CS295 Introduction to Algorithmic Game Theory

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Lecture 12. Monotone Allocations and Myerson's Lemma

1 Introduction

In the previous lecture, we introduce examples of auctions. We want the auctions to have three properties. Dominant Strategy Incentive Compatible (DSIC), social surplus maximization, and implementable in polynomial time. In real life, we have infinitely many mechanisms to choose from. Which mechanism could satisfy our desire? The key question is under what condition a mechanism could induce DSIC properties. We can think about this question in the following way. We first assume truthful bids and identify how to allocate items to bidders to maximize social welfare. Then we derive the appropriate selling prices, to render truthful bidding a dominant strategy. In this lecture, we introduce a necessary and sufficient condition that a mechanism could induce the DSIC properties. And the short answer to this question is that if we have (1) monotone allocation, (2) the bidder bidding zero pay nothing, we could have DSIC mechanism.

The previous short answer is Myerson's lemma. In this lecture, we first define allocation and payment, section 2. Then we introduce Myerso n's lemma, section 3. Finally, we apply Myerson's lemma to the sponsored search auctions as a special case. And we give an example to construct the payment rules inducing DSIC using Myerson's lemma, section 4.

2 Single-Parameter Environments

Generally, the mechanism can be abstracted into three elements. The payers and the goods, the allocation rules, and the payment rules. Here, we would start from a single-parameter environment, a simplified version of players and goods.

Definition 2.1 (Single parameter environments) *A single parameter environment is defined as the following:*

- *n bidders with private value v_i,*
- *Feasible set $\mathcal{X}$, each element of which is an n-dimensional vector $(x_1, \cdots, x_n)$ in which x_i is the amount of "stuff" given to i.*

Here we give some examples of the single parameter environments.

Example 2.1 (Single-Item Auction) *In a single-item auction, X is the set of 0-1 vectors that have at most one 1. That is, $\sum_{i=1}^{n} x_1 \leq 1$.*

Example 2.2 (k-Unit Auction) *With k identical items and the constraint that each bidder gets at most one, the feasible set is the set of 0-1 vectors that satisfy* $\sum_{i=1}^{n} x_1 \leq k$

Example 2.3 (Sponsored Search Auction) *In a sponsored search auction, X is the set of n-vectors corresponding to assignments of bidders to slots, where each slot is assigned to at most one bidder and each bidder is assigned to at most one slot. If bidder i is assigned to slot j, then the component x_i equals the click-through rate α_j of her slot.*

At this level of generality, a mechanism is a general procedure for making a decision when agents have private information (like valuations), whereas an auction is a mechanism specifically for the exchange of goods and money. In the following we define the other two elements of a mechanism, allocation rules and the payment rules.

Definition 2.2 (Allocation Rule) *An allocation rule is a mapping $x : \mathcal{B} :\to \mathcal{X}$, where $\mathcal{B} \subset \mathbb{R}^n$ is the set of all possible bids. An allocation is called monotone if for every bidder i, and bids b_{-i} from the rest of the bidders,*

$$x_i(z, b_{-i}) \text{ is non-decreasing in } z$$

Definition 2.3 (Payment Rule) *A payment rule is a mapping $p : \mathcal{X} \to \mathbb{R}^n$.*

In another word, the allocation rules are a function that assigns each bidder the probability of getting such goods. And the payment rule is how much each bidder will pay if he was assigned a probability of getting a good.

Example 2.4 *A sealed-bid auction consists of the following:*

- *Bidders report their bids. A profile of bid $b = (b_1, \dots, b_n)$.*
- *Auctioneer implements the allocation $x(b) \in \mathcal{X}$.*
- *Auctioneer sets the payment rule $p(b) \in \mathbb{R}^n$.*
- *Bidder i's gets utility $u_i(b) = v_i \cdot x_i(b) - p(b)$.*

3 Myerson's Lemma

In [1], Myerson characterizes the DSIC mechanisms by the following result.

Theorem 3.1 (Myerson's Lemma) *Let (x, p) be a mechanism. We assume that $p_i(b) = 0$ whenever $b_i = 0$ for all bidders i.*

(i) If (x, p) is a DSIC mechanism, then this x is monotone.

(ii) If an allocation rule x is monotone, then there exists a unique payment rule p such that (x, p) is a DSIC mechanism.

