CS295 Topics in Algorithmic Game Theory

Lecture2: Games, Definitions and Existence of Nash equilibrium

Scribe: Parnian Shahkar **Lecturer:** Prof. Ioannis Panageas

1 Normal form games

Definition 1 ***Normal form games*** *are defined as:*

- *A set of* n *players specified by* $[n] = \{1, .., n\}$.
- *Each player* i *has a set of finite strategies/actions* $S_i = \{s_i^1, .., s_i^{k_i}\}$ *to choose from.*
- *By putting the strategies of all players together, we get a set of strategy profiles* $S = S_1 \times ... \times S_n$.
- *We can define the utility of each agent* i *as a function that takes the strategies of all players as its input and gives the payoff of player* i *as output. From now on we denote* S_i *as the realized strategy of player* i*, and* S *as the realized strategy profile of all players. Therefore we can write* $u_i : S \to \mathbb{R}$

Based on the definition above, many games can be characterized as normal form games. To better illustrate the notations we introduced above, the following example is shown.

Example 1 *Rock-Paper-Scissors (RPS) can be characterized as a Normal form game because:*

- *There are two players so* $n = 2$.
- *Each player plays rock, paper, or scissors, therefore* $S_1, S_2 = \{R, P, S\}$.
- *Utility of each player depends on strategies used by both players. Based on the game's rules, if both players play the same strategy, they both get zero utility. If one plays scissors and the other plays rock, rock player wins,* $u_1(R, P) = 1$. *If scissors and paper is played, scissor player wins,* $u_1(S, P) = 1$. *If rock and paper are played, paper player wins,* $u_1(P, R) = 1$. *The winner in each scenario gets utility* 1 *and the loser gets utility* -1. *Therefore we can construct a payoff table for player 1, where player 1 is the row player and player 2 is the column player. The payoff of each player is determined based on the strategies both players choose.*

	R	P	S
R	0	-1	+1
P	+1	0	-1
S	-1	+1	0

Utility of the second player is $u_2 = -u_1$.

2 Mixed strategies and Expected utility

Definition 2 *Each player can choose a vector of probability mass function over its strategy space, to represent with what probability she will choose each strategy. This vector of probabilities that assigns a probability to each strategy of player i is her* ***mixed strategy****. The set of all mixed strategies available to each player can be represented as the following:*

$$\Delta_i = \{x_i : \sum_{s_i \in S_i} x_i(s_i) = 1, x_i \geq 0\}$$

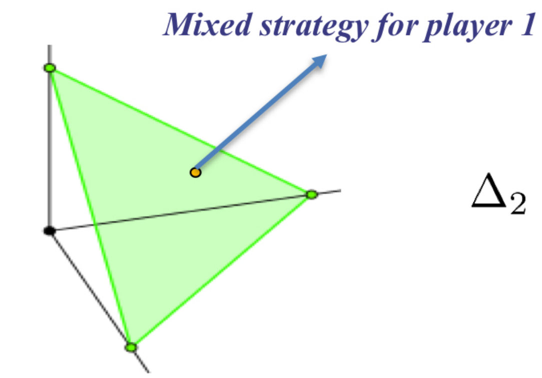
We represent the set of mixed strategies of all players with Δ, where:

$$\Delta = \Delta_1 \times ... \times \Delta_n$$

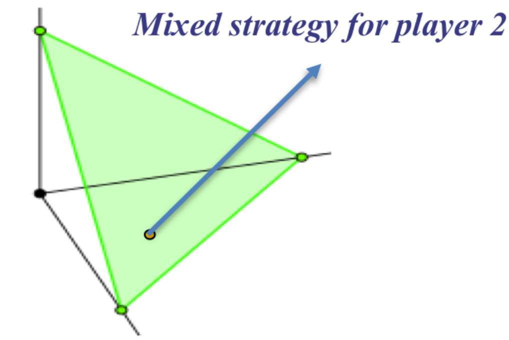
We can also define the notion Δ_{-i} as the set of mixed strategies of all players excluding i.

