

# L10 Other notions of equilibria

CS 280 Algorithmic Game Theory

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# Relaxing Nash equilibrium

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**Question:** Are there other equilibrium notions that are computationally tractable?

**Answer:** Correlated equilibria, i.e., relaxing the **product** distribution assumption.

# Example (Correlated eq.)

	Chicken-out	Dare
Chicken-out	<b>0, 0</b>	<b>-2, 1</b>
Dare	<b>1, -2</b>	<b>-10, -10</b>

Suppose agents are **recommended**  $(C, D)$ ,  $(D, C)$ ,  $(C, C)$  with probability  $\frac{1}{3}$  each.

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- If agent row is **recommended to choose  $C$** , then column is recommended to play  **$C$  or  $D$**  with equal probability. Expected payoff of row is  $\frac{1}{2} \cdot 0 + \frac{1}{2}(-2) = -1$  which is **greater than switching to  $D$**  (expected payoff is -4.5).

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Suppose agents are **recommended**  $(C, D)$ ,  $(D, C)$ ,  $(C, C)$  with probability  $\frac{1}{3}$  each.

- If agent row is **recommended to choose C**, then column is recommended to play **C or D** with equal probability. Expected payoff of row is  $\frac{1}{2} \cdot 0 + \frac{1}{2}(-2) = -1$  which is **greater than switching to D** (expected payoff is -4.5).
- If agent row is **recommended to choose D**, then column is recommended to play **C**. Expected payoff of row is 1 which is **greater than switching to C** (expected payoff is 0).

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Suppose agents are **recommended**  $(C, D)$ ,  $(D, C)$ ,  $(C, C)$  with probability  $\frac{1}{3}$  each.

- If agent row is **recommended to choose C**, then column is recommended to play **C or D** with equal probability. Expected payoff of row is  $\frac{1}{2} \cdot 0 + \frac{1}{2}(-2) = -1$  while expected payoff of column is  $\frac{1}{2} \cdot 1 + \frac{1}{2}(-10) = -4.5$ .
- If agent row is **recommended to choose D**, then column is recommended to play **C**. Expected payoff of row is  $\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (-10) = -9.67$  while expected payoff of column is  $\frac{1}{3} \cdot (-2) + \frac{2}{3} \cdot (-10) = -7.33$ .

**Similarly for column player!**

**$(C, D)$ ,  $(D, C)$  and  $(C, C)$  with probability  $\frac{1}{3}$  each is a correlated eq.**

than switching to

# Definitions

**Definition (Recall).** *A game is specified by*

- *set of  $n$  players  $[n] = \{1, \dots, n\}$*
- *For each player  $i$  a set of strategies/actions  $S_i$ .*
- *set of strategy profiles  $S = S_1 \times \dots \times S_n$ .*
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**Definition (Correlated Equilibrium).** *Correlated equilibrium is a **distribution**  $\chi$  over  $S$  such that for all agents  $i$  and strategies  $b, b'$  of  $i$*

$$\mathbb{E}_{s \sim \chi}[u_i(b, s_{-i}) | s_i = b] \geq \mathbb{E}_{s \sim \chi}[u_i(b', s_{-i}) | s_i = b].$$

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*Similarly for all agents  $i$  and swapping functions  $f : S_i \rightarrow S_i$ ,*

$$\mathbb{E}_{s \sim \chi}[u_i(s_i, s_{-i})] \geq \mathbb{E}_{s \sim \chi}[u_i(f(s_i), s_{-i})].$$

# Correlated equilibrium and Nash

## Remarks:

- Knowing an agent her recommended action, she can **infer** something about other players' moves. Yet she is **better off playing** the **recommended** action.
- Suppose  $\chi$  is a **product distribution**. Then correlated eq. corresponds to Nash eq.







# Correlated equilibrium and Nash

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Set of Nash equilibria  $\subseteq$  Set of correlated equilibria.

# Example (Coarse Correlated eq.)

			
	0, 0	-1, 1	1, -1
	1, -1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

Suppose the actions  $(R, P)$ ,  $(R, S)$ ,  $(P, R)$ ,  $(P, S)$ ,  $(S, R)$ ,  $(S, P)$  are chosen with probability  $\frac{1}{6}$  each.







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- If agent row plays  $R$ , agent column responds with **either  $P$  or  $S$  with equal probability**. If column deviates (say starts responding with paper higher possibility) she will incur more loss when row plays  $S$  (recall row plays  $R$  as well  $S$  with equal probability).

