L09
Introduction to Multi-armed Bandits

50.579 Optimization for Machine Learning
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The framework

Setting. *We are given K arms and time window T (known). At each time step* $t = 1...T$.

- *Player chooses arm* a_t.
- *Observes reward* $r_t \in [0, 1]$ *for the chosen arm.*

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- The algorithm observes only the reward for the selected action, and nothing else.
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Goal: Minimize the regret

$$R(T) = \mu^* T - \sum_{t=1}^{T} \mu(a_t) \text{ or } \mathbb{E}[R(T)].$$

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1. Exploration phase: try each arm N/K times.
2. Select the arm a^* with the highest average reward (break ties arbitrarily).
3. Exploitation phase: Play a^* in all remaining $T - N$ rounds.

Remarks:

- N will be chosen later as a function of T, K.

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Let's analyze the regret for Explore-first algorithm!

Analysis of Explore-First

Remark (Hoeffding Inequality). *Let $\hat{\mu}(a)$ be the empirical (or average) reward for action a after exploration phase. It holds*

$$\Pr\left[|\hat{\mu}(a) - \mu(a)| \leq \sqrt{\frac{2K\log T}{N}}\right] \geq 1 - \frac{1}{T^4}.$$

$$\Pr[|\hat{\mu}(a) - \mu(a)| > \epsilon] \leq 2e^{-2\frac{N}{K}\epsilon^2}.$$

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Let us condition on the "clean" event that the above holds for all arms. By union bound the probability of the "bad" event is at most

$$\frac{K}{T^4} \leq \frac{1}{T^3},$$

hence the "clean" event happens with probability at least $1 - \frac{1}{T^3}$.

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But since we condition on "clean event"

$$\mu(a^*) + \sqrt{\frac{2K \log T}{N}} \geq \hat{\mu}(a^*) \geq \hat{\mu}(a_{best}) \text{ and}$$

$$\hat{\mu}(a_{best}) \geq \mu(a_{best}) - \sqrt{\frac{2K \log T}{N}}.$$

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Hence $\mu(a^*) \geq \mu(a_{best}) - 2\sqrt{\frac{2K \log T}{N}}$.

Analysis of Explore-First

We compute a bound on the regret (conditioned on clean event):

$$R(T) \leq N + (T - N) \times 2\sqrt{\frac{2K \log T}{N}}$$

$$\leq N + \sqrt{\frac{8KT^2 \log T}{N}}$$

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We set $N = 2T^{2/3}(K \log T)^{1/3}$ and we have

$$R(T) \leq 4T^{2/3}(K \log T)^{1/3}$$

Analysis of Explore-First

Using law of total expectation we have

$$\begin{aligned}\mathbb{E}[R(T)] &= \mathbb{E}[R(T)|\text{clean}]\Pr[\text{clean}] + \mathbb{E}[R(T)|\text{bad}]\Pr[\text{bad}] \\ &\leq 4(K\log T)^{1/3}T^{2/3} + T \times \frac{1}{T^3} = O((K\log T)^{1/3}T^{2/3}).\end{aligned}$$

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Namely, we showed:

Theorem (Regret). *Explore-first algorithm achieves regret*

$$O((K\log T)^{1/3}T^{2/3}),$$

where K is the number of arms.

Epsilon-Greedy

Definition (ϵ-greedy)**.** *Consider the following algorithm:*

1. **For** t=1 ... T **do**
2. **Toss** a coin with success prob ϵ_t.
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$$\mathbb{E}[R(t)] \text{ to be } O((K \log t)^{1/3} t^{2/3}),$$

where K is the number of arms and $\epsilon_t \sim t^{-1/3}(K \log t)^{1/3}$.

Remarks:

- Same regret as before but for all rounds!

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Can we do better? Yes, adaptive exploration!

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Upper Confidence Bounds

One natural idea (suppose we have two arms): Alternate them until we find that one arm is much better than the other, at which time we abandon the inferior one.

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Recall (Hoeffding). *Let $n_t(a)$ be the number of samples from arm a in round $1, \ldots, t$, $\hat{\mu}_t(a)$ be the average reward of arm a so far. Hoeffding Inequality suggests*

$$\Pr[|\hat{\mu}_t(a) - \mu(a)| \leq r_t(a)] \geq 1 - \frac{2}{T^4},$$

where $r_t(a) = \sqrt{\frac{2 \log T}{n_t(a)}}$, and $r_t(a)$ is called the confidence radius.

