CS295 Introduction to Algorithmic Game Theory

Instructor: Ioannis Panageas Scribed by: Will Overman, Gilbert Indrajaya

Lecture 12. Monotone Allocations and Myerson's Lemma.

1 Introduction

When designing an auction, there are three desirable properties that the designer would like to satisfy.

1. DSIC (Dominant-Strategy Invenctive Compatibility). No matter what other agents do, the dominant strategy of each agent should be to play truthfully with respect to their valuation.
2. Social surplus maximization. The allocation should maximize the sum $\sum_{i=1}^{n} x_i v_i$.
3. The auction should be implementable in polynomial time.

Example 1.1 *Sponsored Search Auctions*

In this auction there is a search engine which is essentially our auctioneer. When a user arrives and inputs a query, an auction is conducted to decide which of the advertiser's links will be shown and in which order in the search results. Also a corresponding price is determined for each advertiser. Specifically,

- There are k slots
- The bidders are the advertisers
- Each slot j has a click through rate a_j, such that $a_1 \geq a_2 \geq ... \geq a_k$
- Each bidder i has a private valuation v_i and gets value $a_j v_i$ if they are assigned slot j

We will return to this example at the end and show how we can determine an auction satisfying our three above conditions of interest.

Definition 1.1 Single parameter environments. *A single parameter environment is defined by the following conditions*

- *There are n bidders with private v_i*
- *There is a feasible set $\mathcal{X}$, each element of which is a n-dimensional vector $(x_1, ..., x_n)$ in which x_i is the amount of stuff given to bidder i*

Example 1.2 *Examples of single parameter environments*

1. Single-item auctions: is 0-1 vectors with at most one 1, i.e., $\sum x_i \leq 1$
2. k identical goods, each bidder gets at most one: $\mathcal{X}$ is 0-1 vectors with $\sum x_i \leq k$
3. In sponsored search, $\mathcal{X}$ is the set of n-vectors with x_i being a_j if slot j is assigned to bidder i.

Definition 1.2 Allocations and Payments. *A sealed-bid auction is defined by the following conditions*

- *Bidders report bids* $b = (b_1, ..., b_n)$
- *Auctioneer chooses feasible allocation* $x(b) \in \mathcal{X}$
- *Auctioneer chooses payments* $p(b) \in \mathbb{R}^n$
- *Bidder* i *gets utility* $u_i = v_i \cdot x_i(b) - p_i(b)$

Definition 1.3 Monotone Allocations. *An allocation rule* x *for a single-parameter environment is monotone if for every bidder* i *and bids* b_{-i} *by the rest of bidders, the allocation* $x_i(z, b_{-i})$ *is nondecreasing in* z.

2 Myerson's Lemma

Theorem 2.1 *Myerson's Lemma*

Let (x, p) be a mechanism, which is the allocation and payments. We assume that $p_i(b) = 0$ whenever $b_i = 0$ for all bidders i.

1. It holds that if (x, p) is DSIC mechanism then x is monotone.
2. If x is a monotone allocation, then there is a unique payment rule such that (x, p) is DISC. Essentially if given the allocation, there is a unique payment rule, depending on the allocation, so that the pair is DSIC.

Remark 2.1 *This lemma characterizes all single-parameter environments.*

Proof: We start by proving point 1. We assume that we are given a mechanism (x, p) that is DSIC. Let $0 \leq y \leq z$.

If bidder i has private valuation z, to avoid reporting y, DSIC requries the condition

$$u_i(z) \geq u_i(y) \text{ for all } i$$
$$z \cdot x_i(z) - p_i(z) \geq z \cdot x_i(y) - p_i(y) \text{ for all } i$$

In particular, the utility of agent i at z has to be larger than the utility of player i at y. The utility of playing the truth has to be at least the utility of not playing the truth.