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- If agent column is instructed to play  $P$  then she knows that **other player is playing either  $R$  or  $S$**  and column has **average payoff 0**. She can change then to  $R$  and **improve payoff to  $1/2$**  compared to zero if she plays recommended action. In this case, column could exploit knowledge of action recommendation to improve her payoff.

# Definitions

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Remark: The difference between coarse correlated and correlated is that we can choose a “smart” **swap function**, namely  $f$  “knows” the distribution  $\chi$ .



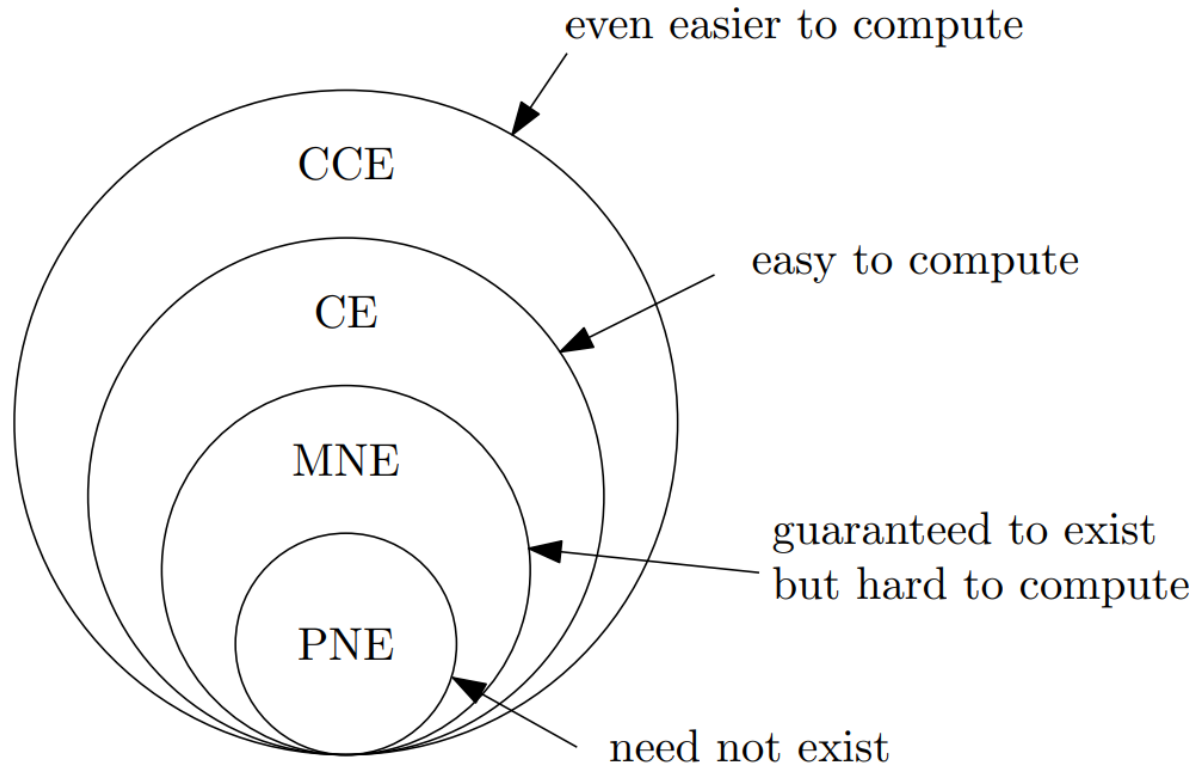
# Definitions

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Set of correlated equilibria  $\subseteq$  Set of coarse correlated equilibria.

# Full picture of Inclusions



# Online learning in Games

**Definition.** *At each time step  $t = 1 \dots T$ .*

- Each *player  $i$*  chooses  $x_i^{(t)} \in \Delta_i$  (simplex).
- Each *player* experiences payoff  $u_i(x^{(t)})$  and observes all players strategies  $x_j^{(t)}$ .

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Player's  $i$  goal is to minimize the (time average) **Regret**, that is:

$$\frac{1}{T} \left[ \max_{a \in S_i} \sum_{t=1}^T u_i(a, x_{-i}^{(t)}) - \sum_{t=1}^T u_i(x^{(t)}) \right].$$

If  $\text{Regret} \rightarrow 0$  as  $T \rightarrow \infty$ , the algorithm is called **no-regret**.

# A no-regret Algorithm

**Definition (Online Gradient Descent).** Let  $\ell_t : \mathcal{X} \rightarrow \mathbb{R}$  be family of convex functions, differentiable and  $L$ -Lipschitz in some compact convex set  $\mathcal{X}$  of diameter  $D$ . Online GD is defined:

Initialize at some  $x_0$ .

