conditions**.

**KKT** are **first-order conditions for constrained Optimization**
\[ L(x, p) = \sum_{j=1}^{n} \ln u_i - \sum_{j=1}^{m} p_j \left( \sum_{i=1}^{n} x_{ij} - 1 \right) \]

**Remark:** Langrangian involves objective and constraints!
Eisenberg-Gale Convex Program

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Remark: Langrangian involves objective and constraints!

KKT conditions: $x$ are primal variables, $p$ are dual variables.

Primal feasibility: 
$x_{ij} \geq 0$ for all $i \in B, j \in G$.

Dual feasibility:
$p_j \geq 0$ for all $j \in G$. 

Intro to AGT
Eisenberg-Gale Convex Program

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$$L(x, p) = \sum_{j=1}^{n} \ln u_i - \sum_{j=1}^{m} p_j \left( \sum_{i=1}^{n} x_{ij} - 1 \right)$$

- **objective**
- **constraint for good j**

Remark: Langrangian involves objective and constraints!

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**Primal feasibility**: $x_{ij} \geq 0$ for all $i \in B$, $j \in G$.

**Dual feasibility**: $p_j \geq 0$ for all $j \in G$.

\[
\begin{aligned}
\frac{\partial L(x,p)}{\partial x_{ij}} &= \frac{u_{ij}}{u_i} - p_j = 0 \text{ if } x_{ij} > 0. \\
\frac{\partial L(x,p)}{\partial x_{ij}} &= \frac{u_{ij}}{u_i} - p_j \leq 0 \text{ if } x_{ij} = 0.
\end{aligned}
\]

Complementary Slackness

\[
\begin{aligned}
\frac{\partial L(x,p)}{\partial p_j} &= 1 - \sum_{i=1}^{n} x_{ij} = 0 \text{ if } p_j > 0. \\
\frac{\partial L(x,p)}{\partial p_j} &= 1 - \sum_{i=1}^{n} x_{ij} \geq 0 \text{ if } p_j = 0.
\end{aligned}
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Intro to AGT
Eisenberg-Gale Convex Program

Let \((x^*, p^*)\) satisfy the KKT conditions. Then \((x^*, p^*)\) solves

$$\min_{p \geq 0} \max_{x \geq 0} L(x, p) = \max_{x \geq 0} \min_{p \geq 0} L(x, p)$$

since it is convex – concave,

where \(L(x, p) = \sum_{j=1}^{n} \ln u_i - \sum_{j=1}^{m} p_j (\sum_{i=1}^{n} x_{ij} - 1)\).
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\textbf{Remark:} Observe that dual variables \(p\) \textit{penalize if a constraint is violated.}
Eisenberg-Gale Convex Program

Let \((x^*, p^*)\) satisfy the **KKT conditions**. Then \((x^*, p^*)\) solves

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**Remark:** Observe that dual variables \(p\) penalize if a constraint is violated.

**Theorem (Fisher Market).** For the linear case of Fisher Market and assuming that for each good \(j\), there exists a buyer \(i\) with \(u_{ij} > 0\) then:

- The set of equilibrium allocations is convex.
- Equilibrium utilities and prices are unique.
- If all \(u_{ij}\)'s are rational then allocations and prices are rational.
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Proof. Let \( x^* \) be an optimum of EG program and let \( p^* \) be the dual variables so that \( (x^*, p^*) \) satisfy the KKT constraints. We shall show that \( (x^*, p^*) \) is a market equilibrium.
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By assumption we have $p_j^* > 0$ for all $j \in \mathcal{G}$ (why?)
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By KKT we have there exists buyer $i$ so that $u_{ij} > 0$. We conclude from KKT $p_{j}^* \geq \sum_{j' = 1}^{m} \frac{u_{ij}}{u_{ij'}} x_{ij'}^* > 0$. 

Intro to AGT
Proof cont. Let $x^*$ be an optimum of EG program and let $p^*$ be the dual variables so that $(x^*, p^*)$ satisfy the KKT constraints. We shall show that $(x^*, p^*)$ is a market equilibrium.

1) We showed that $p_j^* > 0$ for all $j \in G$.