Example 2 *In the Rock-Paper-Scissors game, the set of mixed strategies for each player is the green triangle shown below in the strategy space. Each player chooses a point of its mixed strategy set. For example if player 1 chooses $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$, this means that she has chosen to play the first strategy(Rock) with probability $\frac{2}{3}$, the second strategy(Paper) with probability $\frac{1}{6}$, and the third strategy(Scissors) with probability $\frac{1}{6}$. Corner points of the triangles represent the pure strategies.*

$\Delta_1 \equiv$



$\Delta_2 \equiv$



Definition 3 *Given a mixed strategy $x \in \Delta$,* ***expected utility*** *of player i is defined as the following:*

$$u_i(x) = \mathbb{E}_{s \sim x} u_i(s) = \sum_{(s_1,..,s_n) \in S} u_i(s_1, .., s_n) \Pi_{j=1}^n x_j(s_j)$$

Example 3 *In Rock-Paper-Scissors, if player 1 chooses her strategy to be $x_1 = (\frac{3}{6}, \frac{2}{6}, \frac{1}{6})$ and player 2 chooses his strategy to be $x_2 = (\frac{1}{2}, \frac{1}{2}, 0)$, the expected utility of player one can be computed(using the payoff table drawn in example 1) as below:*

$$u_1(x_1, x_2) = \frac{3}{6}\frac{1}{2} u_1(R, R) + \frac{3}{6}\frac{1}{2} u_1(R, P) + \frac{3}{6} 0 u_1(R, S) + ... + \frac{1}{6} 0 u_1(S, S) = -\frac{1}{12}$$

As the expected utility of the second player is minus the expected utility of the first player, $u_2(x_1, x_2) = \frac{1}{12}$.

3 Nash Equilibrium

Definition 4 *Suppose each player chooses her mixed strategy x_i. The mixed strategy of all players can be described as $x \equiv (x_1, .., x_n) \in \Delta$. x is a **Nash Equilibrium** if given the mixed strategies of other players(x_{-i}), agent i does not have any incentive to deviate from x_i. In other words $\forall x'_i \in \Delta_i$:*

$$u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i}).$$

*We can extend this definition by defining x to be **ϵ-approximate Nash Equilibrium**(in the additive form) if and only if the following holds:*

$$u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i}) - \epsilon.$$

*We can even go further and define x to be **ϵ-approximate Nash Equilibrium**(in the multiplicative form) if and only if the following holds:*

$$u_i(x_i, x_{-i}) \geq (1 - \epsilon) u_i(x'_i, x_{-i}).$$

Theorem 1 *(**Nash**).Every game with finite number of players and actions has a Nash Equilibrium.*

Before proving the above theorem, we need to understand the Brouwer theorem from topology.

Theorem 2 *(**Brouwer**). Let D be a convex, compact subset of $\mathbb{R}^d$ and $f : D \to D$ a continuous function. There always exists $x \in D$ such that $f(x) = x$.*

As this theorem states, in any continuous function operated on a convex and compact domain that maps the outputs to the same domain as input, a fixed point always exists. Now let's prove theorem 1 using Brouwer's theorem.

Proof 1 *Consider any finite game and the following continuous function that gets any point in the mixed strategies space as its input.*

$$f_{i,s_i}(x) = \frac{x_i(s_i) + max\{u_i(s_i; x_{-i}) - u_i(x), 0\}}{1 + \sum_{s' \in S_i} max\{u_i(s'; x_{-i}) - u_i(x), 0\}}$$

For each agent i, we can define the above function for all of her pure strategies $s_i \in \{s^i_1, .., s^i_m\}$. By vertically concatenating $f_{i,s_i}(x)$ for all pure strategies of player i, we get vector $f_i(x)$. Such vector is defined for all agents. Based on these notations, the following observations can be made:

- *In $f_{i,s_i}(x)$, the denominator is determined in a way that the sum of $f_{i,s_i}(x)$ over all pure strategies of player i is 1:*

$$\begin{aligned} \sum_{s_i \in S_i} f_{i,s_i}(x) &= \frac{\sum_{s_i \in S_i} [x_i(s_i) + max\{u_i(s_i; x_{-i}) - u_i(x), 0\}]}{1 + \sum_{s' \in S_i} max\{u_i(s'; x_{-i}) - u_i(x), 0\}} \\ &= \frac{1 + \sum_{s_i \in S_i} [max\{u_i(s_i; x_{-i}) - u_i(x), 0\}]}{1 + \sum_{s' \in S_i} max\{u_i(s'; x_{-i}) - u_i(x), 0\}} \\ &= 1 \end{aligned} \quad (1)$$

