CS295 Introduction to Algorithmic Game Theory

Instructor: Ioannis Panageas Scribed by: Poulin, Sam

Lecture 7. Price of Anarchy.

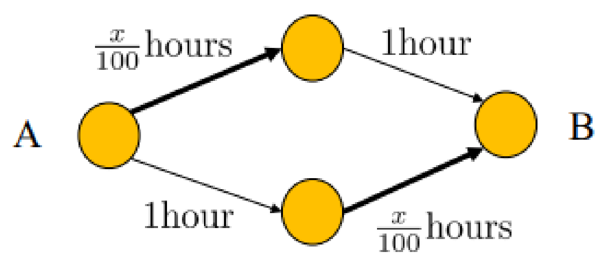
1 Introduction

Definition 1.1 *Price of Anarchy (PoA)* $= \frac{\textit{performance of worst case Nash Equillibrium}}{\textit{optimal performance if agents do not decide on their own}}$

The Price of Anarchy can be understood as the worst case "price" of letting agents rationally (selfishly) decide their own actions in a game. It is the ratio of the cost of the worst performing NE and the cost of the best outcome possible if agents are "told" what to do (i.e. not neccessarily rational choices). A PoA > 1 means that agents could rationally choose to play one or more strategies which give outcomes worse than the optimal outcome. Many theorems which bound the PoA for specific types of games have been proven, a number of which are presented below.

Example 1.1 *Braess's Paradox*

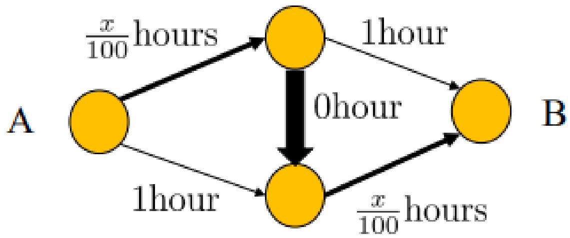
To illustrate the idea of PoA, take the following graph for example:



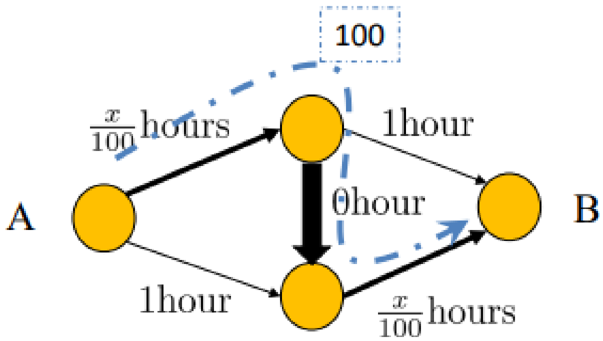
Suppose 100 drivers want to get from vertice A to vertice B. They can either take the top or bottom path. The time it takes for a driver to travel either path is $1 + \frac{x}{100}$ hours, where x is the number of drivers who choose that path. Each driver wants to minimize their own commute time.

If 50 drivers choose each path, than the commute time for all drivers is 1.5 hours. In this case, no driver has any incentive to deviate, as doing so would increase their commute to 1.51 hours. As such, 50 drivers routed along each path is the unique equilibrium flow for this example. The "cost" of this flow (later defined as the *social cost*) is 150 total hours of driving.

Now, suppose that an additional edge is added to the graph as in the following:



With this new “fast link”, the previous equilibrium flow no longer holds: each driver from the top path is incentivized to take the fast link and decrease their commute to 1.01 hours. The new equilibrium flow is for all of the drivers to traverse from A to B as such:



The cost of this new equilibrium is 200 hours of driving, as each driver’s commute is 2 hours. Although the previous routing would still result in a better commute time for all drivers (150 hours), because of the new fast link this now is the only routing such that no driver is incentivized to change their route. The Price of Anarchy captures the ratio between the cost of the worst-case rational scenario and the cost best case scenario. The PoA in this routing game is

$$\frac{200}{150} = \frac{4}{3}.$$

2 Non-atomic Selfish Routing

The above game is an example of a *non-atomic sefish routing game.*

Definition 2.1 *A non-atomic selfish routing game is defined by:*

- *Graph* $G(V, E)$
- *Source destination pairs* $(s_1, t_1), ..., (s_k, t_k)$
- r_i *traffic from* $s_i \to t_i$
- $c_e(f) \geq 0$ *cost function for edge* e*, continuous and non-descreasing*

