# L06 Potential and Congestion Games

CS 280 Algorithmic Game Theory

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# Potential Games

**Definition** (Potential Games)**.** *A normal form game is specified by*

- *set of n players* $[n] = \{1, ..., n\}$
- *For each player i a set of strategies/actions* $S_i$ *and a utility* $u_i : \times_{j=1}^{n} S_j \to \mathbb{R}$ *denoting the payoff of i.*
- *set of strategy profiles* $S = S_1 \times ... \times S_n$.
- *There exists a potential function* $\Phi : S \to \mathbb{R}$ *so that for all agents i and* $s_i, s_i'$

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) = u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}).$$

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**Example** (Battle of sexes)**.**

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|---|---|
| **5, 2** | **-1,-2** |
| **-5, -4** | **1,4** |

$\Rightarrow$

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| **-6** | **2** |

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> **Weighted Potential Games:**
>
> $$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) = w_i \cdot \left(u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})\right),$$
>
> where $w_i > 0$.

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$$\Phi(s_i^*, s_{-i}^*) - \Phi(s_i', s_{-i}^*) = u_i(s_i^*, s_{-i}^*) - u_i(s_i', s_{-i}^*) < 0.$$

Contradiction!

# Potential Games

**Algorithm** (Greedy)**.**

1. Initialize $s^{(0)}$ arbitrarily.
2. **Loop**
3. **Find** agent $i, s'_i$ so that $u_i(s'_i, s^{(t)}_{-i}) > u_i(s^{(t)})$
4. **Set** $s^{(t+1)} = (s'_i, s^{(t+1)}_{-i})$.
5. $t = t + 1$
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- The graph has no cycles.
- The algorithm reaches a sink vertex (no outgoing edges).

# Congestion Games

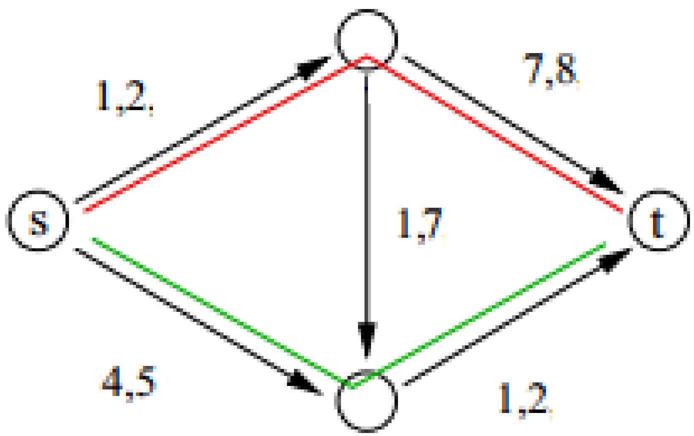
A congestion game is defined by:

- $n$ set of players.
- $E$ set of edges/facilities/ bins.
- $S_i \subset 2^E$ the set of strategies of player $i$.
- $c_e : \{1, ..., n\} \to \mathbb{R}^+$ cost function of edge $e$.

For any $s = (s_1, ..., s_n)$

- $l_e(s)$ number of players (load) that use edge $e$.
- $c_i(s) = \sum_{e \in s_i} c_e(l_e)$ the cost function of player $i$.

# Congestion Games




For this game:

$n = \{1, 2\}$ (red, green)
$E$ are the edges of the network.
$S_i$ is all $s - t$ paths.
$c_e$ on edges.

Remark: Defined by Rosenthal in 1973. Capture atomic routing **games!**

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Fix agent $i$ strategy $s, s'$, where $s, s'$ differ on agent's $i$ strategy.

- $\Phi(s) = \sum_{e\in s\cap s'} \sum_{j=1}^{l_e(s)} c_e(j) + \sum_{e\in s\setminus s'} \sum_{j=1}^{l_e(s)} c_e(j) + \sum_{e\in s'\setminus s} \sum_{j=1}^{l_e(s)} c_e(j) +$
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Missing terms

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$l_e(s) = l_e(s') + 1$

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**Theorem** (Rosenthal 73′). *Congestion Games are potential games.*

$$\Phi(s) - \Phi(s') = \sum_{e\in s\setminus s'} c_e(l_e(s)) - \sum_{e\in s'\setminus s} c_e(l_e(s'))$$

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Remark: Monderer and Shapley showed that potential games can be reduced to congestion games!

# An Algorithm for symmetric network congestion games

Assumption: All players have the same endpoints $S$ and $T$ (and thus they all have the same set of paths/strategies).

