L02 Games: Definitions and Existence of Nash Equilibrium

CS 280 Algorithmic Game Theory

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Definitions

Definition (Normal Form Games). *A normal form game is specified by*

- *set of n players* $[n] = \{1, ..., n\}$
- *For each player i a set of strategies/actions* S_i *and a utility* $u_i : \times_{j=1}^{n} S_j \to \mathbb{R}$ *denoting the payoff of i.*
- *set of strategy profiles* $S = S_1 \times ... \times S_n$.

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Example (Rock-Paper-Scissors). *We have that:*

- $n = 2$
- $S_1, S_2 = \{R, P, S\}$.
- $u_1(R, R) = 0$, $u_1(R, P) = -1$, $u_1(R, S) = 1$, $u_1(P, R) = 1$, $u_1(P, P) = -1$, $u_1(P, S) = -1$ $u_1(S, R) = -1$, $u_1(S, P) = 1$, $u_1(S, S) = 0$.
- $u_2 = -u_1$

Definitions

Definition (Mixed strategies). *The set of mixed strategies available to player i are all distributions over* S_i

$$\Delta_i = \{x_i : \sum_{s_i \in S_i} x_i(s_i) = 1 \textit{ and } x_i \geq \mathbf{0}\}$$

We also denote $\Delta = \Delta_1 \times ... \times \Delta_n$ *the set of mixed strategies of all players and* Δ_{-i} *the set of mixed strategies of all players excluding i.*

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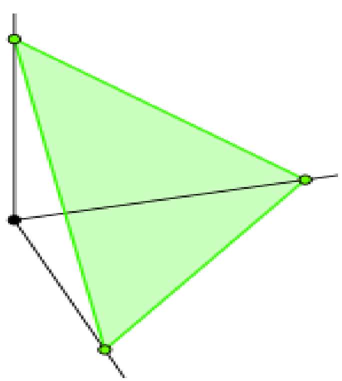
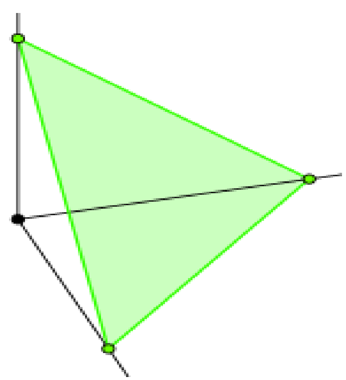


$\Delta_2 \equiv$



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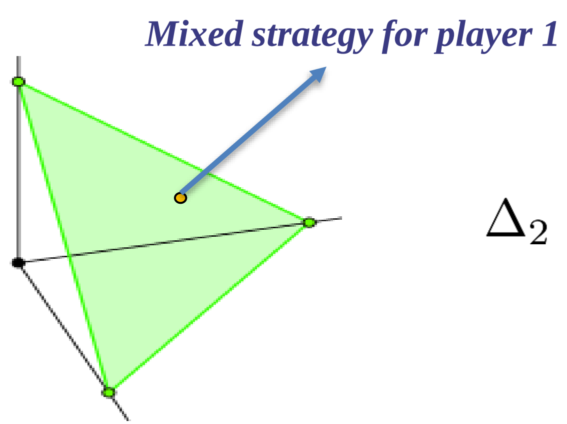

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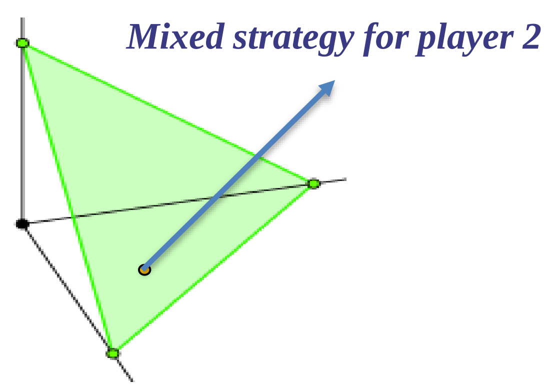

Definitions

Definition (Expected utility). *Given a mixed strategy $x \in \Delta$, the expected utility of player i is*

$$u_i(x) = \sum_{(s_1,\ldots,s_n)\in S} u_i(s_1, \ldots, s_n) \prod_{j=1} x_j(s_j)$$

or (in a more compact way)

$$u_i(x) = \mathbb{E}_{s\sim x} u_i(s).$$

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Example (Rock-Paper-Scissors). *We have that:*