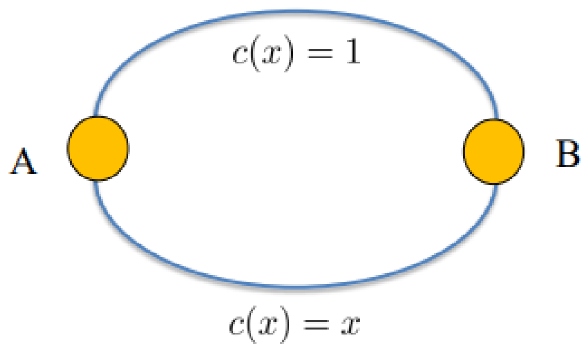
- *Social cost*

$$\sum_{e \in E} c_e(f_e)$$

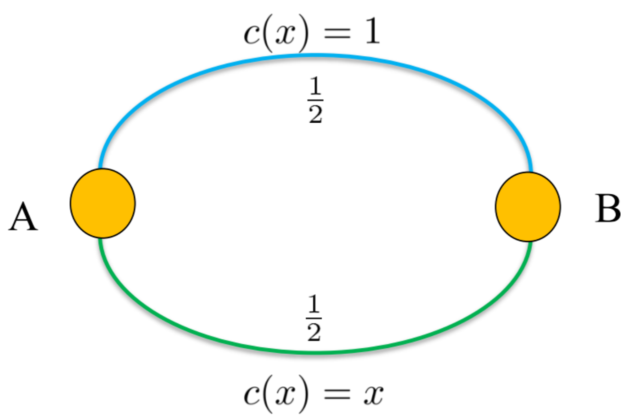
Equilibrium flow is routed such that the social cost is minimized. In non-atomic selfish routing games, equilibrium flow exists and is unique.

Example 2.1 *Pigou network*

The Pigou network is a simple non-atomic selfish routing game illustrated in the graph below:



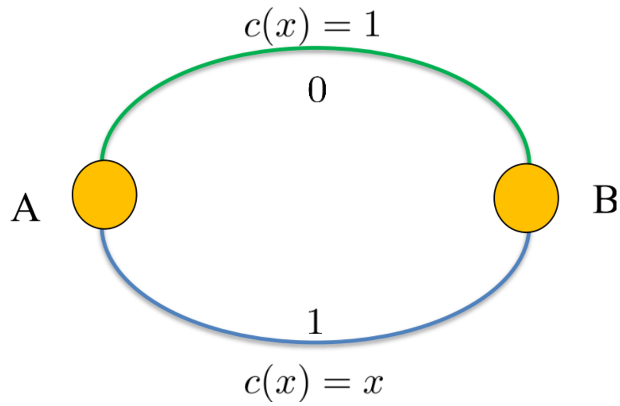
The traffic from A to B is 1. The cost of the top edge is always 1, whereas the cost of the bottom edge is equal to the flow through that edge. The optimal flow is $\frac{1}{2}$ on each edge:



This gives a social cost of

$$\frac{1}{2} \cdot c_{top}(\frac{1}{2}) + \frac{1}{2} \cdot c_{bottom}(\frac{1}{2}) =$$
$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

However, this is not an equilibrium flow, because the cost of the bottom edge is greater than the cost of the top edge. In fact, until all 1 flow is directed through the bottom edge, it's cost function is cheaper than the top edge. As such, this is the equilibrium flow; there is no longer any cheaper edge to route flow along.



This gives a social cost of

$$0 \cdot c_{top}(0) + 1 \cdot c_{bottom}(1) =$$

$$0 \cdot 1 + 1 \cdot 1 = 1$$

The PoA in this game is $\frac{1}{\frac{3}{4}} = \frac{4}{3}$, same as in *Example 1.1*. This is no coincidence!

Definition 2.2 *Linear costs: costs of the form $c_e(f) = a_e \cdot f + b_e$*

Theorem 2.2 *Roughgarden-Tardos 00', PoA for linear costs.*
For every network with linear costs:

$$\textit{cost of Nash flow} \leq \tfrac{4}{3} \cdot \textit{ cost of optimal flow.}$$

This means that in a non-atomic selfish routing game with linear cost functions, the Price of Anarchy is bounded: no matter what network is used for the game, the cost of any Nash flow is no more than $\frac{4}{3}$ the cost of the optimal flow. Both *Example 1.1* and *Example 2.1* show the worst possible PoA for non-atomic selfish routing games with linear cost functions.