Now , if bidder i has private valuation y, to avoid reporting z, DSIC demands

$$y \cdot x_i(y) - p_i(y) \geq y \cdot x_i(z) - p_i(z) \text{ for all } i$$

We can combine the above two inequalities, moving the p_i to same side of both inequalities, to obtain

$$z(x_i(y) - x_i(z)) \leq p(y) - p(z) \leq y(x_i(y) - x_i(z)) \tag{1}$$

Since $y \leq z$, the only way this inequality can hold is if the left hand side is nonpositive, which requires the inequality $x_i(y) \leq x_i(z)$, which implies monotonicity.

Now we prove point 2 of Myerson's Lemma. We now know that x is monotone. Assume x is piecewise constant. If there is a jump at point z, say of magnitude h, then as $y \to z$ from the left we get

$$z \cdot h \leq p(y) - p(z) \leq y \cdot h$$

Hence there exists a jump in p so that

$$\text{jump in } p \text{ at } z = z \cdot \text{ jump in } x_i \text{ at } z$$

So we know the payments are also piece wise constant and in particular, using our assumption $p_i(0) = 0$ the payments are given by the formula

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot \text{ jump in } x_i(\cdot, b_{-i}) \text{ at } z_j \tag{2}$$

where $z_1, ..., z_l$ are the breakpoints of $x_i(\cdot, b_{-i})$ in $[0, b_i]$.

Similarly we can assume x is monotone and suppose that x is differentiable. Divide both sides of (1) by $y - z$ and let $y \to z$, giving us

$$p_i'(z) = z \cdot x_i'(z) \tag{3}$$

$$p_i(b_i, b_{-i}) = \int_0^{b_i} z \cdot \frac{dx_i(z, b_{-i})}{dz} dz \tag{4}$$

Now we need to show that the resulting payment structure indeed gives a DISC mechanism (x, p). We show this in proof by picture, see Figure 1. Essentially we can compare what happens when we truthfully bid, when we overbid, and when we underbid, comparing the product utility we derive subtracted by the money we spend and hence the resulting utility we obtain. Truthfully bidding clearly gives the highest reuslting utility as can be see in the figure.

Thus the allocation x along with the payments p either given formula (2) or (3), corresponding to the piecewise constant or differentiable assumptions, respectively, give a DISC mechanism (x, p).