- *f is a continuous function. Moreover, since both $x_i(s_i)$ and $max\{u_i(s_i; x_{-i}) - u_i(x), 0\}$ are non-negative, the output of f is positive.*
- *$f_i(x)$ is a m-dimensional vector where m is the dimension of strategy space of player i.*

According to the observations described, f is a continuous function that maps each point in Δ_i to a point in Δ_i(Δ_i is convex and compact). Based on Brouwer's theorem, this function has a fixed point. Let x^* be the fixed point of f. In what follows, we show that x^* is a Nash Equilibrium.

Since x^* is a fixed point, $f(x^*) = x^*$. Therefore, for each agent i $f_i(x^*) = x_i^*$. In other words, for each agent i and $s \in S_i$, $f_{i,s}(x^*) = x_i^*(s)$. Hence:

$$x_i^*(s) = \frac{x_i^*(s) + max\{u_i(s; x_{-i}^*) - u_i(x^*), 0\}}{1 + \sum_{s' \in S_i} max\{u_i(s'; x_{-i}^*) - u_i(x^*), 0\}} \tag{2}$$

Therefore we have that:

$$x_i^*(s)[1 + \sum_{s' \in S_i} max\{u_i(s'; x_{-i}^*) - u_i(x^*), 0\}] = x_i^*(s) + max\{u_i(s; x_{-i}^*) - u_i(x^*), 0\} \tag{3}$$

Which leads to the following:

$$x_i^*(s)[\sum_{s' \in S_i} max\{u_i(s'; x_{-i}^*) - u_i(x^*), 0\}] = max\{u_i(s; x_{-i}^*) - u_i(x^*), 0\} \tag{4}$$

Now two cases might happen:

- *$x_i^*(s) = 0$, In this case, the left hand side of equation 4 becomes 0. Therefore $max\{u_i(s; x_{-i}^*) - u_i(x^*), 0\} = 0$. This means that $u_i(s; x_{-i}^*) \leq u_i(x^*)$.*
- *$x_i^*(s) > 0$, then if $u_i(s; x_{-i}^*) < u_i(x^*)$, this means that the right hand side of equation 4 is 0. Therefore $\sum_{s' \in S_i} max\{u_i(s'; x_{-i}^*) - u_i(x^*), 0\} = 0$. Since all the terms in the sum are non-negative, all the terms should be 0. So for all $s' \in S_i$, $max\{u_i(s'; x_{-i}^*) - u_i(x^*), 0\} = 0$. Hence $u_i(s'; x_{-i}^*) \leq u_i(x^*)$.*
  Since $x_i^* \in \Delta_i$, $\sum_{s'} x_i^*(s') = 1$. Therefore utility of player i by choosing strategy x_i^* can be written as the following: $u_i(x^*) = 1 \times u_i(x^*) = \sum_{s'} x_i^*(s') u_i(x^*)$. In the case where $u_i(s; x_{-i}^*) < u_i(x^*)$, we have shown that $\forall s' \in S_i$, $u_i(s'; x_{-i}^*) \leq u_i(x^*)$. Therefore,
  $$u_i(x^*) = \sum_{s'} x_i^*(s') u_i(x^*) > \sum_{s'} x_i^*(s') u_i(s'; x_{-i}^*)$$
  The Right hand side of the above equation is the expected utility of player i if she chooses to play x_i^*. Therefore $u_i(x^*) > \sum_{s'} x_i^*(s') u_i(s'; x_{-i}^*) = u_i(x^*)$. This is a contradiction. Therefore, the initial assumption that $u_i(s; x_{-i}^*) < u_i(x^*)$ is never true. ***Hence if $x_i^*(s) > 0$, then $u_i(s; x_{-i}^*) \geq u_i(x^*)$.***