Proof: To show (i), suppose (x, p) is a DSIC. Fix an agent i and b_{-i} from the rest of the bidders. Let $0 \leq y \leq z$. Suppose bidder i has private valuation y. DSIC condition requires that bidder i cannot deviate to reporting z. Formally,

$$yx_i(y, b_{-i}) - p_i(y, b_{-i}) \geq yx_i(z, b_{-i}) - p_i(z, b_{-i}) \tag{1}$$

Similarly, if bidder i has private valuation of z, then

$$zx_i(z, b_{-i}) - p_i(z, b_{-i}) \geq zx_i(y, b_{-i}) - p_i(y, b_{-i}) \tag{2}$$

Rearranging and combing (1) and (2) gives

$$y[x_i(y, b_{-i}) - x_i(z, b_{-i})] \geq p_i(y, b_{-i}) - p_i(z, b_{-i}) \geq z[x_i(y, b_{-i}) - x_i(z, b_{-i})] \tag{3}$$

Further rearrangement of (3) gives

$$(y - z)[x_i(y, b_{-i}) - x_i(z, b_{-i})] \geq 0$$

Since, by assumption $y \geq z$, we can conclude that $x_i(y, b_{-i}) \geq x_i(z, b_{-i})$ and thus x is a monotone allocation rule.

Now we show (ii). Assume x is a monotone allocation. Our first goal is to derive a payment rule p so that (x, p) is DSIC. Suppose that x_i is a non-decreasing piecewise constant left-continuous function.[1] Fix z in (3), and let y tend to z from above. If $x_i(\cdot, b_{-i})$ is continuous at z, as $y \to z$, both sides of (3) tends to zero, so $p_i(\cdot, b_{-i})$ is also continuous at such z. Now suppose there is the jump of $x_i(\cdot, b_{-i})$ at z with size $h(z)$. Then, as $y \to z$, both sides of (3) equals to $z \cdot h(z)$. It follows that

$$\text{jump in } p(\cdot, b_{-i}) \text{ at } z \;=\; z \cdot [\text{jump in } x_i(\cdot, b_{-i}) \text{ at } z].$$

This implies that $p_i(\cdot, b_{-i})$ is also a right-continuous non-decreasing piecewise constant function that has the same discontinuity points as $x_i(\cdot, b_{-i})$. As we normalize $p_i(0) = 0$, the explicit formula for this payment rule is

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot [\text{jump in } x_i(\cdot, b_{-i}) \text{ at } z_j], \tag{4}$$

where $z_1, \ldots, z_l$ is the breakpoints of $x_i(\cdot, b_{-i})$ that lies in $[0, b_i]$.

Now consider the case where $x_i(\cdot, b_{-i})$ is an arbitrary non-decreasing function. A result from real analysis states that every non-decreasing real-valued function is differentiable almost everywhere. Suppose $x_i(\cdot, b_{-i})$ is differentiable at z. Dividing (3) by $y - z$ and taking limits $y \to z$ gives

$$p_i'(z, b_{-i}) = zx_i'(z, b_{-i}). \tag{5}$$

[1] If we assume this step function right-continuous, a similar argument will go through if we modify this proof by finding some y tends to z from below.

By the fundamental theorem of calculus[2] in that case.

$$p_i(b_i, b_{-i}) = p_i(b_i, b_{-i}) - p_i(0, b_{-i}) = \int_0^{b_i} p_i'(z, b_{-i})dz = \int_0^{b_i} zx_i'(z, b_{-i})dz. \quad (6)$$

Note that this unique payment rule depends on the normalization of $p_i(0, b_{-i}) = 0$. Another remark here is that the payment rule (6) is essentially the limit version of the payment rule (4).

The rest of the proof is to show that (x, p) is indeed a DSIC mechanism. We will give a proof by picture for the case where $x_i(\cdot, b_{-i})$ is piecewise constant.

