Recent advances in computing Nash equilibria in Markov Games

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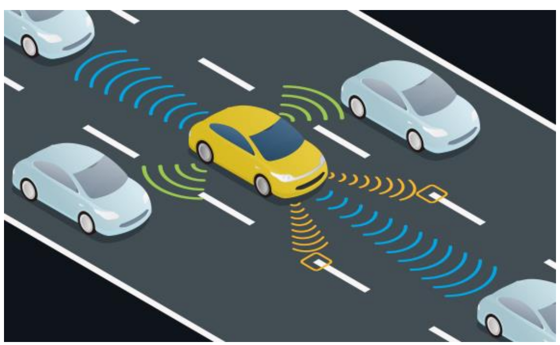
Based on joint works with F.Kalogiannis, I.Anagnostides, S.Leonardos, W.Overman, M.Vlatakis, V.Chatziafratis, G.Piliouras and S.Stavroulakis

Multi-agent systems and RL

Decentralized systems

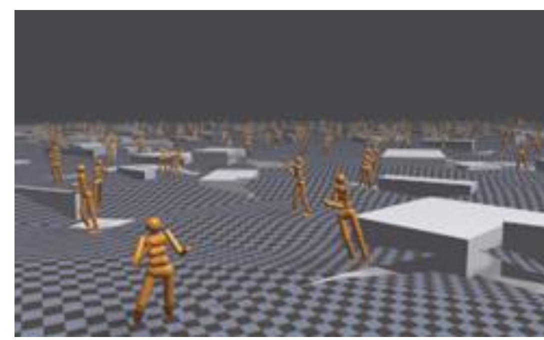
Individual interests (rational agents, cooperation/competition etc)

Distributed optimization



Self-driving cars

Auctions



Robotics

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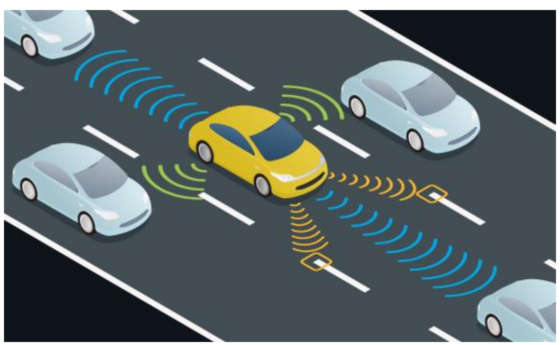

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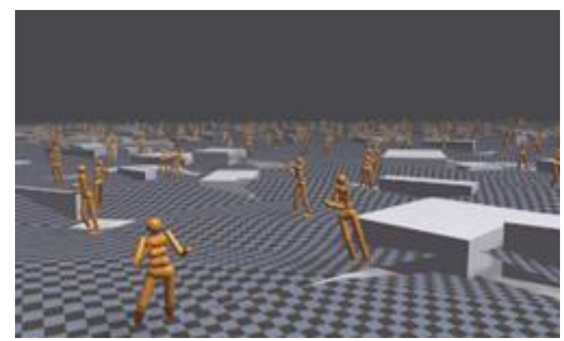

Robotics

How these systems evolve? Predictions?

Outline

- Basics on single agent RL and policy gradient
- Definitions and basics on two-player zero-sum games
- Potential Markov games
- Adversarial Markov team games
- Polymatrix Markov games
- Open Questions/Future projects

- *Single agent RL*

The framework

A finite Markov Decision Process (MDP) is defined as follows:

- A finite state space $\mathcal{S}$.
- A finite action space $\mathcal{A}$.
- A transition model $\mathbb{P}$ where $\mathbb{P}(s'|s, a)$ is the probability of transitioning into state s' upon taking action a in state s. $\mathbb{P}$ is a matrix of size $(S \cdot A) \times S$.
- Reward function $r : \mathcal{S} \times \mathcal{A} \to [-1, 1]$.
- A discounted factor $\gamma \in [0, 1)$.
- $\boldsymbol{\rho} \in \Delta(\mathcal{S})$, an initial state distribution.

