CS295 Introduction to Algorithmic Game Theory

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Lecture 11. Intro to Mechanism Design.

1 Motivation

Definition 1.1 *Single-item allocation problem*

- one indivisible item
- n agents competing for the item
- each agent i has a personal valuation v_i for the item

The goal of the single-item allocation problem is to maximize the social surplus, which is defined as the valuation of the agent that receives the item.

Protocol 1.1 *Highest bidder*

Suppose the agents are asked to report bids b_i, and the item is given to the agent i^* with the highest bid (i.e. $i^* = \text{argmax}_i\ b_i$).

With this mechanism, the agents have no incentive to bid truthfully. In particular, any agent with a $v_i > 0$ is incentivized to bid as highly as possible. Therefore, this mechanism will produce very unpredictable outcomes and will not do a good job of maximizing the social surplus.

Protocol 1.2 *Lottery*

Suppose an agent is chosen uniformly at random and given the item.

Clearly, this mechanism also does a poor job of maximizing the social surplus. To reason about how poor, we must define the concept of approximation ratios for mechanisms.

Definition 1.2 *Approximation ratio of a mechanism*

$$\text{approximation ratio} = \frac{\text{optimal social surplus}}{\mathbb{E}[\text{social surplus of mechanism outcome}]}$$

Theorem 1.3 *Lottery is an n-approximation*

Proof: The worst-case scenario is that one agent has a high valuation and all others have valuations of almost 0.

Suppose $v_1 >> \epsilon$ and $v_i = \epsilon$ for all $i \geq 2$. The optimal social surplus is clearly v_1. The expected social surplus of the lottery mechanism is:

$$\frac{1}{n}(v_1 + (n-1)\epsilon)$$

The approximation ratio is:

$$\frac{v_1}{\frac{1}{n}(v_1 + (n-1)\epsilon)}$$

which approaches n as $\epsilon \to 0$. ■

2 Auctions

In order to create mechanisms with predictable and desirable behavior, we must introduce **payments**, which will discourage agents from making untruthfully high bids.

Protocol 2.1 *First-price auction*

Just as in Protocol 1.1, agents report bids b_i, and the item is given to the agent i^* with the highest bid (i.e. $i^* = \text{argmax}_i\, b_i$). However, i^* must also pay the amount they bid, b_{i^*}.

Definition 2.1 *Agent utility in an auction*

If agent i gets the item and pays p_i, then their utility is $u_i = v_i - p_i$. Otherwise, their utility is $u_i = 0$.

This protocol represents a significant improvement over Protocol 1.1, since agents are at least never incentivized to bid higher than their valuations. Other than that, though, it is hard to reason about first-price auctions. Agents want to bid lower than their valuations to maximize their utility, but they don't want to bid lower than the other agents and miss out on winning the item. For the auction designer, it is hard to predict what will happen.

Protocol 2.2 *Second-price auction, aka Vickrey auction*

- agents report bids b_i
- the item is given to the highest bidder $i^* = \text{argmax}_i\, b_i$
- i^* pays the amount of the second highest bid $p_{i^*} = \max_{j \neq i^*} b_j$

Theorem 2.3 *Second-price auctions are truthful*

In a second-price auction, it is a dominant strategy for every agent to bid exactly their true valuation (i.e. $b_i = v_i$). Recall the definition of dominant: an agent's strategy is dominant if it always maximizes the utility of that agent regardless of what the other agents do. Dominant strategies are better than Nash equilibria, because they obviate the need for agents to predict other agents' behaviors.

Proof:

Consider any agent i and define $B = \max_{j \neq i} b_j$.

- Case 1. $v_i < B$:
  - If $b_i = v_i$ (i.e. i bids truthfully), then agent i will not win the item. $u_i = 0$.
  - If $b_i < v_i$, then agent i will still not win. $u_i = 0$.
  - If $B > b_i > v_i$, then agent i will still not win. $u_i = 0$.
  - If $b_i \geq B > v_i$, then agent i will win, but their utility will be negative. $u_i = v_i - B < 0$.
- Case 2. $v_i \geq B$:
  - If $b_i = v_i$ (i.e. i bids truthfully), then agent i will win the item. $u_i = v_i - B \geq 0$.
  - If $bi > v_i$, then agent i will still win and pay the same amount. $u_i = v_i - B \geq 0$.
  - If $v_i > b_i \geq B$, then agent i will still win and pay the same amount. $u_i = v_i - B \geq 0$.
  - If $v_i \geq B > b_i$, then agent i will not win. $u_i = 0$.

In no case can agent i increase their utility by doing something other than bidding truthfully. ■

Below we provide an alternative proof of the second-price auctions are DSIC.