■

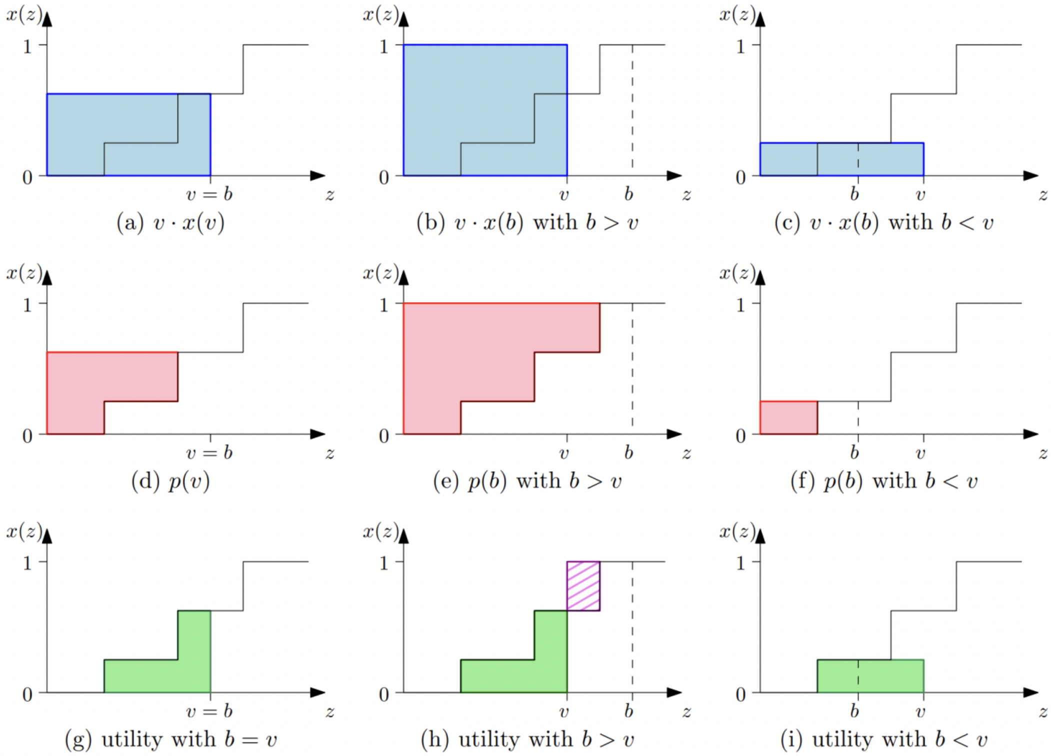

Figure 1: Allocation is in the first row, price is in the second, and resulting utility is in the third. The first column corresponds to the player truthfully bidding, while second is overbidding, and third is underbidding. The highest utility results from truthfully bidding.

Remark 2.2 *Myserson's Lemma regenerates the Vickrey auction as a special case. To see this, fix* i, b_{-i} *and set* $B = \max_{j \neq i} b_j$. *Then* $x_i(z, b_{-i})$ *is 0 for* $0 \leq z < B$ *and 1 for* $z \geq B$. *Moreover,* $p_i(z, b_i) = B$ *for* $z \geq B$ *and 0 for* $0 \leq z < B$.

Now we consider solving the sponsored search auction problen of Example 1.1.

1. Assume, without justification, that bidders bid truthfully. How should we assign bidders slots so that we can maximize surplus?

2. Given our answer to 1, how should we set selling prices so that DSIC holds?

The approach is to assign the t-th highest bidder to the jth highest slot for $j = 1, ..., k$. Note that this can clearly be done in polynomial time with sorting. Moreover, the allocation is monotone since the higher you bid the better the slot you are given. So by Myerson's Lemma we know that there are payments that make this mechanism DSIC as long as we determine how to set the prices.

Consider $b_1 \geq ... \geq b_n$. Focus on the first bidder (fixing the others), and assume that the bid ranges from 0 to b_i. Thus the allocation $x_1(z, b_{-1})$ ranges from 0 to a_1 with a jump as b_{j+1} of $a_j - a_{j+1}$ (when bidder 1 becomes j-th highest effectively). Hence for the i-th highest bidder, we get the payment

$$p_i(b) = \sum_{j=1}^{k} b_{j+1}(a_j - a_{j+1})$$

3 Knapsack Approximation

Some types of auctions fail the third desirable property of auctions. One such auction is a knapsack auction.

Definition 3.1 Knapsack auction. *A knapsack auction is defined by the following conditions:*

- *Each bidder i has a publicly known size w_i and a private valuation v_i.*
- *The seller has capacity W.*
- *Feasibility set X is all zero-one vectors $(x_1, x_2, ..., x_n)$ so that $\sum x_i w_i \leq W$.*

Remark 3.2 *Note that k-identical item auctions (e.g. sponsored search auctions) are a special case of knapsack auctions if all bidders have the same size $w_1 = w_2 = ... = w_n = 1$.*

As before can use the payment rule from Myerson's Lemma above to set the payments such that DSIC holds, given an allocation of bidders. However, if we assume bidders bid truthfully, to find the allocation of bidders that maximize surplus we would need to solve

$$\begin{aligned} \max_x \quad & \sum_{i=1}^{n} x_i b_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i w_i \leq W \\ & \forall_i \, x_i \in \{0, 1\} \end{aligned}$$

which is integer programming, an NP-complete problem. To make the problem tractable, we would relax the problem by computing the approximate surplus instead; this is done greedily by removing all $w_i > W$ for all i, then sorting and re-indexing bidders such that $\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq ... \geq \frac{b_n}{w_n}$. We then choose as many bidders as possible (say S) so that $\sum_{i=1}^{S} w_i \leq W$ and $\sum_{i=1}^{S+1} w_i > W$, allocating to highest feasible bidder or first S bidders, whichever gives higher surplus.