Basic idea: Min-cost flow reduction

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Basic idea: Min-cost flow reduction

**Definition (Min-cost flow).** *Given a graph $G(V, E)$, a source $s$ and a sink $t$ we would like to send flow $d$ from $s$ to $t$.*

- *Each edge $(u, v)$ has capacity $c(u, v)$ and cost per flow unit $a(u, v)$.*

$$\min \sum_{e:(u,v)} f(u,v) \cdot a(u,v)$$

$$\text{s.t } f(u,v) \leq c(u,v) \text{ for all edges } (u,v) \quad \text{capacity cosntraints}$$

$$f(u,v) = -f(v,u) \text{ for all edges } (u,v)$$

$$\sum_w f(u,w) = 0 \;\; \forall u \neq s,t \text{ flow conservation}$$

$$\sum_w f(s,w) = d \text{ and } \sum_w f(w,t) = d$$

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**Definitio**[...] *and a sink t we w*[...]

- *Each* [...] $a(u,v)$.

**Min-cost flow via LP!**

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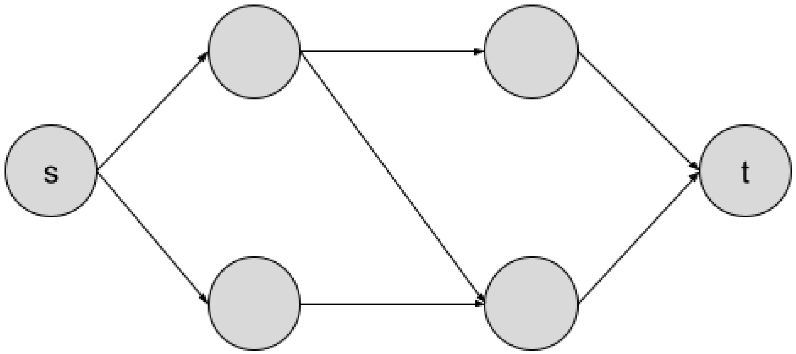
Initial graph in the Congestion Game.



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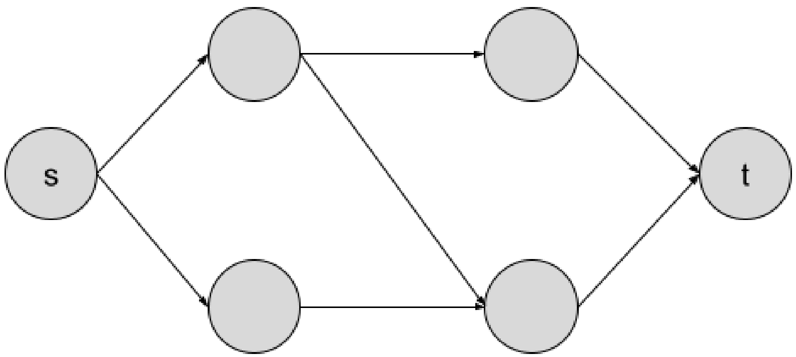
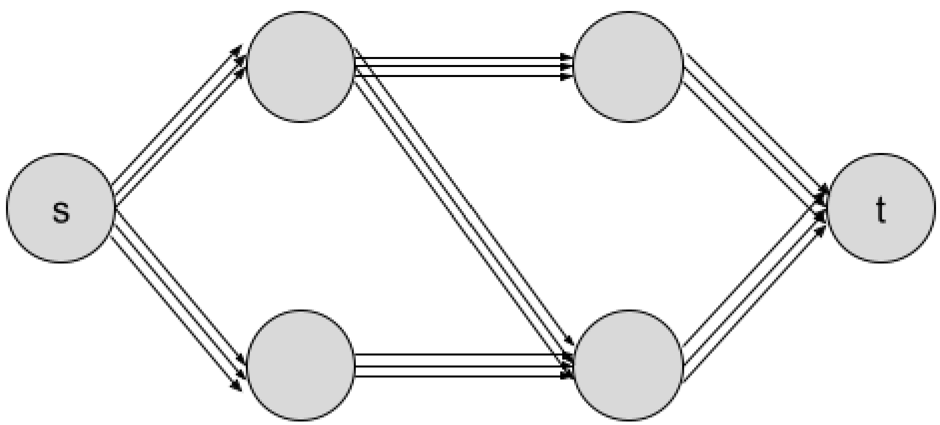



Create another graph with same vertices and for each edge e $:= (u, v)$ add $n$ parallel edges of capacity one and costs in increasing order $c_e(1), \ldots, c_e(n)$



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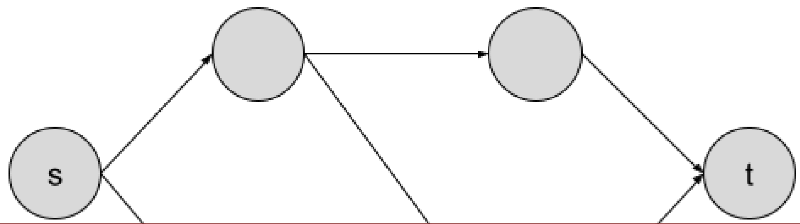


**The min-cost flow minimizes the potential $\Phi$! HW2**

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