- *Say* $x_1 = (\frac{3}{6}, \frac{2}{6}, \frac{1}{6})$ *and* $x_2 = (\frac{1}{2}, \frac{1}{2}, 0)$.
- *Then* $u_1(x_1, x_2) = \frac{3}{12}u_1(R,R) + \frac{3}{12}u_1(R,P) + 0u_1(R,S) + \frac{2}{12}u_1(P,R) + \frac{2}{12}u_1(P,P) + 0u_1(P,S) + \frac{1}{12}u_1(S,R) + \frac{1}{12}u_1(S,P) + 0u_1(S,S) = -\frac{3}{12} + \frac{2}{12} - \frac{1}{12} + \frac{1}{12} = -\frac{1}{12}$
- $u_2(x_1, x_2) = \frac{1}{12}$

Existence of Nash Equilibrium

Definition (Nash equilibrium)**.** *A mixed strategy* $x \equiv (x_1; \ldots; x_n) \in \Delta$ *is a Nash equilbrium if and only if for all agents* i *and* $x'_i \in \Delta_i$ *we have*

$$u_i(x_i; x_{-i}) \geq u_i(x'_i; x_{-i}).$$

Definition (ϵ-approximate Nash equilibrium)**.** *A mixed strategy* $x \equiv (x_1; \ldots; x_n) \in \Delta$ *is a* ϵ*-approximate Nash equilbrium if and only if for all agents* i *and* $x'_i \in \Delta_i$ *we have*

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Theorem (Nash 51′). *Every game with a finite number of players and actions has a Nash equilibrium.*

Proof of existence of NE

Before we proceed with the proof, an important theorem from topology…

Theorem (Brouwer)**.** *Let D be a convex, compact subset of $\mathbb{R}^d$ and $f : D \to D$ a continuous function. There always exists $x \in D$ such that*

$$f(x) = x.$$

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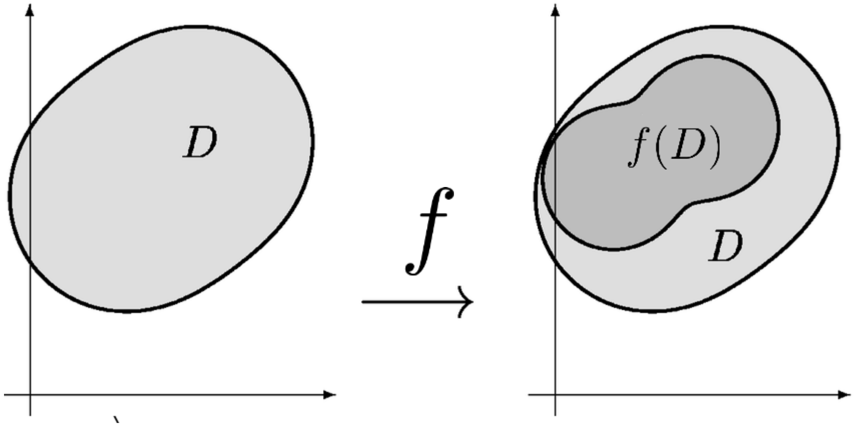
Remark:

x is called a fixed point of f. Uniqueness is not true!



Proof of existence of NE

Proof. Consider any finite game. Define function $f : \Delta \to \Delta$ as follows

$$f_{is_i}(x) = \frac{x_i(s_i) + \max\{u_i(s_i; x_{-i}) - u_i(x), 0\}}{1 + \sum_{s' \in S_i} \max\{u_i(s'; x_{-i}) - u_i(x), 0\}}$$

Gain if i were to switch to pure strategy s_i instead of mixed x_i

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For RPS

$$f(x) = (f_{1R}(x), f_{1P}(x), f_{1S}(x), f_{2R}(x), f_{2P}(x), f_{2S}(x))$$

with $f_{1R}(x) = \left(\frac{x_{1R}+\max\{-x_{2P}+x_{2S}-u_1(x),0\}}{1+\max\{-x_{2P}+x_{2S}-u_1(x),0\}+\max\{x_{2R}-x_{2S}-u_1(x),0\}+\max\{-x_{2R}+x_{2P}-u_1(x),0\}}\right)$

etc…

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Observations:

- For each player $i \Rightarrow \sum_{s' \in S_i} f_{is'}(x) = 1$. why?

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- f is continuous mapping from Δ to Δ (which is convex and compact).

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Gain if i were to switch to pure strategy s_i instead of mixed x_i

Observations:

- For each play

Fixed point always exists!!

- f is continuous mapping from Δ to Δ (which is convex and compact).

Proof of existence of NE

Proof cont.

$$f_{is_i}(x) = \frac{x_i(s_i) + \max\{u_i(s_i; x_{-i}) - u_i(x), 0\}}{1 + \sum_{s' \in S_i} \max\{u_i(s'; x_{-i}) - u_i(x), 0\}}$$

Let x^* be a fixed point of f. We will show that x^* is a Nash Equilibrium!

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Proof cont.