Theorem 3.1 2-approximation *Assuming truthful bids, the surplus of the greedy allocation rule is at least 50% of the maximum possible surplus.*

Proof: Let S be the number of agents so that

$$\sum_{i=1}^{S} w_i \leq W \text{ and } \sum_{i=1}^{S+1} w_i > W$$

It holds that

$$\sum_{i=1}^{S+1} v_i \geq \text{OPT}$$

Hence,

$$\max\left(\sum_{i=1}^{S} v_i, v_{S+1}\right) \geq \frac{1}{2}\text{OPT}$$

■

4 Bayesian Setting

Definition 4.1 Bayesian Single Parameter Setting. *Bayesian setting single parameter environment is defined as:*

- *n bidders with private valuation v_i*
- *Feasible set X, each element of which is an n-dimensional vector $(x_1, x_2, ..., x_n)$ in which x_i is the amount of stuff given to i.*
- *v_i is assumed to be drawn from a distribution F_i with density f_i and support $[0, v_{\max}]$*

Suppose we have one item with post price r and one person. The revenue would then be $r \cdot (1 - F(r))$ where $1 - F(r)$ is the probability for the person to have valuation higher than r. A reserve price of r means that the bidder needs to bid at least r. If F is uniform in $[0, 1]$, then the r that maximizes revenue is

$$\max_{r \in [0,1]} r - r^2 \rightarrow r = \frac{1}{2}, \text{rev} = \frac{1}{4}$$

Definition 4.2 Payments. *Assume bidders are truthful, i.e. $b = v$. By Myerson's lemma,*

$$p_i(v_i, v_{-i} = \int_0^{v_i} z \cdot \frac{dx_i(z, v_{-i})}{dz} dz$$

Since valuations are random variables in this case, we care about the expectation:

$$E_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} p_i(v_i, v_{-i}) f(v_i) dv$$

Thus from the first equation we have

$$E_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \int_0^{v_i} z \cdot x_i'(z, v_{-i}) dz f(v_i) dv$$

Reversing the integration,

$$E_{v_i\sim F_i}[p_i(v_i,v_{-i})] = \int_0^{v_{\max}}\int_z^{v_{\max}} f(v_i)dv\, z\cdot x_i'(z,v_{-i})dz$$

$$E_{v_i\sim F_i}[p_i(v_i,v_{-i})] = \int_0^{v_{\max}}(1-F_i(z))z\cdot x_i'(z,v_{-i})dz$$

$$= -\int_0^{v_{\max}} x_i'(z,v_{-i})\frac{(1-F_i(z)-zf_i(z)}{f_i(z)}f_i(z)dz$$

Let $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ be called *virtual valuations.* We would then get

$$E_{v_i\sim F_i}[p_i(v_i,v_{-i})] = \int_0^{v_{\max}} x_i'(z,v_{-i})\phi(z)f_i(z)\,dz$$

$$E_{v_i\sim F_i}[p_i(v_i,v_{-i})] = E_{v_i\sim F_i}[\phi(v_i)x_i(v_i,v_{-i})]$$

$$\text{Revenue} = E_{v\sim F_1,\ldots,F_n}\left[\sum_i p_i(v)\right] = E_{v\sim F_1,\ldots,F_n}\left[\sum_i x_i(v)\phi_i(v)\right]$$

As in the previous auction models, we return to our original two questions of surplus-maximizing allocation and DSIC-compliant payment designs. In the Bayesian setting we would maximize the virtual social welfare, namely $\sum x_i(v)\phi_i(v)$. A distribution F_i determines whether the allocation is *monotone*, where a higher valuation v_i gives higher x_i.

Definition 4.3 Regular F. *A distribution F is regular if the corresponding virtual value function $v - \frac{1-F(v)}{f(v)}$ is strictly increasing.*

Example 4.1 Uniform distribution is regular. *Let F be uniform in $[0,1]$. The valuation function would be $2v-1$, which is strictly increasing. Therefore uniform distributions are regular.*

Consider a single item with n bidders and regular F. We would then give the item to the bidder with the highest positive virtual valuation; since the virtual valuation function is strictly increasing, that same person must be the highest bidder. The winner pays $\phi_i(v_i)$. Note that this auction mimics a Vickrey auction with reserve price $\phi^{-1}(0)$.