Proof: Let f^* be a Nash flow and f be another flow. We first show the variational inequality

$$\sum_e f_e^* c_e(f_e^*) \leq \sum_e f_e c_e(f_e^*)$$

Observe that if f^* is an equilibrium flow, if $f_p^* > 0$ then $c_{p'}(f^*) \leq c_p(f^*)$ for all paths $p^{'}$. Therefore all paths p so that $f_p^* > 0$ have the same cost, say L. Hence $\sum_p f_p^* c_p(f^*) = L \cdot F$ where $F = \sum_p f_p^*$ is the total flow. Since $c_p(f^*) \geq L$ we conclude $\sum_p f_p c_p(f^*) \geq L \sum_p f_p = L \cdot F$
Combining the above we get the variational inequality:

$$\sum_e f_e c_e(f^*) = \sum_p f_p c_p(f^*) \geq L \cdot F = \sum_p f_p^* c_p(f^*) = \sum_e f_e^* c_e(f^*) \equiv$$

$$\sum_e f_e^* c_e(f_e^*) \leq \sum_e f_e c_e(f_e^*)$$

We then get that

$$\sum_e f_e^* c_e(f_e^*) \leq \sum_e f_e c_e(f_e) + \sum_e f_e(c_e(f_e^*) - c_e(f_e))$$

Additionally, we have that

$$\sum_e f_e(c_e(f_e^*) - c_e(f_e)) \leq \frac{1}{4}\sum_e f_e^* c_e(f_e^*)$$

based off the following two cases:

- Case 1: $c_e(f^*) < c_e(f)$. Trivially, $f_e(c_e(f^* - c_e(f)) \leq \frac{1}{4} f_e^* c_e(f^*)$
- Case 2: $c_e(f^*) \geq c_e(f) \to f_e^* \geq f_e$. Because edge costs are linear, LHS $= a_e f_e(f_e^* - f_e)$ and RHS $\geq \frac{1}{4} a_e f_e^{*2}$ since $xy - y^2 \leq \frac{x^2}{4}$

Thus, we conclude that

$$\sum_e f_e^* c_e(f_e^*) \leq \sum_e f_e c_e(f_e) + \frac{1}{4}\sum_e f_e^* c_e(f_e^*) \equiv$$

$$\sum_e f_e^* c_e(f_e^*) \leq \frac{4}{3}\sum_e f_e c_e(f_e)$$

■

Theorem 2.3 *Roughgarden 02', PoA for polynomial costs*

For every network with polynomial costs of maximum degree d:

$$\textit{cost of Nash flow} \leq \theta(\tfrac{d}{\log d}) \cdot \textit{ cost of optimal flow.}$$

3 Congestion Games

Definition 3.1 *A congestion game is defined by:*

- n *set of players*
- E *set of edges/facilities/bins*
- $S_i \subset 2^E$*: the set of strategies of player* i.
- $c_e : \{1, ..., n\} \to \mathbb{R}^+$ *cost function of edge* e
- *for each strategy* $s = (s_1, ..., s_n)$
  - $l_e(s)$*: number of players (load) that use edge* e
  - $c_i(s) = \sum_{e \exists s_i} c_e(l_e)$*: the cost function of player* i

Congestion games capture atomic routing games.

Theorem 3.1 *Chistodoulou-Koutsoupias, PoA for linear costs*

For every congestion game with linear costs:

cost of Nash flow $\leq \frac{5}{2}\cdot$ *cost of optimal flow.*

Proof: Let l^* be a Nash equilibrium in which i uses path P_i and assume i deviates to path $\tilde{P}_i$. The following variational inequality holds:

$$\sum_{e\in P_i} c_e(l_e^*) \leq \sum_{e\in P_i\cap\tilde{P}_i} c_e(l_e^*) + \sum_{e\in \tilde{P}_i P_i} c_e(l_e^*+1) \equiv$$

$$\sum_{e\in P_i} c_e(l_e^*) \leq \sum_{e\in P_i\cap\tilde{P}_i} c_e(l_e^*+1) + \sum_{e\in \tilde{P}_i P_i} c_e(l_e^*+1) \equiv$$

$$\sum_{e\in P_i} c_e(l_e^*) \leq \sum_{e\in \tilde{P}_i} c_e(l_e^*+1)$$

Consider any configuration $\tilde{l}$, where each agent j uses path $\tilde{P}_j$. Summing for all agents i

$$\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \leq \sum_{i\in[n]}\sum_{e\in \tilde{P}_i} c_e(l_e^*+1)$$

$$\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \leq \sum_{e} \tilde{l}_e c_e(l_e^*+1)$$

Because costs are linear we have

$$\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \leq \sum_{e} a_e\tilde{l}_e(l_e^*+1) + b_e\tilde{l}_e$$

Since $y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2$ for naturals y, z

$$\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \leq \sum_{e} a_e(\frac{5}{3}\tilde{l}_e^{\,2} + \frac{1}{3}l_e^{*2}) + b_e\tilde{l}_e$$