Theorem 2.4 ***Vickrey is truthful.*** *In second price auctions, every bidder i has a dominant strategy that is bid truthfully (set the bid $b_i = v_i$). Dominant means the bid maximize the utility $b_i = \arg\max_{\hat{b}_i}(v_i - p_i)\mathbb{I}(\hat{b}_i = b_{i^*})$.*

Proof: Fix an agent i and set $B = \max_{j \neq i} b_j$ to be the second highest bid. Consider the two cases:

1. $b_i < B$ then the utility $(v_i - p_i)\mathbb{I}(b_i = b_{i^*}) = 0$.
   Therefore no matter what b_i is, the utility will be the same.
2. $b_i > B$ then the utility $(v_i - p_i)\mathbb{I}(b_i = b_{i^*}) = (v_i - B)$.
   - Suppose $b_i > B > v_i$, the utility will be negative
   - Suppose $b_i > v_i > B$, the utility will be $v_i - B \geq 0$.
   - Suppose $b_i < v_i$, the utility will be $v_i - B \geq 0$.
   - Suppose truthful $b_i = v_i$, the utility will be $v_i - B \geq 0$

Therefore, the utility of truthful report is $(v_i - B)\mathbb{I}(v_i > B)$, optimistic report $b_i > v_i$ is $(v_i - B)\mathbb{I}(b_i > B)$, conservative report $b_i < v_i$ is $(v_i - B)\mathbb{I}(b_i > B)$. To see why truthful report gets the max utility, considering any fixed B, v_i, $(v_i - B)\mathbb{I}(v_i > B) \geq (v_i - B)\mathbb{I}(b_i > B)$ and the equality holds if and only if $v_i = b_i$. ■

3 Quasi-Linear Environments

Inspired by the properties that the second-price auction preserved, now we are ready to introduce the framework for general auction. In fact, the general auction should preserve three properties:

- Dominant strategy incentive compatible (DSIC), i.e., truthtelling is dominant strategy.

- If bidders are truthful, auction maximizes surplus $\sum_{i=1}^{n} v_i x_i$, where $x_i = \mathbb{I}(\text{i wins the item})$
- The auction can be implemented in polynomial time.

Example 3.1 ***Problem:*** *Consider a society of n citizens and public good G.*

- *Each agent has valuation v_i for the good.*
- *Cost of building G is C.*
- *G should be built if $\sum_{i=1}^{n} v_i > C$*

Goal: *Design a mechanism that charges citizens in a way the G is built only if $\sum_{i=1}^{n} v_i > C$.*
Solution: *Charge citizen i the amount $p_i := \max(0, C - \sum_{j \neq i} b_i)$. Similarly can be shown that is DSIC.*

Definition 3.1 ***Quasi-linear environment****, aka Vickrey-Clarke-Groves (VCG) environment:*

- *n agents,*
- *Set of outcomes $\mathcal{X}$*
- *Each agent i has a valuation $v_i : \mathcal{X} \to \mathbb{R}^+$*
- *Each agent has utility $u_i = v_i - p_i$ where p_i is the receive payment (positive or negative).*

Definition 3.2 ***Vickrey-Clarke-Groves (VCG) mechanism****. The family of mechanisms is defined as follows:*

- *Agents have valuations v_i and report their bids b_i.*
- *Set $x^* = \arg\max_{x \in \mathcal{X}} \sum_{i=1}^{n} b_i(x)$.*
- *Each agent pays $p_i := h_i(b_{-i}) - \sum_{j \neq i} b_j(x^*)$*
- *Each agent has utility $u_i = v_i(x^*) - p_i(x^*)$.*

Theorem 3.2 *VCG is DSIC. Every VCG mechanism is DSIC*

Proof: Fix an agent i and let $x^* = \arg\max_{i=1}^{n} v_i(x)$. Assume that i reports $b_i \neq v_i$ and x' be the maximum if i report untruthfully b_i. $X' = \arg\max \sum_{j \neq i} v'_j(x)$. Considering the utility of different allocation outcomes $u_i(x^*), u_i(x')$

$$u_i(x^*) = v_i(x^*) + \sum_{j \neq i} v_j(x^*) - h_i(v_{-i}) u_i(x') = v_i(x') + \sum_{j \neq i} v_j(x') - h_i(v_{-i})$$

It is not hard to see $u_i(x^*) - u_i(x') \geq 0$. Therefore, we proofed the truthful report is the dominant strategy in VCG mechanism. ■

Corollary 3.3 *The payment method* $p_i := \max(0, C - \sum_{j \neq i} b_i)$ *is VCG mechanism, since if we change the* $h_i(b_{-i}) = C$ *in Definition 3.2 the results hold.*

Next we will show hold to choose an appropriate h_i for the mechanism design. And the intuitive idea lies in the utility should not be negative.

Definition 3.3 ***Clarke pivot*** *This suggest we use* $h_i(b_{-i}) = \max_x \sum_{j \neq i} b_i(x)$

Remark 3.4 .

1. *The utility of the highest valuation is non-negative. This could be seen from* $u_i(x^*) = \sum_{j=1}^{n} v_i(x^*) - \max_x \sum_{j \neq i} b_i(x) = \max_x \sum_{j \in [n]} v_i(x) - \max_x \sum_{j \neq i} v_i(x) \geq 0$
2. *The second-price auction is a special case of Clarke pivot.*