Let x^* be a fixed point of f. We will show that x^* is a Nash Equilibrium!

Since $f(x^*) = x^*$ it holds that for all agents i and $s \in S_i$ that $f_{is}(x^*) = x_i^*(s)$

$$\Rightarrow x_i^*(s) \sum_{s' \in S_i} \max\{u_i(s'; x_{-i}^*) - u_i(x^*), 0\} = \max\{u_i(s; x_{-i}^*) - u_i(x^*), 0\}$$

Proof of existence of NE

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Cases:

- $x_i^*(s) = 0 \Rightarrow u_i(s; x_{-i}^*) \leq u_i(x^*)$.

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- $x_i^*(s) = 0 \Rightarrow u_i(s; x_{-i}^*) \leq u_i(x^*)$.
- $x_i^*(s) > 0$ then if $u_i(s; x_{-i}^*) < u_i(x^*)$ we get $u_i(s'; x_{-i}^*) \leq u_i(x^*)$ for all $s' \in S_1$. But then $u_i(x^*) = \sum_{s'} x_i^*(s') u_i(x^*) > \sum_{s'} u_i(s'; x_{-i}^*) x_i^*(s') = u_i(x^*)$.

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Contradiction!

Proof of existence of NE

Proof cont.

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- $x_i^*(s) > 0$ then if $u_i(s; x_{-i}^*) \geq u_i(x^*)$.

In particular since $u_i(x^*) = \sum_{s'} u_i(s'; x_{-i}^*) x_i^*(s)$ we have $u_i(s; x_{-i}^*) = u_i(x^*)$ whenever $x_i^*(s) > 0$.

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Proof cont.

It holds for all agents i:

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Consider any $\tilde{x}_i$, we will show that

$$u_i(x_i^*; x_{-i}^*) \geq u_i(\tilde{x}_i; x_{-i}^*).$$

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Consider any $\tilde{x}_i$, we will show that

$$u_i(x_i^*; x_{-i}^*) \geq u_i(\tilde{x}_i; x_{-i}^*).$$

From two bullets we get $\tilde{x}_i(s) u_i(s, x_{-i}^*) \leq \tilde{x}_i(s) u_i(x^*)$ so taking the summation

$$u_i(\tilde{x}_i, x_{-i}^*) = \sum_{s'} \tilde{x}_i(s') u_i(s'; x_{-i}^*) \leq \sum_{s'} \tilde{x}_i(s') u_i(x^*) = u_i(x^*)$$

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Consider any $\tilde{x}_i$, we will show that

$$u_i(x_i^*; x_{-i}^*) \geq u_i(\tilde{x}_i; x_{-i}^*).$$

So finding Nash Equilibria is like computing fixed points! Can it be computationally hard? YES…

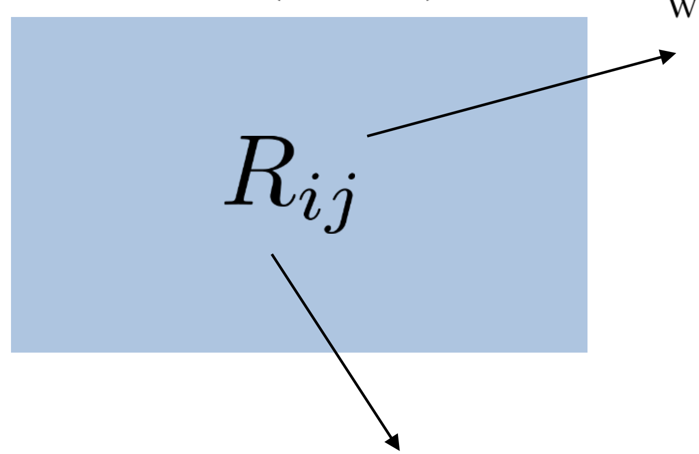
From two bullets … ummation

$$u_i(\tilde{x}_i, x_{-i}^*) = \sum_{s'} \tilde{x}_i(s') u_i(s'; x_{-i}^*) \leq \sum_{s'} \tilde{x}_i(s') u_i(x^*) = u_i(x^*)$$

Zero-sum Games

- 2 players: Row and Column
- n, m strategies available
- Payoff matrix R of size $n \times m$.

$$\mathcal{G} = (R_{n \times m})$$



money the column player pays
for playing j
when row player plays i.

money the row player gets for playing i
when column player plays j.

Zero-sum Games

Column player chooses $y \in \Delta_m$

Row player chooses $x \in \Delta_n$

$$R_{ij}$$

Row gets $x^\top R y$.

Column pays $x^\top R y$.

Example: Two candidates are aiming for presidency.

	Tax-cuts	Society
Economy	**3, -3**	**-1,1**
Education	**-2, 2**	**1,-1**

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How should they play?