Definitions

Definition (Markovian stationary policy). *Policy is called a function*

$$\pi : \mathcal{S} \to \mathcal{A}.$$

Definition (Value function). *Given a policy π the value function is given by*

$$V^{\pi}(\boldsymbol{\rho}) = \mathbb{E}_{\pi,\mathbb{P}}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) | s_0 \sim \boldsymbol{\rho}\right]$$

The goal is to solve

$$\max_{\pi} V^{\pi}(\boldsymbol{\rho}).$$

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Remarks

- The **max** operator is over all (possibly non-stationary and randomized) policies.
- It suffices to focus on deterministic.
- *V is not concave in π.*

Example

Example (Navigation). *Suppose you are given a grid map. The state of the agent is their current location. The four actions might be moving 1 step along each of east, west, north or south. The transitions in the simplest setting are deterministic. There is a goal g that is trying to reach. Reward is one if the agent reaches the goal and zero otherwise.*

0.729	0.81	0.9	☆
0.656		0.81	0.9
0.590	0.656	0.729	0.81

→	→	→	☆
↑		↑	↑
↑	→	↑	↑

Remark

- What is V?
- What is γ in the example?

Bellman operator

Definition (Bellman Operator). *Let's define the following operator $\mathcal{T}$:*

$$\mathcal{T}\, W(s) = \max_{a \in \mathcal{A}} \{ r(s,a) + \gamma \sum_{s'} \mathbb{P}(s'|s,a) W(s') \}$$

Set $V^*(s) := \max_{\pi} V^{\pi}(s)$.

Claim (Bellman Operator). *V^* is the unique fixed point of the operator.*

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Proof. Easy to see V^* is a fixed point. We will show that $\mathcal{T}$ is contracting! (Banach Fixed point Theorem).

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$$\|\mathcal{T}V - \mathcal{T}V'\|_\infty = \left\| \max_a \{ r(s,a) + \gamma \sum_{s'} \mathbb{P}(s'|a,s) V(s') \} - \max_{a'} \{ r(s,a') + \gamma \sum_{s'} \mathbb{P}(s'|a',s) V'(s') \} \right\|_\infty$$

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Bellman operator

$||x - y||_\infty \geq |||x||_\infty - ||y||_\infty|$

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$$= \gamma\left\|\max_a\{\mathbb{P}_a(V - V')\}\right\|_\infty$$

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$$||Ax||_\infty \leq ||A||_\infty ||x||_\infty$$

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$$= \gamma \left\| \max_a \{\mathbb{P}_a(V - V')\} \right\|_\infty$$

$$\leq \gamma \|V - V'\|_\infty \qquad \text{since } \|\mathbb{P}_a\|_\infty = 1.$$

Remarks

- Bellman operator is contracting for infinity norm.
- Applying the operator does not give a polynomial time algorithm. Why?
- Linear programming can give optimal policies in polynomial time.

- *Definition of Markov games and solution concepts*

n-player Markov game: Formal definition

Markov games or *stochastic* games are established as a framework for multi-agent reinforcement learning [Littman, 1994]

- $\mathcal{S}$, a finite state space,
- $\mathcal{N}$, a finite set of agents with $n := |\mathcal{N}|$,
- $\mathcal{A}_k$, a finite action space each player k, and $\mathcal{A} = \times_{k=1}^{n} \mathcal{A}_k$
- $r_k : \mathcal{S} \times \mathcal{A} \to [-1, 1]$, a reward function for each agent k,
- $\mathbb{P} : \mathcal{S} \times \mathcal{A} \to \mathcal{S}$ a transition probability function,
- $\gamma \in [0, 1)$, a discount factor,
- $\boldsymbol{\rho} \in \Delta(\mathcal{S})$, an initial state distribution.

Solution Concept

- Every agent k picks a *policy* $\pi_k : \mathcal{S} \to \Delta(\mathcal{A}_k)$ (*do not share randomness*)
- The goal of each agent k is to *maximize* their own value function:

$$V_k^{(\pi_k, \pi_{-k})}(\boldsymbol{\rho}) = \mathbb{E}_{\pi, \mathbb{P}} \left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \;\middle|\; s_0 \sim \boldsymbol{\rho} \right].$$

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An ϵ-approximate *Nash equilibrium (NE)* $\pi^* = (\pi_1^*, \ldots, \pi_n^*)$ means that no agent can unilaterally increase their expected value more than ϵ,

$$V_k^{\pi^*}(\boldsymbol{\rho}) \geq V_k^{(\pi'_k, \pi^*_{-k})}(\boldsymbol{\rho}) - \epsilon, \;\; \forall k \in \mathcal{N}, \forall \pi'_k.$$

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Remarks

- Fixing all agents but *i*, induces a classic MDP. Every agent aims at (approximate) best response.
- Generalizes notion of Nash Equilibrium.
- Nash (stationary, Markovian) policy always exist (Fink 64).
- Policies are defined to be *Markovian* and *stationary.*

Policy Gradient Iteration

Definition (Direct Parametrization). *Every agent uses the following:*

$$\pi_k(a \mid s) = x_{k,s,a}$$

with $x_{k,s,a} \geq 0$ *and* $\sum_{a \in A_k} x_{k,s,a} = 1$.