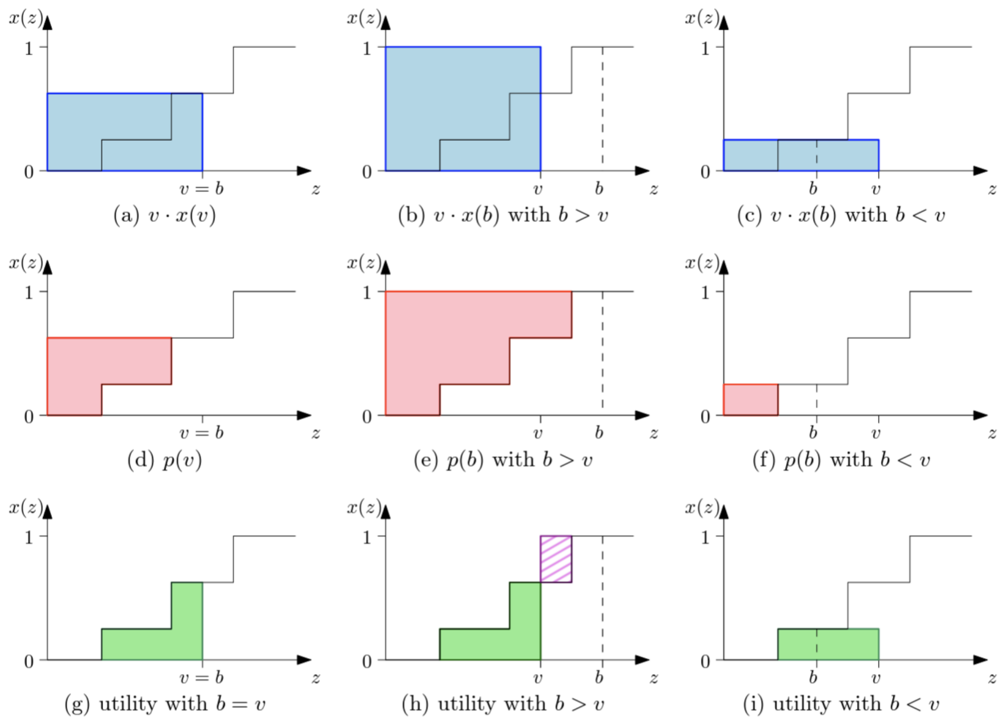

Figure 1: A pictorial proof of monotone allocation x is DSIC when x_i is piecewise constant and p_i is defined by (4). The allocation rule is in the first row, the payment rule is in the second row, and the resulting utility is in the third row. The first column corresponds to the case that the bidder bids truthfully, the second column is when the bidder overbids and the third column is when the bidder underbids. The highest utility results from truthfully bidding.

[2]In particular, we require functions in (5) to be *absolutely continuous* (see: `https://en.wikipedia.org/wiki/Absolute_continuity#Equivalent_definitions`) so that the fundamental theorem of calculus could be applied. Essentially, monotone functions that are not absolutely continuous are very similar to piecewise continuous functions so that we can construct a payment rule similar to (4)

It is not hard to show that (p, x) is a DSIC mechanism for the case where x_i that is differentiable and associates with a payment rule (6). By integration by parts, (6) is equivalent to

$$p_i(b_i, b_{-i}) = zx_i(z, b_{-i})\Big|_0^{b_i} - \int_0^{b_i} x_i(z, b_{-i})dz = b_i x_i(b_i, b_{-i}) - \int_0^{b_i} x_i(z, b_{-i})dz.$$

Denote the utility of bidder i with private valuation z by $u_i(z, b_{-i}) = zx_i(z, b_{-i}) - p(z, b_{-i})$. One consequence of (3) is

$$\begin{aligned} u_i(y, b_{-i}) - u_i(z, b_{-i}) &= yx_i(y, b_{-i}) - p(y, b_{-i}) - [zx_i(z, b_{-i}) - p(z, b_{-i})] \\ &= yx_i(y, b_{-i}) - zx_i(z, b_{-i}) + p(z, b_{-i}) - p(y, b_{-i}) \\ &= yx_i(y, b_{-i}) - zx_i(z, b_{-i}) + \left[\left(zx_i(z, b_{-i}) - \int_0^z x_i(s, b_{-i})ds\right) - \left(yx_i(y, b_{-i}) - \int_0^y x_i(s, b_{-i})ds\right)\right] \\ &= \int_z^y x_i(s, b_{-i})ds. \end{aligned} \tag{7}$$

