CS295 Introduction to Algorithmic Game Theory

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Lecture 3. LP Duality and Zero-sum games

1 Introduction to Linear Programming

The standard form of a Linear Program (LP) is shown as 2

$$\begin{aligned} \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{1}$$

The goal of Equation 1 is to find a feasible solution x^* if there exist one, which is called **feasibility problem**. Here, we assume x is $m \times 1$ vector, A is $n \times m$ matrix, b is $n \times 1$ vector. In other words, we have n constraints and m variables.

Now suppose our goal is finding the optimal solution of the linear program, or return infeasible if there doesn't exist a feasible solution. We can introduce the following standard form:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{2}$$

In this form, $c^T x$ is what we can define and this form is called **optimization problem**.

1.1 Feasibility and Optimization

Lemma 1.1 *The feasibility problem$^{(i)}$ and the optimization problem$^{(ii)}$ are polynomial equivalent.*

Proof: Suppose we know the solution of (ii) x^*, then x_1^* satisfies all the constraints and thus is a feasible solution of problem (i).

On the contrary, we want to show if the problem could be reduced to a form that: does $A'x \leq b$ has at least one solution? The problem (ii) is equivalent to ask: if there exists the largest q such that we satisfy $c^T x \geq q$ and $Ax \leq b$, where we could binary search to find the best q. Therefore, the feasibility and optimality problems are polynomial time equivalent. ■

1.2 Primal and Dual Formulation

Every LP is associated with another LP, called the *dual* (in this case, the original LP is called the *primal*). The standard form of primal and dual is shown as Table(1. In summary, each inequality constraint in primal is associated with a variable in dual; each variable coefficient in the primal objective function is associated with one constraint in dual. If the original problem is not in the standard form, the common practice is to transform min/max or the inequality sign by multiplying (-1).

	Primal	**Dual**
Objective function	Max $Z = c^T x$	Min $W = b^T y$
Row (i)	$a_{i1}x_n + ... + a_{in}x_n = b_i$	no sign constraint on y_i
Row (i)	$a_{i1}x_n + ... + a_{in}x_n \leq b_i$	$y_i \geq 0$
Variable (j)	$x_j \geq 0$	$a_{1j}y_1 + ... + a_{mj}y_m \geq c_j$
Variable (j)	no sign constraint on x_j	$a_{1j}y_1 + ... + a_{mj}y_m = c_j$

Table 1: Transformation between dual and primal [1]

1.3 Strong and Weak Duality

There are four possible scenarios with respect to the feasibility of primal-dual pairing shown as follows, where we focus on the 1st case (**bold**):

- **The Primal is bounded and feasible $\Rightarrow$ The Dual is bounded and feasible.**
- The Primal is unbounded and feasible $\Rightarrow$ The Dual is infeasible.
- The Primal is infeasible $\Rightarrow$ The Dual is unbounded and feasible.
- The Primal is infeasible $\Rightarrow$ The Dual is infeasible.

But why do we care about the feasibility/optimality relationship between primal and dual? The underlying intuition is that, if the primal LP is a maximization problem, the dual can be used to find upper bounds on its optimal value.

Theorem 1.2 *(**Weak duality**). Assume that primal is feasible and bounded. It holds that*

$$\max_{x \in P} c^T x \leq \min_{y \in D} b^T y$$

Proof: Suppose P and D are the domain of primal and dual problems, respectively. For any feasible $x \in P$ and $y \in D$. Then we have

$$Ax \leq b \Rightarrow y^T(Ax) \leq y^T b \Rightarrow x^T A^T y \leq y^T b \quad \text{(re-arrange the scaler term)}$$

Similarly, based on the feasibility constraint of dual that $A^T x \geq c$, we have $x^T A^T y \geq x^T c$ Using transitivity, we have $c^T x \leq x^T A^T y \leq y^T b$ for $\forall x \in P$ and $\forall y \in D$. Under the assumption that primal is bounded and feasible, we could conclude $\max_{x \in P} c^T x \leq \min_{x \in D} b^T y$ ■

Theorem 1.3 *(**Strong duality**). Assume that primal is feasible and bounded. It holds that*

$$\max_{x \in P} c^T x = \min_{y \in D} b^T y$$

We use the zero-sum presidency example (refer to Lecture 1) to verify the strong duality theorem.

In the contexts of *zero-sum* game, x means the mixed strategy (the probability distribution) of row player. Accordingly, primal problem represents what row player should act to maximize his/her utility z if he/she plays first. The dual problem objective function w turns out to be the negation of how much column player would get. For the specific example below, the optimal solution is $x^* = (\frac{3}{7}, \frac{4}{7}), y = (\frac{2}{7}, \frac{5}{7})$, and the optimally matches at $z = w = \frac{1}{7}$.

$$\begin{aligned} \max \quad & 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot z \\ \text{s.t.} \quad & \begin{pmatrix} -3 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix} \leq 0 \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned} \qquad \begin{aligned} \min \quad & 0 \cdot y_1 + 0 \cdot y_2 + 1 \cdot w \\ \text{s.t.} \quad & \begin{pmatrix} -3 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ w \end{pmatrix} \leq 0 \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

2 Zero-Sum Games as Linear Programs

2.1 Computing Nash in zero-sum game by LP

What do we care about LP apart from its theoretic characteristics? One of the benefits is to help represent and compute Nash in zero-sum games. Recall the pay-off table $R_{ij} \in \mathbf{R}^{n \times m}$ in zero-sum game, where the row player chooses the strategy $x \in \Delta_n$ and column player chooses the strategy $y \in \Delta_m$. For given value of x and y, the row play will gets the payoff $x^T R y$ while the column player gets the negation.