From the definition of expected utility, we know that $u_i(x^*) = \sum_{s'} u_i(s'; x^*_{-i}) x^*_i(s')$*. Whenever* $x^*_i(s') = 0$*, the* $u_i(s'; x^*_{-i}) x^*_i(s')$ *term in the sum becomes zero. In other cases where* $x^*_i(s') > 0$*, by using* $u_i(s; x^*_{-i}) \geq u_i(x^*)$*, we get the following:*

$$
\begin{aligned}
u_i(x^*) &= \sum_{s'} u_i(s'; x^*_{-i}) x^*_i(s') \\
&\geq \sum_{s'} u_i(x^*) x^*_i(s') \\
&= u_i(x^*) \sum_{s'} x^*_i(s') \\
&= u_i(x^*)
\end{aligned}
\tag{5}
$$

Therefore, the inequality used in 5 is equality and we have that $u_i(s; x^*_{-i}) = u_i(x^*)$ *whenever* $x^*_i(s') > 0$*. To summarize, for each player* i*, the following holds:*

- *When* $x^*_i(s) = 0$*,* $u_i(s; x^*_{-i}) \leq u_i(x^*)$*.*
- *When* $x^*_i(s) > 0$*,* $u_i(s; x^*_{-i}) = u_i(x^*)$*.*

In order to show that x^* *is a Nash Equilibrium, we need to show that* $\forall \tilde{x}_i \in \Delta_i$*,* $u_i(x^*_i; x^*_{-i}) \geq u_i(\tilde{x}_i; x^*_{-i})$*. From the two bullet points above, it can be easily seen that* $u_i(s; x^*_{-i}) \leq u_i(x^*)$*. Hence* $\tilde{x}_i(s) u_i(s; x^*_{-i}) \leq \tilde{x}_i(s) u_i(x^*)$*. By taking the summation we get the following:*

$$
\sum_s \tilde{x}_i(s) u_i(s; x^*_{-i}) \leq \sum_s \tilde{x}_i(s) u_i(x^*) \tag{6}
$$

From the definition of expected utility, we know that $u_i(\tilde{x}_i, x^*_{-i}) = \sum_s \tilde{x}_i(s) u_i(s; x^*_{-i})$ *and since* $\tilde{x}_i \in \Delta_i$*,* $\sum_s \tilde{x}_i(s) = 1$*, so we have* $u_i(x^*) = \sum_s \tilde{x}_i(s) u_i(x^*)$*, Therefore we get the following:*

$$
\begin{aligned}
u_i(\tilde{x}_i, x^*_{-i}) &= \sum_s \tilde{x}_i(s) u_i(s; x^*_{-i}) \\
&\leq \sum_s \tilde{x}_i(s) u_i(x^*) \\
&= u_i(x^*)
\end{aligned}
\tag{7}
$$

To sum up, we proved that $\forall \tilde{x}_i \in \Delta_i$*,* $u_i(x^*_i; x^*_{-i}) \geq u_i(\tilde{x}_i; x^*_{-i})$*, therefore,* x^* *which was the fixed point of* f*, is indeed a Nash Equilibrium.*

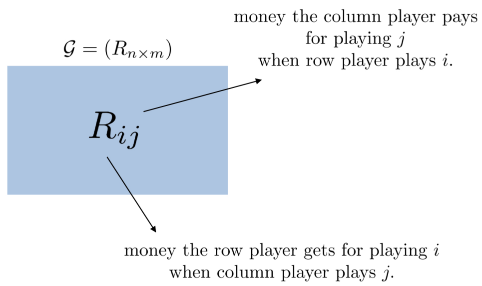
According to theorem 1, In order to find a Nash Equilibrium of a finite game, we only need to find the fixed points of the function f as defined above. Finding such fixed points can generally be computationally hard. In the next section, we introduce a special form of games where computing a Nash Equilibrium is easy for them by using linear programming.

