L16 Introduction to Markets

CS 295 Introduction to Algorithmic Game Theory

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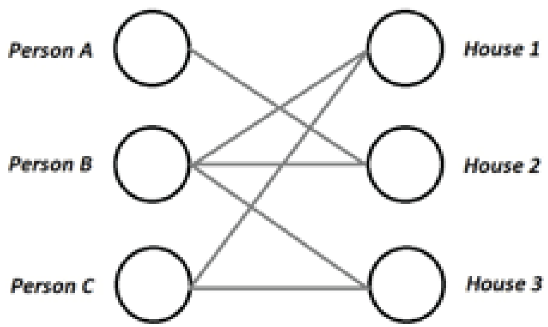
Food Markets

Stock Markets



Matching Markets

Driven by a rule: Supply meets demand!

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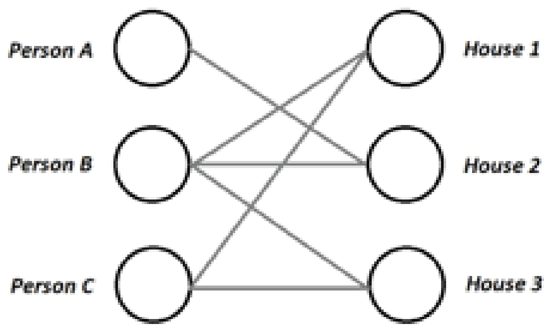

Matching Markets

Definitions

Definition (Market). *A market consists of:*

- *A set $\mathcal{B}$ of n buyers/traders.*
- *A set $\mathcal{G}$ of m goods.*
- *Each buyer i has e_i amount of \$. W.l.o.g assume $e_i = 1$.*
- *b_j denotes the amount of each good. W.l.o.g $b_j = 1$.*
- *u_{ij} denotes the utility derived by i on obtaining a unit amount of good of j.*
- *Each good j is associated with a price p_j.*

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Definition (Fisher Market). *A market so that the utilities are linear:*
Let x_{ij} *be the amount of units buyer* i *gets of good* j *then*

$$u_i = \sum_{j \in \mathcal{G}} x_{ij} u_{ij}.$$

Definitions

Definition (Market clearance). *A vector of price* (x^*, p^*) *is called* ***market equilibrium*** *if for given prices* p^*, *each buyer is assigned an optimal basket of goods relative the prices and buyer's badget and there is no surplus or deficiency of any of the goods*

Goal: Compute allocations and prices in polynomial time!

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$$\begin{aligned} \max \textstyle\sum_{j=1}^{m} x_{ij} u_{ij} & \\ \text{s.t} \textstyle\sum_{j=1}^{m} p_j x_{ij} \leq 1 & \\ x_i \geq 0 & \end{aligned}$$

Budget constraint.

Eisenberg-Gale Convex Program

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From the perspective of good j:

Demand for good j.

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Supply for good j.

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Can we find (x, p) s.t all are satisfied simultaneously?

Eisenberg-Gale **Convex** Program

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$$\max \sum_{j=1}^{n} \ln u_i$$

$$\text{s.t } u_i = \sum_{j=1}^{m} u_{ij} x_{ij} \text{ for all buyers } i \in \mathcal{B},$$

$$\sum_{i=1}^{n} x_{ij} \leq 1 \text{ for all goods } j \in \mathcal{G},$$

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- The domain above is compact hence there is an optimal solution x^*.

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Is x^* an equilibrium? What are the prices?

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$$L(x,p) = \underbrace{\sum_{j=1}^{n} \ln u_i}_{\text{objective}} - \sum_{j=1}^{m} \underbrace{p_j(\sum_{i=1}^{n} x_{ij} - 1)}_{\text{constraint for good } j}$$

Remark: Langrangian involves objective and constraints!

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KKT conditions: x are primal variables, p are dual variables.

Primal feasibility:

$x_{ij} \geq 0$ for all $i \in \mathcal{B},\ j \in \mathcal{G}$.

Dual feasibility:

$p_j \geq 0$ for all $j \in \mathcal{G}$.