Suppose row player plays first and wants to get at least z for all pure strategies of column player, we have

$$x^T R \vec{e}_j \geq z \Rightarrow x^T R \geq z \cdot \mathbf{1}^T \tag{3}$$

where $\vec{e}_j$ is the unit vector $(0, ...1, .., 0)^T$ with only the j^{th} position being 1, representing when column play chooses the j^{th} strategy, $\mathbf{1}$ is the vector with all elements being 1. Eq (3) implies that no matter what strategy y plays, the payoff of x is at least z. In addition, we have $x^T \mathbf{1} = 1$ and non-negative constraints because the probability of each strategy should sum up to 1.

2.2 Interpretation of Primal and Dual

Based on the interpretation above, we have LP for player x as

$$\begin{aligned} \max \quad & z \\ & x^T R \geq z \cdot \mathbf{1}^T \\ & x^T \mathbf{1} = 1 \\ & x \geq \mathbf{0} \end{aligned} \tag{4}$$

Equivalently, the above maximum is also a maxmin problem that

$$\max_{x \in \Delta_n} \min_{y \in \Delta_m} x^T R y$$

Consider the dual of the previous LP (4), we will find it is formulated as

$$\begin{aligned} \min \quad & z' \\ & y^T(-R)^T \geq -z' \cdot \mathbf{1}^T \\ & y^T\mathbf{1} = 1 \\ & y \geq \mathbf{0} \end{aligned} \tag{5}$$

If we set $z'' = -z'$, we could transform the objective function of (5) from **min** z' to **-max** z''. That is exactly the LP (6 as if y play first with sign flipped:

$$\begin{aligned} -\max \quad & z'' \\ & y^T(-R)^T \geq z'' \cdot \mathbf{1}^T \\ & y^T\mathbf{1} = 1 \\ & y \geq \mathbf{0} \end{aligned} \tag{6}$$

3 Nash Equilibrium and Linear Programs

A zero-sum game can be formulated by a series of linear programs, where each player can be represented by a LP. Assume the two players use LP_1 and LP_2 as their corresponding linear programs, which is presented in Equation 7 and Equation 8. Here, we introduce the following Theorem 3.1:

$$\begin{aligned} LP_1 : \max z_1 & \\ & x_1^T R \geq z_1 \cdot \mathbf{1}^T \\ & x_1^T\mathbf{1} = 1 \\ & x_1 \geq \mathbf{0} \end{aligned} \tag{7}$$

$$\begin{aligned} LP_2 : \max z_2 & \\ & x_2^T(-R)^T \geq z_2 \cdot \mathbf{1}^T \\ & x_2^T\mathbf{1} = 1 \\ & x_2 \geq \mathbf{0} \end{aligned} \tag{8}$$

Theorem 3.1 *Suppose for player 1, the payoff matrix is R. Let $(x_1^*,\ z_1^*)$ be optimal for LP_1 and $(x_2^*,\ z_2^*)$ be optimal for LP_2. Then (x_1^*, x_2^*) is a Nash Equilibrium of this zero-sum game, and the payoff of player 1 is z_1 and for player 2 is $z_2 = -z_1$*

Proof: Since the game is zero-sum, it indicates that the payoff matrix for player 2 is $-R$. Since $(x_1^*,\ z_1^*)$ is optimal, according to the definition, it is feasible to LP_1, which indicates a new equation, as shown in Equation 9. Similarly, because $(x_2^*,\ z_2^*)$ is optimal and feasible to LP_2, we observe that

Equation 10.

$$x_1^{*T} R x_2^* \geq z_1 \tag{9}$$

$$x_2^{*T} (-R)^T x_1^* \geq z_2 \implies x_2^{*T} R^T x_1^* \leq -z_2 \tag{10}$$

Now, by Strong duality Theorem 1.3, we can figure out that $z_1 = -z_2$. Moreover, Equation 9 implies that if player 1 plays x_1^* against player 2, his payoff is at least z_1. With Equation 10, this is exactly z_1. So x_1^* is the best choice for player 1 to play against player 2. Similarly by using the same argument, we can obtain that if player 2 plays x_2^* against player 1, his payoff is exactly z_2 and x_2^* is the best response. Hence, $(x_1^*,\ x_2^*)$ is a Nash equilibrium and the players' payoffs are z_1 and $z_2 = -z_1$ respectively[2].

■

Theorem 3.2 *Let (x_1^*, x_2^*) be a Nash equilibrium and set $z^* = x_1^{*T} R x_2^*$. (x_1^*, z^*) is optimal solution for LP1 and $(x_2^*, -z^*)$ is optimal solution for LP2.*

4 Corollaries

Corollary 4.1 *Von Neuman minimax theorem:*

$$\max_{x \in \Delta_n} \min_{y \in \Delta_m} x^\intercal R y = \min_{y \in \Delta_m} \max_{x \in \Delta_n} x^\intercal R y$$

Corollary 4.2 *Uniqueness of payoffs: For both the row player and column player of a zero-sum game, the payoff is the same at all the Nash equilibria points.*

Corollary 4.3 *Convexity of Nash Equilibria: In a zero-sum game, the set of the Nash equilibria points is convex.*

References

[1] Howard Karloff *Linear Prgramming (pp 52-53)*. Birkhäuser Basel

[2] Constantinos Daskalakis *Games, Decision, and Computation, Lecture 3*