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Definition (Policy Gradient Ascent). *PGA is defined iteratively:*

$$x_k^{(t+1)} := \Pi_{\Delta(A_k)^S}(x_k^{(t)} + \eta \nabla_{x_k} V_k^{x^{(t)}}(\rho),$$

where Π denotes projection on product of simplices.

- *Two-player zero sum Markov games*

2-player zero-sum Markov games

– $\mathcal{N} = \{1, 2\}$, i.e., $n = 2$,

– $\mathcal{A}, \mathcal{B}$, the finite action space of players $1, 2$ respectively.

– $r_2 = -r_1$,

– rest the same.

Conventions

- We call player 2 the maximizer and player 1 the minimizer.
- The value of maximizer is $V^{(\pi_1, \pi_2)}(\rho)$.

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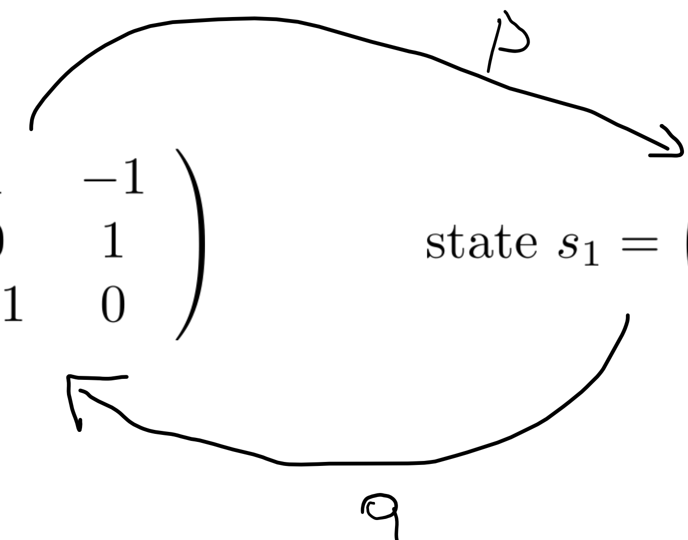
$$\text{state } s_0 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \text{state } s_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$



2-player zero-sum Markov games

A crucial property:

Theorem (Shapley 53). *In any two-player zero-sum Markov game*

$$\min_{\pi_1} \max_{\pi_2} V^{\pi_1,\pi_2}(\boldsymbol{\rho}) = \max_{\pi_2} \min_{\pi_1} V^{\pi_1,\pi_2}(\boldsymbol{\rho})$$

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Remark

- The game has a unique value V^* (recall Von Neumann for normal form two player zero-sum games).
- The theorem implies it does not matter who plays first.
- The function is **not** convex-concave!
- The proof of Shapley uses a contraction argument.
- The complexity of finding a Nash equilibrium is *unknown*.

2-player zero-sum Markov games

Proof. Similar to Bellman, different operator.

Let val(.) be the operator applied to a payoff matrix that returns the value of the corresponding zero-sum game.

$$\text{e.g., val}\left(\begin{bmatrix}-1, 1 \\ 1, -1\end{bmatrix}\right) = 0.$$

2-player zero-sum Markov games

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$$\text{e.g., } \mathrm{val}\left(\begin{bmatrix} -1, 1 \\ 1, -1 \end{bmatrix} \right) = 0.$$

Fact: $|val(A) - val(B)| \leq max_{i,j} |A_{ij} - B_{ij}|$

Given a value vector $V(s)$, we define the operator $\mathcal{T}$

$$\mathcal{T}V(s) := \mathrm{val}(r_2(s,.,.) + \gamma \sum_{s'} \mathbb{P}(s'|s,.,.)V(s')).$$

2-player zero-sum Markov games

$$
\begin{aligned}
\|\mathcal{T}V - \mathcal{T}V'\|_\infty &= \left\| \mathrm{val}\{r(s,.,.) + \gamma \sum_{s'} \mathbb{P}(s'|s,.,.)V(s')\} - \mathrm{val}\{r(s,.,.) + \gamma \sum_{s'} \mathbb{P}(s'|s,.,.)V'(s')\} \right\|_\infty \\
&\leq \left\| \max_{a,b}\{r(s,a,b) + \gamma \sum_{s'} \mathbb{P}(s'|s,a,b)V(s') - r(s,a,b) - \gamma \sum_{s'} \mathbb{P}(s'|s,a,b)V'(s')\} \right\|_\infty \\
&= \gamma \left\| \max_{a,b}\{\mathbb{P}_{a,b}(V - V')\} \right\|_\infty \\
&\leq \gamma \|V - V'\|_\infty
\end{aligned}
$$