Positive prices $\implies$

By complementary slackness we have $\sum_{i=1}^{n} x_{ij}^* = 1$. 
Proof cont. Let $x^*$ be an optimum of EG program and let $p^*$ be the dual variables so that $(x^*, p^*)$ satisfy the KKT constraints. We shall show that $(x^*, p^*)$ is a market equilibrium.

1) We showed that $p^*_j > 0$ for all $j \in \mathcal{G}$. \hspace{4cm} \text{Positive prices}

2) We showed that $\sum_{i=1}^{n} x^*_{ij} = 1$ for all $j \in \mathcal{G}$. \hspace{4cm} \text{Goods sold out}
Eisenberg-Gale Convex Program

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2) We showed that \( \sum_{i=1}^{n} x^*_{ij} = 1 \) for all \( j \in \mathcal{G} \). \textbf{Goods sold out}

Using KKT conditions for fixed buyer \( i \) we also have for \( x^*_{ij} > 0 \)

\[
\frac{u_{ij}}{\sum_{j'=1}^{m} x^*_{ij'} u_{ij'}} = p^*_j \Rightarrow \frac{u_{ij} x^*_{ij}}{\sum_{j'=1}^{m} x^*_{ij'} u_{ij'}} = x^*_i p^*_j
\]
Eisenberg-Gale Convex Program

Proof cont. Let $x^*$ be an optimum of EG program and let $p^*$ be the dual variables so that $(x^*, p^*)$ satisfy the KKT constraints. We shall show that $(x^*, p^*)$ is a market equilibrium.

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Using KKT conditions for fixed buyer $i$ we also have for $x^*_{ij} > 0$

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Summing over all goods $j \in G$ the above we have

$$1 = \frac{\sum_{j=1}^{m} u_{ij} x^*_{ij}}{\sum_{j'=1}^{m} x^*_{ij'} u_{ij'}} = \sum_{j=1}^{m} x^*_{ij} p^*_j$$
Eisenberg-Gale Convex Program

Proof cont. Let $x^*$ be an optimum of EG program and let $p^*$ be the dual variables so that $(x^*, p^*)$ satisfy the KKT constraints. We shall show that $(x^*, p^*)$ is a market equilibrium.

1) We showed that $p^*_j > 0$ for all $j \in G$.  
Positive prices

2) We showed that $\sum_{i=1}^{n} x^*_{ij} = 1$ for all $j \in G$.  
Goods sold out

3) We showed that $\sum_{j=1}^{m} x^*_{ij} p^*_j = 1$ for all $i \in B$.  
Buyers spent all their money
Eisenberg-Gale Convex Program

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Positive prices

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Goods sold out

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Hence $(x^*, p^*)$ is a market equilibrium. Since EG is a convex program, the set $x^*$ of optimal solutions to EG is a convex set.
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2) We showed that $\sum_{i=1}^{n} x^*_{i,j} = 1$ for all $j \in G$.  
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3) We showed that $\sum_{j=1}^{m} x^*_{i,j} p^*_j = 1$ for all $i \in B$.  
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Uniqueness of utilities is derived since $ln$ is a strictly concave function.
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Hence $(x^*, p^*)$ is a market equilibrium. Since EG is a convex program, the set $x^*$ of optimal solutions to EG is a convex set.

Uniqueness of utilities is derived since $\ln$ is a strictly concave function.

By doing the transformation $q_j = \frac{1}{p_j}$ the prices should satisfy a linear system (by KKT conditions) with rational coefficients.
Other utility functions

CES (Constant elasticity of substitution) utility functions:

\[ u_i(x) = \left( \sum_{j=1}^{m} u_{ij}x_{ij}^\rho \right)^{\frac{1}{\rho}}, \text{ for } -\infty < \rho \leq 1. \]

Remark:
• \( u_i(x) \) is concave function.
• If \( u_{ij} = 0 \), then the corresponding term in the utility function is always 0.
• If \( u_{ij} > 0, x_{ij} = 0 \), and \( \rho < 0 \) then \( u_i(x) = 0 \) no matter what the other \( x_{ij} \)'s are.
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\[
\rho = 1 \quad \text{ Linear utility form}
\]

\[
\rho \to -\infty \quad \text{ Leontief utility form}
\]

\[
\rho \to 0 \quad \text{ Cobb-Douglas form}
\]

**Elasticity of substitution** \(\sigma = \frac{1}{1-\rho}\).