Observe that

$$\frac{5}{3}C(\tilde{l}) = \frac{5}{3}\sum_{i\in[n]}\sum_{e\in \tilde{P}_i} c_e(\tilde{l}_e) = \sum_{e} \frac{5}{3}a_e\tilde{l}_e^{\,2} + \frac{5}{3}b_e\tilde{l}_e \geq \sum_{e} \frac{5}{3}a_e\tilde{l}_e^{\,2} + b_e\tilde{l}_e$$

Therefore

$$C(l^*) \leq \frac{5}{3}C(\tilde{l}) + \frac{1}{3}\sum_{e} a_e l_e^{*2}$$

$$C(l^*) \leq \frac{5}{3}C(\tilde{l}) + \frac{1}{3}C(l^*)$$

$$C(l^*) \leq \frac{5}{2}C(\tilde{l})$$

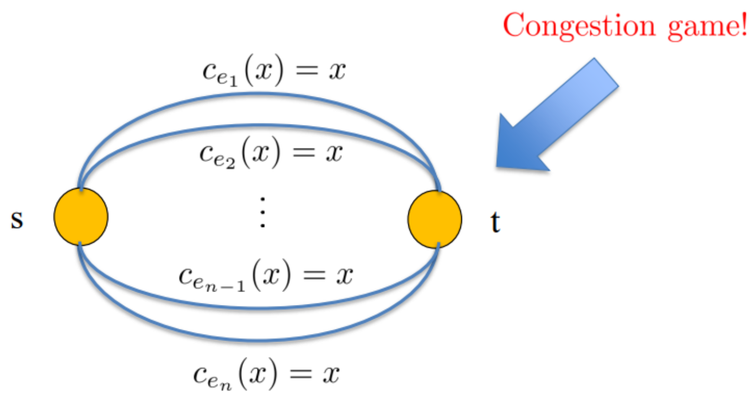
■

4 PoA of Balls & Bins

Definition 4.1 *Balls & Bins is defined as follows:*

- *set of n balls and n bins $\{e_1, ...e_n\}$.*
- *each ball i chooses a bin j and pays the load of the bin j, l_j.*
- *social cost = maximum bin load =* $\max_{j\in[n]}(l_j)$

Balls & Bins is a congestion game:



Theorem 4.1 *Koutsoupias-Papadimitriou, PoA for balls & bins*

The PoA is $\Omega(\frac{\ln n}{\ln \ln n})$

Proof: We will use the second moment method.

- The optimal social cost is one: each ball is in its own bin.
- A uniform strategy $(\frac{1}{n}, ..., \frac{1}{n})$ for each ball is a Nash equilibrium.
- With high probability, we how that the uniform strategy gives max load $\Omega(\frac{\ln n}{\ln \ln n})$ which implies the expected max load is $\Omega(\frac{\ln n}{\ln \ln n})$.

Claim: Bin i had at least $k << n$ balls with probability at least:

$$\binom{n}{k}\frac{1}{n^k}(1-\frac{1}{n})^{n-k} \geq \frac{1}{n^k}(\frac{n}{k})^k\frac{1}{e} = \frac{1}{en^{1/3}}$$

Suppose $k = \frac{\ln n}{3 \ln \ln n}$. We then have $k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln n}} = n^{1/3}$ Thus, bin i has at least $\frac{\ln n}{3 \ln \ln n}$ balls with probability at least $\frac{1}{en^{1/3}}$.

Let X_i be the indicator that bin i has at least $\frac{\ln n}{3 \ln \ln n}$ balls and X be the expected number of all bins with at least $\frac{\ln n}{3 \ln \ln n}$ balls.

$$X = \sum_i X_i \rightarrow E[X] = \sum_i E[X_i]$$

Observe that $E[X] \geq \frac{n^{2/3}}{e} \gg 1$ but this does not impy $X \geq 1$ with high probability. We need to argue about the variance (second moment).

Chebyshev's inequality gives

$$Pr[|X - E[X]| \geq tE[X]] \leq \frac{Var[X]}{t^2 E^2[X]}$$

thus

$$Pr[X = 0] \leq Pr[|X - E[X]| \geq tE[X]] \leq \frac{Var[X]}{t^2 E^2[X]}$$

From negative correlation we have that $Var[X] \leq \sum_i Var[X_i]$.

Moreover, $Var[X_i] = E[X_i^2] - E^2[X_i] \leq E[X_i^2] = E[X_i] \leq 1$. We conclude that

$$Pr[X = 0] \leq \frac{n}{e^2 n^{4/3}} = \frac{n^{-1/3}}{e^2}.$$

■