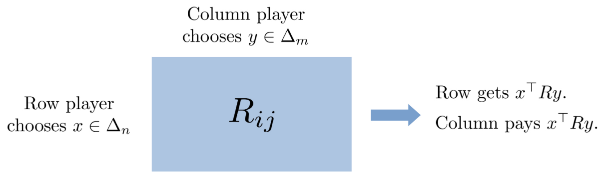
4 Zero-sum games

Definition 5 ***Zero-sum games*** *can be characterized as below:*

- *There are two players, namely row-player and column-player.*
- *Both players have finite strategies to choose from. Without loss of generality the row-player has n available strategies and the column-player has m available strategies.*
- *The payoff of two players sum up to zero.*

The payoff matrices are $n \times m$ matrices that represent the payoff of each player based on the chosen strategies. Since the payoff of two players sum up to zero, it is enough to show the payoff matrix of the row-player. The payoff matrix of the column player is derived from negating the payoff matrix of the row-player($C = -R$). In the following figure, R is the $n \times m$ payoff matrix of the row-player.





Example 4 *Consider the following zero-sum game: there are two candidates aiming for presidency. The row-player should choose one strategy between Economy and Education and the column-player should choose between Tax-cut and Society. Based on different combinations, each of the candidates receive a specific utility depicted in the following payoff matrix. Note that in each cell of the matrix, the first value is the utility of the row-player and the second value is the utility of the column-player.*

	Tax-cuts	Society
Economy	**3, -3**	**-1,1**
Education	**-2, 2**	**1,-1**

Consider the following cases:

- *Assume that the row player plays Economy with probability x_{11} and plays Education with probability x_{12}. Now if the column player plays Tax-cuts, his expected utility will be $-3x_{11} + 2x_{12}$, and if he chooses Society, his expected utility will be $x_{11} - x_{12}$. The column player will choose the strategy that maximizes his utility, therefore his* *utility will be* $\max\{-3x_{11} + 2x_{12}, x_{11} - x_{12}\}$*. Because this is a zero-sum game, the utility of the row-player is negation of the utility of the column-player which is* $-\max\{-3x_{11}+2x_{12}, x_{11}-x_{12}\} = \min\{3x_{11}-2x_{12}, -x_{11}+x_{12}\}$*. Now the row-player will choose* (x_{11}, x_{12}) *in a way that she maximizes her own utility, therefore:*

$$(x_{11}^*, x_{12}^*) = \arg\max_{x_{11},x_{12}} \min\{3x_{11} - 2x_{12}, -x_{11} + x_{12}\} \tag{8}$$

This optimization problem can be written in the form of the following LP:

Linear Program for Row player

$$\begin{aligned} &\max z \\ \text{s.t } &3x_{11} - 2x_{12} \geq z \\ &-x_{11} + x_{12} \geq z \\ &x_{11} + x_{12} = 1 \\ &x_{11}, x_{12} \geq 0 \end{aligned}$$

- *Assume that the column-player plays Tax-cuts with probability* x_{21} *and Society with probability* x_{22}*. If row-player plays Economy her expected utility will be* $3x_{21} - x_{22}$ *and if she plays Education her expected utility will be* $-2x_{21}+x_{22}$*. Row-player's best response would result in the following utility for her:* $\max\{3x_{21} - x_{22}, -2x_{21} + x_{22}\}$*. Because this is a zero-sum game, column player's utility is the negation of the row-player's utility, which is* $\min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}$*. The column-player chooses his strategy in a way to maximize his utility, therefore:*

$$(x_{21}^*, x_{22}^*) = \arg\max_{x_{21},x_{22}} \min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\} \tag{9}$$

This optimization problem can be written in the form of the following LP:

Linear Program for column player

$$\begin{aligned} &\max w \\ \text{s.t } &-3x_{21} + x_{22} \geq z \\ &2x_{21} - x_{22} \geq z \\ &x_{21} + x_{22} = 1 \\ &x_{21}, x_{22} \geq 0 \end{aligned}$$

In the next lecture, we will see that the two linear programs shown above are dual and they will both give us the same solution, which is the Nash Equilibrium.

The Nash Equilibrium of other Zero-sum games with even higher dimension of strategy spaces can be found using similar techniques used in the example above. In other words, for all zero-sum games, the problem of finding the Nash Equilibrium can be reduced to solving a linear program. Since we can solve these LPs fast, finding Nash Equilibrium in Zero-sum games are not computationally hard.