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$$\left.\begin{aligned}
\frac{\partial L(x,p)}{\partial x_{ij}} &= \frac{u_{ij}}{u_i} - p_j = 0 \text{ if } x_{ij} > 0.\\
\frac{\partial L(x,p)}{\partial x_{ij}} &= \frac{u_{ij}}{u_i} - p_j \leq 0 \text{ if } x_{ij} = 0.\\
\frac{\partial L(x,p)}{\partial p_j} &= 1 - \sum_{i=1}^{n} x_{ij} = 0 \text{ if } p_j > 0.\\
\frac{\partial L(x,p)}{\partial p_j} &= 1 - \sum_{i=1}^{n} x_{ij} \geq 0 \text{ if } p_j = 0.
\end{aligned}\right\} \text{Complementary Slackness}$$

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Let (x^*, p^*) satisfy the KKT conditions. Then (x^*, p^*) solves

$$\boxed{\min_{p\geq 0}\max_{x\geq 0} L(x,p) = \max_{x\geq 0}\min_{p\geq 0} L(x,p)} \text{ since it is } \mathit{convex-concave},$$

where $L(x,p) = \sum_{j=1}^{n} \ln u_i - \sum_{j=1}^{m} p_j(\sum_{i=1}^{n} x_{ij} - 1)$.

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Remark: Observe that dual variables p penalize if a constraint is violated.

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Theorem (Fisher Market). *For the linear case of Fisher Market and assuming that for each good j, there exists a buyer i with $u_{ij} > 0$ then:*

- *The set of equilibrium allocations is convex.*
- *Equilibrium utilities and prices are unique.*
- *If all u_{ij}'s are rational then allocations and prices are rational.*

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By KKT we have there exists buyer i so that $u_{ij} > 0$. We conclude from KKT $p_j^* \geq \frac{u_{ij}}{\sum_{j'=1}^{m} u_{ij'} x_{ij'}^*} > 0$.

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Proof cont. Let x^* be an optimum of EG program and let p^* be the dual variables so that (x^*, p^*) satisfy the KKT constraints. We shall show that (x^*, p^*) is a market equilibrium.

1) We showed that $p_j^* > 0$ for all $j \in \mathcal{G}$.

Positive prices $\Longrightarrow$

By complementary slackness we have $\sum_{i=1}^{n} x_{ij}^* = 1$.

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Goods sold out

Using KKT conditions for fixed buyer i we also have for $x_{ij}^* > 0$

$$\frac{u_{ij}}{\sum_{j'=1}^m x_{ij'}^* u_{ij'}} = p_j^* \Rightarrow \frac{u_{ij} x_{ij}^*}{\sum_{j'=1}^m x_{ij'}^* u_{ij'}} = x_{ij}^* p_j^*$$

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Summing over all goods $j \in \mathcal{G}$ the above we have

$$1 = \frac{\sum_{j=1}^{m} u_{ij} x_{ij}^*}{\sum_{j'=1}^{m} x_{ij'}^* u_{ij'}} = \sum_{j=1}^{m} x_{ij}^* p_j^*$$

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3) We showed that $\sum_{j=1}^m x_{ij}^* p_j^* = 1$ for all $i \in \mathcal{B}$. Buyers spent all their money

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By doing the transformation $q_j = \frac{1}{p_j}$ the prices should satisfy a linear system (by KKT conditions) with rational coefficients.

Other utility functions

CES (Constant elasticity of substitution) utility functions:

$$u_i(x) = \left(\sum_{j=1}^{m} u_{ij} x_{ij}^{\rho} \right)^{\frac{1}{\rho}}, \text{ for } -\infty < \rho \leq 1.$$

Remark:

- $u_i(x)$ is concave function.
- If $u_{ij} = 0$, then the corresponding term in the utility function is always 0.
- If $u_{ij} > 0$, $x_{ij} = 0$, and $\rho < 0$ then $u_i(x) = 0$ no matter what the other x_{ij}'s are.

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$\rho = 1$ $\longrightarrow$ Linear utility form

$\rho \to -\infty$ $\longrightarrow$ Leontief utility form

$\rho \to 0$ $\longrightarrow$ Cobb-Douglas form

Elasticity of substitution $\sigma = \frac{1}{1-\rho}$.