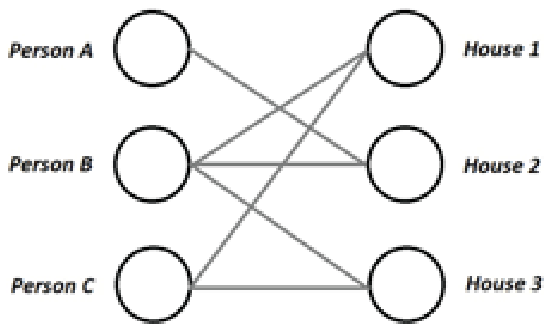
L15 Introduction to Markets

CS 280 Algorithmic Game Theory

Ioannis Panageas

Food Markets

Stock Markets



Matching Markets

Driven by a rule: Supply meets demand!

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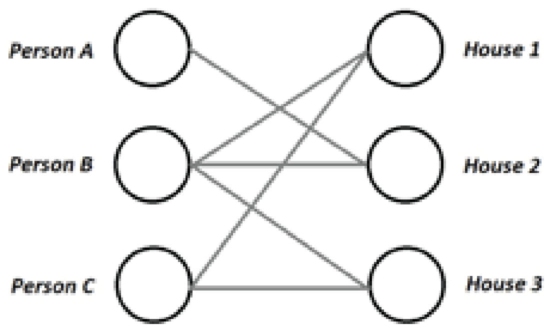

Matching Markets

Definitions

Definition (Market). *A market consists of:*

- *A set* $\mathcal{B}$ *of* n *buyers/traders.*
- *A set* $\mathcal{G}$ *of* m *goods.*
- *Each buyer* i *has* e_i *amount of \$. W.l.o.g assume* $e_i = 1$.
- b_j *denotes the amount of each good. W.l.o.g* $b_j = 1$.
- u_{ij} *denotes the utility derived by* i *on obtaining a unit amount of good of* j.
- *Each good* j *is associated with a price* p_j.

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Definition (Fisher Market). *A market so that the utilities are linear: Let x_{ij} be the amount of units buyer i gets of good j then*

$$u_i = \sum_{j \in \mathcal{G}} x_{ij} u_{ij}.$$

Definitions

Definition (Market clearance). *A vector of price* (x^*, p^*) *is called* ***market equilibrium*** *if for given prices* p^*, *each buyer is assigned an optimal basket of goods relative the prices and buyer's budget and there is no surplus or deficiency of any of the goods*

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$$\text{s.t} \sum_{j=1}^{m} p_j x_{ij} \leq 1$$
$$x_i \geq 0$$

→ Budget constraint.

Eisenberg-Gale Convex Program

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Demand for good j.

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Can we find (x, p) s.t all are satisfied simultaneously?

Eisenberg-Gale **Convex** Program

Consider the following **convex** program:

$$\max \sum_{j=1}^{n} \ln u_i$$

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- The domain above is compact hence there is an optimal solution x^*.

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Is x^* an **equilibrium**? What are the **prices**?

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$$L(x,p) = \underbrace{\sum_{j=1}^{n} \ln u_i}_{\text{objective}} - \textstyle\sum_{j=1}^{m} \underbrace{p_j(\sum_{i=1}^{n} x_{ij} - 1)}_{\text{constraint for good } j}$$

Remark: Langrangian involves objective and constraints!

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KKT conditions: x are primal variables, p are dual variables.

Primal feasibility:

$x_{ij} \geq 0$ for all $i \in \mathcal{B}$, $j \in \mathcal{G}$.

Dual feasibility:

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$$\left.\begin{array}{l}
\frac{\partial L(x,p)}{\partial x_{ij}} = \frac{u_{ij}}{u_i} - p_j = 0 \text{ if } x_{ij} > 0. \\
\frac{\partial L(x,p)}{\partial x_{ij}} = \frac{u_{ij}}{u_i} - p_j \leq 0 \text{ if } x_{ij} = 0. \\
\frac{\partial L(x,p)}{\partial p_j} = 1 - \sum_{i=1}^{n} x_{ij} = 0 \text{ if } p_j > 0. \\
\frac{\partial L(x,p)}{\partial p_j} = 1 - \sum_{i=1}^{n} x_{ij} \geq 0 \text{ if } p_j = 0.
\end{array}\right\} \text{Complementary Slackness}$$

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Let (x^*, p^*) satisfy the KKT conditions. Then (x^*, p^*) solves

$$\min_{p \geq 0} \max_{x \geq 0} L(x, p) = \max_{x \geq 0} \min_{p \geq 0} L(x, p)$$ since it is *convex − concave*,

where $L(x, p) = \sum_{j=1}^{n} \ln u_i - \sum_{j=1}^{m} p_j (\sum_{i=1}^{n} x_{ij} - 1)$.