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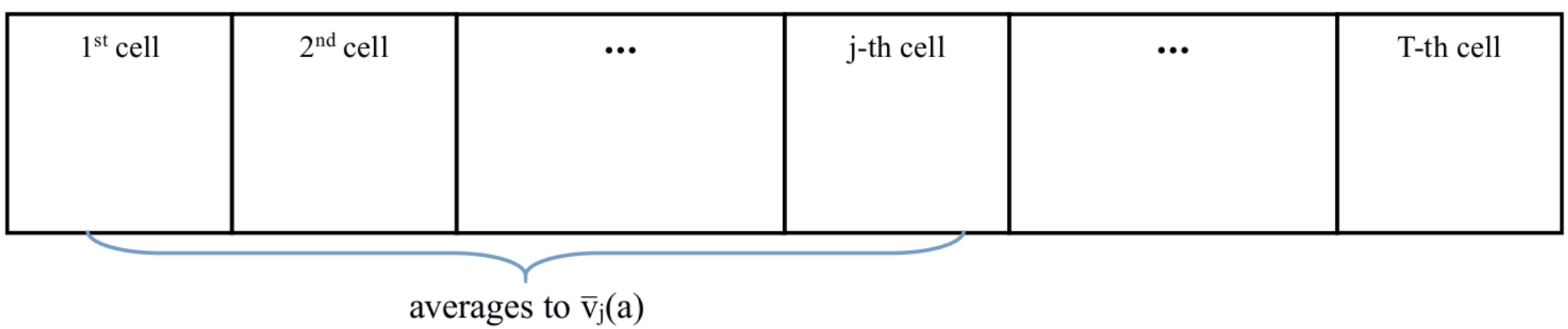
However $n_t(a)$ should not be fixed (r.v)... Samples Are not independent anymore!

Upper Confidence Bounds

For each arm a, imagine there is a reward tape $1 \times T$ table with each cell independently sampled from D_a. The j-th time a given arm a is chosen by the algorithm, its reward is taken from the j-th cell in this arm's tape.



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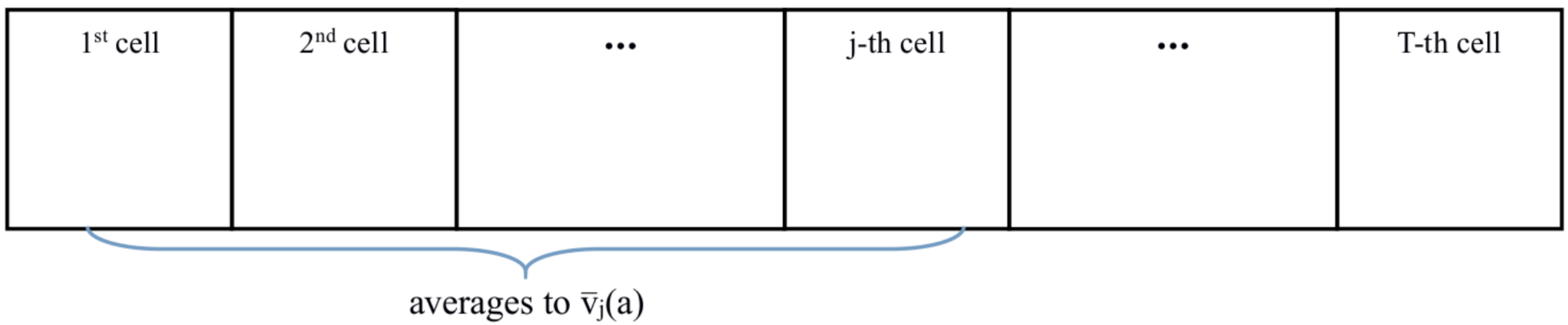

Now we can use Hoeffding Inequality hence for all j

$$\Pr[|\hat{v}_j(a) - \mu(a)| \leq r_j(a)] \geq 1 - \frac{2}{T^4},$$

therefore by union bound on j and arms we get

$$\Pr[\forall j, a \; |\hat{v}_j(a) - \mu(a)| \geq r_j(a)] \geq 1 - \frac{1}{T^2},$$

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Definition (Confidence bounds). *We define upper/lower confidence bounds for every arm a and round t*

$$UCB_t(a) = \hat{\mu}_t(a) + r_t(a), \;\; LCB_t(a) = \hat{\mu}_t(a) - r_t(a).$$

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Much better than before!

Analysis of UCB Elimination

Let us define the "clean" event (we condition on that)

$$\mathcal{E} = \{\forall j, a \; |\hat{\mu}_j(a) - \mu(a)| \le r_j(a)\}.$$

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Observe that the disqualified arm cannot be the best arm. How long did it take to disqualify it?

Let τ be the last round when we did not invoke the stopping rule, namely when the confidence intervals of the two arms still overlap. It holds that

$$|\mu(a) - \mu(a')| \leq 2(r_\tau(a) + r_\tau(a'))$$

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Moreover because we alternated we have $n_\tau(a) = n_\tau(a') = \frac{\tau}{2}$ hence

$$r_\tau(a) \text{ and } r_\tau(a') \text{ are } O\left(\sqrt{\frac{\log T}{\tau}}\right).$$

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\mathbb{E}[R(T)] &= \mathbb{E}[R(T)|\mathcal{E}]\Pr[\mathcal{E}] + \mathbb{E}[R(T)|\not\mathcal{E}]\Pr[\not\mathcal{E}] \\
&\leq \Delta \times \tau + T \times O\left(\frac{1}{T^2}\right).
\end{aligned}
$$

The above gives $O(\sqrt{T \log T})$.

More than two arms

Definition (UCB Elimination). *Consider the following algorithm:*

1. Initially all arms are set "active";
2. Try all active arms once.
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Theorem (Regret). *UCB Elimination algorithm achieves regret*

$$\mathbb{E}[R(T)] \textit{ to be } O(\sqrt{KT \log T}).$$

Remarks:

- The proof is almost the same as before. Try to prove it alone.

Conclusion

- Introduction to Multi-armed bandits.
  - Explore-first.
  - Epsilon-greedy
  - UCB Elimination

- Next lecture we will talk more about Exploration-Exploitation tradeoff.