Suppose bidder i has private valuation z. The gain in deviation to reporting y is

$$\begin{aligned} zx_i(y, b_{-i}) - p(y, b_{-i}) - u_i(z, b_i) &= yx_i(y, b_{-i}) - (y - z)x_i(y, b_{-i}) - p(y, b_{-i}) - u_i(z, b_{-i}) \\ &= u_i(y, b_{-i}) - u_i(z, b_i) - (y - z)x_i(y, b_{-i}) \\ &= \int_z^y x_i(s, b_{-i})ds - \int_z^y x_i(y, b_{-i})ds. \end{aligned} \tag{8}$$

Suppose $y \geq z$. Since x_i is monotone, for all $z \leq s \leq y$, we have $x_i(s, b_{-i}) \leq x_i(y, b_{-i})$. Therefore, (8) is negative since the integrand is negative. It follows that bidder i will not deviate to reporting y due to negative gain.
Now suppose $y \leq z$. By a similar argument, after rewriting (8), we can conclude that

$$gain = \int_y^z -x_i(s, b_{-i})ds - \int_y^z -x_i(y, b_{-i})ds = \int_y^z x_i(y, b_{-i})ds - \int_y^z x_i(s, b_{-i})ds \leq 0.$$

Hence, in all cases, there is no incentive for bidder not to report truthfully, i.e., (x, p) in this case is DSIC. This completes the proof of Myerson's lemma. ■

4 Applied Myerson's Lemma to Sponsored Search Auctions

The explicit payment rule given by Myerson's lemma is easy to understand and apply in many applications. In this section, we applied Myerson's Lemma to argue that such a payment rule in sponsored search auction will produce the DSIC. But let's first check the single-item auctions.

Lemma 4.1 *In the single-item auction, Myerson's lemma regenerates the Vickrey auction (second-price payment rule) as a special case.*

Proof: Fixing a bidder i and bids $\mathbf{b}_{-i}$ by the other bidders. Let the function $x_i(z, \mathbf{b}_{-i})$ is 0 up to $B = \max_{j \neq i} b_j$ and 1 thereafter. Myerson's payment formula for such piecewise constant functions is the following

$$p_i\left(b_i,\ \mathrm{b}_{-i}\right)=\sum_{j=1}^{\ell} z_j \cdot\left[\text { jump in } x_i\left(\cdot, \mathrm{b}_{-i}\right) \text { at } z_j\right],$$

where $z_1, \ldots, z_l$. are the breakpoints of the allocation function $xi(\cdot, \mathbf{b}_{-i})$ in the range $[0, b_i]$. For the highest-bidder single-item allocation rule, this is either 0 (if $b_i < B$) or, if $b_i > B$, there is a single breakpoint (a jump of 1 at B) and the payment is $p_i(b_i, \mathbf{b}_{-i}) = B$. Thus, Myerson's lemma regenerates the second-price payment rule as a special case. ■

Then we want to check a more complicated case, sponsored search auction. Suppose the sponsored search auction is the following settings, with k, slots with click-through rates $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. We want to ask two questions (1) How should we assign bidders to slots so that we can maximize surplus? (2) Given step 1, how we should set selling prices so that DSIC holds?

Lets first answer the first question. Let $\mathbf{x}(\mathbf{b})$ be the allocation rule that assigns the ith highest bidder to the ith highest slot, for $i = 1, 2, \ldots, k$. We claim that this rule is monotone, and assumes truthful bids, welfare-maximizing. This rule is monotone by definition since the ith highest bidder was assigned to the ith highest slot. This is welfare maximizing since if the bidder bids truthfully, the highest slot was assigned to the highest-value bidder.

Then we want to answer the question, how we should set the selling price so that DSIC holds. By Myerson's lemma, if we have the monotone allocation, we have a unique payment rule such that (x, p) is DSIC. Let's consider a payment rule, $b_1 \geq \cdots \geq b_n$. We first focus on the highest bidder and increase her bid from 0 to b_1, holding the bother bids fixed. The allocation $x_1(z, \mathbf{b}_{-1})$ increases from 0 to α_1 as z increase from 0 to b_1, with a jump of $\alpha_j - \alpha_{j+1}$ at the point where z becomes the jth highest bid in the profile $(z, \mathbf{b}_{-1})$, namely b_{j+1}. Then the payment function is given by

$$p_i(b)=\sum_{j=i}^k b_{j+1}\left(a_j-a_{j+1}\right)$$

for the ith highest bidder.

References

[1] Myerson, R.B., 1981. Optimal auction design. *Mathematics of operations research*, 6(1), pp.58-73.