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Theorem (Fisher Market). *For the linear case of Fisher Market and assuming that for each good j, there exists a buyer i with $u_{ij} > 0$ then:*

- *The set of equilibrium allocations is convex.*
- *Equilibrium utilities and prices are unique.*
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By KKT we have there exists buyer i so that $u_{ij} > 0$. We conclude from KKT $p_j^* \geq \frac{u_{ij}}{\sum_{j'=1}^{m} u_{ij'} x_{ij'}^*} > 0$.

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1) We showed that $p_j^* > 0$ for all $j \in \mathcal{G}$.

Positive prices $\Longrightarrow$

By complementary slackness we have $\sum_{i=1}^{n} x_{ij}^* = 1$.

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Using KKT conditions for fixed buyer i we also have for $x_{ij}^* > 0$

$$\frac{u_{ij}}{\sum_{j'=1}^m x_{ij'}^* u_{ij'}} = p_j^* \Rightarrow \frac{u_{ij} x_{ij}^*}{\sum_{j'=1}^m x_{ij'}^* u_{ij'}} = x_{ij}^* p_j^*$$

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Summing over all goods $j \in \mathcal{G}$ the above we have

$$1 = \frac{\sum_{j=1}^m u_{ij} x_{ij}^*}{\sum_{j'=1}^m x_{ij'}^* u_{ij'}} = \sum_{j=1}^m x_{ij}^* p_j^*$$

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3) We showed that $\sum_{j=1}^{m} x_{ij}^* p_j^* = 1$ for all $i \in \mathcal{B}$. Buyers spent all their money

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By doing the transformation $q_j = \frac{1}{p_j}$ the prices should satisfy a linear system (by KKT conditions) with rational coefficients.

Other utility functions

CES (Constant elasticity of substitution) utility functions:

$$u_i(x) = \left(\sum_{j=1}^{m} u_{ij} x_{ij}^{\rho} \right)^{\frac{1}{\rho}}, \text{ for } -\infty < \rho \leq 1.$$

Remark:

- $u_i(x)$ is concave function.
- If $u_{ij} = 0$, then the corresponding term in the utility function is always 0.
- If $u_{ij} > 0$, $x_{ij} = 0$, and $\rho < 0$ then $u_i(x) = 0$ no matter what the other x_{ij}'s are.

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$\rho = 1$ ⟶ Linear utility form

$\rho \to -\infty$ ⟶ Leontief utility form

$\rho \to 0$ ⟶ Cobb-Douglas form

Proportional Response Dynamics

Market dynamics:

Each time step the buyers face the same market parameters, (goods, budget constraint, utility function) while the buyers make their bidding decisions according to the previous market actions

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Notation:

- $b_{ij}^{(t)}$ the bid of buyer i for good j at time t.
- $p_j^{(t)} = \sum_{i \in \mathcal{B}} b_{ij}^{(t)}$ price for good j.
- Allocation $x_{ij}^{(t)} = \frac{b_{ij}^{(t)}}{p_j^{(t)}}$.
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For each agent i and good j set

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Theorem (Convergence). *The proportional response dynamics converges to a market equilibrium in the Fisher market with linear utility functions.*
For linear functions, it converges to an ϵ-market equilibrin in $O\left(\frac{1}{\epsilon^2}\right)$ iterations.

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Remark:

- The convergence result holds for CES utilities with a different rate.
- Similar rate to Multiplicative Weights Method (not a coincidence).

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The potential function will be (show it is decreasing)

$$\Phi^{(t)} = \sum_{i \in \mathcal{B}} \mathrm{KL}(b^*_i || b^{(t)}_i).$$

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Remark:

- KL divergence $\mathrm{KL}(x||y) = \sum x_i \log \frac{x_i}{y_i}$ for distributions x, y.
- $\mathrm{KL}(x||y) \geq 0$, pseudo-distance, symmetry not satisfied.