Zero-sum Games

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Suppose row player plays (x_{11}, x_{12}). How should column player respond?

Answer: If she chooses Tax-cuts she gets in expectation $u_2(x_1,' Tax - cuts') = -3x_{11} + 2x_{12}$ and if she chooses Society, she gets $u_2(x_1,' Society') = x_{11} - x_{12}$.

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Column plays best response: Column should get

$$\max\{-3x_{11} + 2x_{12}, x_{11} - x_{12}\}.$$

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Row gets (zero-sum):

$$\min\{3x_{11} - 2x_{12}, -x_{11} + x_{12}\}.$$

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If row wants to maximize her utility, she should play then

$$(x_{11}^*, x_{12}^*) = \arg\max_{x_{11}, x_{12}} \min\{3x_{11} - 2x_{12}, -x_{11} + x_{12}\}$$

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Linear Program for Row player

$$\begin{aligned} &\max z \\ \text{s.t } &3x_{11} - 2x_{12} \geq z \\ &-x_{11} + x_{12} \geq z \\ &x_{11} + x_{12} = 1 \\ &x_{11}, x_{12} \geq 0 \end{aligned}$$

Zero-sum Games

Column plays best response: Column should get

$$\max\{-3x_{11} + 2x_{12}, x_{11} - x_{12}\}.$$

Row gets (zero-sum):

$$\min\{3x_{11} - 2x_{12}, -x_{11} + x_{12}\}.$$

If row wants to maximize her utility, she should play then

$$(x_{11}^*, x_{12}^*) = \arg\max_{x_{11}, x_{12}} \min\{3x_{11} - 2x_{12}, -x_{11} + x_{12}\}$$

Linear Program for Row player

Sol $x_1 = (\frac{3}{7}, \frac{4}{7}), z = \frac{1}{7}$

$$\begin{aligned} &\max z \\ \text{s.t } &3x_{11} - 2x_{12} \geq z \\ &-x_{11} + x_{12} \geq z \\ &x_{11} + x_{12} = 1 \\ &x_{11}, x_{12} \geq 0 \end{aligned}$$

Row gets at least 1/7!

Zero-sum Games

Suppose now that column player plays (x_{21}, x_{22}). How should row player respond?

Answer: If she chooses Economy she gets in expectation $u_1('Economy', x_2) = 3x_{21} - x_{22}$ and if she chooses Education, she gets $u_1('Education', x_2) = -2x_{21} + x_{22}$.

Row plays best response: Row should get

$$\max\{3x_{21} - x_{22}, -2x_{21} + x_{22}\}.$$

Zero-sum Games

Suppose now that column player plays (x_{21}, x_{22}). How should row player respond?

Answer: If she chooses Economy she gets in expectation $u_1('Economy', x_2) = 3x_{21} - x_{22}$ and if she chooses Education, she gets $u_1('Education', x_2) = -2x_{21} + x_{22}$.

Row plays best response: Row should get

$$\max\{3x_{21} - x_{22}, -2x_{21} + x_{22}\}.$$

Column gets (zero-sum):

$$\min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}.$$

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If Column wants to maximize her utility, she should play then

$$(x_{21}^*, x_{22}^*) = \arg\max_{x_{21}, x_{22}} \min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}$$

Zero-sum Games

Supp … er re-
sponc …

Linear Program for column player

$$\max w$$
$$\text{s.t } -3x_{21} + x_{22} \geq w$$
$$2x_{21} - x_{22} \geq w$$
$$x_{21} + x_{22} = 1$$
$$x_{21}, x_{22} \geq 0$$

Answ … $_2) =$
$3x_{21}$ … $x_{21} +$
x_{22}.

Row plays best response: Row should get

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Zero-sum Games

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Sol $x_2 = (\frac{2}{7}, \frac{5}{7}), w = -\frac{1}{7}$

Column gets at least -1/7!

Row plays best response: Row should get

$$\max\{3x_{21} - x_{22}, -2x_{21} + x_{22}\}.$$

Column gets (zero-sum):

$$\min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}.$$

If Column wants to maximize her utility, she should play then

$$(x_{21}^*, x_{22}^*) = \arg\max_{x_{21}, x_{22}} \min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}$$

Zero-sum Games

Linear Program for column player Sol $x_2 = (\frac{2}{7}, \frac{5}{7}), w = -\frac{1}{7}$

$$\max w$$

Since zero sum $(\frac{3}{7}, \frac{4}{7}), (\frac{2}{7}, \frac{5}{7})$ must be a Nash equilibrium!!

Next lecture LP duality to get NE and minimax theorem!

$$(x_{21}^*, x_{22}^*) = \arg\max_{x_{21}, x_{22}} \min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}$$