For  $t:=1$  to  $T$  do

1.  $y_t = x_t - \alpha_t \nabla \ell_t(x_t)$ .

2.  $x_{t+1} = \Pi_{\mathcal{X}}(y_t)$ .

Regret:  $\frac{1}{T} \left( \sum_{t=1}^T \ell_t(x_t) - \min_x \sum_{t=1}^T \ell_t(x) \right)$ .

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For  $t:=1$  to  $T$  do step-size

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$\ell_t = -u_i(x^{(t)})$

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# Analysis of Online GD for $L$ -Lipschitz

**Theorem (Online Gradient Descent).** Let  $\ell_t : \mathcal{X} \rightarrow \mathbb{R}$  be family of convex functions, differentiable and  $L$ -Lipschitz in some compact convex set  $\mathcal{X}$  of diameter  $D$ . It holds

$$\left( \frac{1}{T} \sum_{t=1}^T \ell_t(x_t) - \min_x \sum_{t=1}^T \ell_t(x) \right) \leq \frac{3LD}{2\sqrt{T}},$$

with appropriately choosing  $\alpha = \frac{D}{L\sqrt{t}}$ .

Remarks:

- If we want error  $\epsilon$ , we need  $T = \Theta\left(\frac{L^2 D^2}{\epsilon^2}\right)$  iterations.
- I could have written **Multiplicative Weights Update**. This is **another** no-regret

algorithm! Same regret guarantees, i.e.,  $O\left(\frac{1}{\sqrt{T}}\right)$ .

# Analysis of Online GD for $L$ -Lipschitz

*Proof.* Let  $x^*$  be the argmin of  $\sum \ell_t(x)$ .

$$\begin{aligned}\ell_t(x_t) - \ell_t(x^*) &\leq \nabla \ell_t(x_t)^\top (x_t - x^*) \text{ convexity,} \\ &= \frac{1}{\alpha_t} (x_t - y_t)^\top (x_t - x^*) \text{ definition of GD,}\end{aligned}$$



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# Analysis of Online GD for $L$ -Lipschitz

*Proof cont.* Since

$$\ell_t(x_t) - \ell_t(x^*) \leq \frac{1}{2\alpha_t} \left( \|x_t - x^*\|_2^2 - \|x_{t+1} - x^*\|_2^2 \right) + \frac{\alpha_t L^2}{2},$$

taking the telescopic sum we have

$$\begin{aligned} \sum_{t=1}^T (\ell_t(x_t) - \ell_t(x^*)) &\leq \sum_{t=1}^T \|x_t - x^*\|_2^2 \left( \frac{1}{2\alpha_t} - \frac{1}{2\alpha_{t-1}} \right) + \frac{L^2}{2} \sum_{t=1}^T \alpha_t. \\ &\leq \frac{D^2}{2} \sum_{t=1}^T \left( \frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}} \right) + \frac{L^2}{2} \sum_{t=1}^T \alpha_t. \end{aligned}$$

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where we used the fact  $\sum \frac{1}{\sqrt{t}} \leq 2\sqrt{T}$  and  $\alpha_t = \frac{D}{\sqrt{t}L}$ .

# Computing coarse correlated equilibria

Suppose that each agent  $i$  uses **no-regret dynamics (online GD)**, with  $l_t = -u_i(x^{(t)})$  where  $x^{(t)}$  is the mixed strategy profile at **iterate  $t$** .

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- Let  $\sigma^t$  be the **product distribution** on  $S$  induced by  $x^{(t)}$ .
- Let  $\sigma$  be the uniform distribution over  $\{\sigma^1, \dots, \sigma^T\}$ .

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We conclude that for each agent  $i$

$$\mathbb{E}_{s \sim \sigma} [u_i(s)] = \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{s \sim \sigma^t} [u_i(s)]$$
$$\min_{b \in S_i} \mathbb{E}_{s \sim \sigma} [u_i(b, s_{-i})] = \min_{b \in S_i} \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{s \sim \sigma^t} [u_i(b, s_{-i})]$$

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Suppose  
where  $s$

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We can

$$\min_{b \in S_i} \mathbb{E}_{s \sim \sigma} [u_i(b, s_{-i})] - \mathbb{E}_{s \sim \sigma} [u_i(s)] \leq \frac{3}{2} \frac{\sqrt{n}}{\sqrt{T}}$$

Lipschitz

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Choosing  $T = \frac{9n}{4\epsilon^2}$  we conclude  $\sigma$  is  $\epsilon$ -approximate CCE!

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If we use MWUA, it gives  $O\left(\frac{\ln n}{\epsilon^2}\right)$ .

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