2-player zero-sum Markov games

$$\|\mathcal{T}V - \mathcal{T}V'\|_\infty = \left\| \mathrm{val}\{r(s,.,.) + \gamma \sum_{s'} \mathbb{P}(s'|s,.,.)V(s')\} - \mathrm{val}\{r(s,.,.) + \gamma \sum_{s'} \mathbb{P}(s'|s,.,.)V'(s')\} \right\|_\infty$$

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$$= \gamma \left\| \max_{a,b}\{\mathbb{P}_{a,b}(V - V')\} \right\|_\infty$$

$$\leq \gamma \|V - V'\|_\infty$$

Remarks

- Bellman operator is contracting for infinity norm.
- Applying the operator does not give a polynomial time algorithm. Why?

Some facts about Policy Gradient

Definition (Policy Gradient Ascent). *PGA is defined iteratively:*

$$x_k^{(t+1)} := \Pi_{\Delta(A_k)^S}(x_k^{(t)} + \eta \nabla_{x_k} V_k^{x^{(t)}}(\rho)),$$

where Π denotes projection on product of simplices.

Theorem (Policy Gradient Ascent [Agarwal et al 2020]). *It can be shown for one agent that after $O(1/\epsilon^2)$ iterations, an ϵ-optimal policy can be reached.*

Theorem (Policy Gradient Descent/Ascent [Daskalakis et al 2020]). *It can be shown a two-time scale Policy Gradient Descent/Ascent can give an ϵ-Nash equilibrium in poly$(1/\epsilon)$ time.*

Remarks

- No guarantees for more than two players (only very specific settings).
- Can we find other classes of Markov games that PGA converges?
- In general, approximating even stationary CCE is PPAD-complete [Daskalakis et al 2022].

- *Potential Markov games*

Markov Potential Games

Definition (Markov Potential Game). *A Markov Decision Process (MDP) is called a* Markov Potential Game (MPG) *if there exists a (state-dependent) function* $\Phi_s : \Pi \to \mathbb{R}$ *for* $s \in S$ *so that*

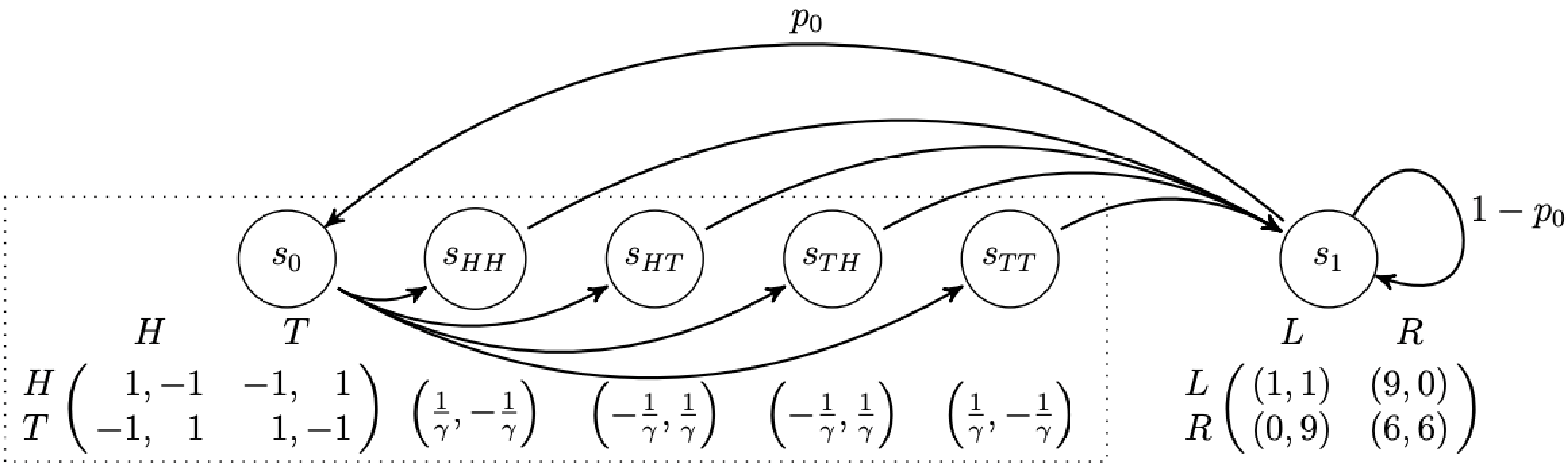
$$\Phi^{(\pi_k, \pi_{-k})}(s) - \Phi^{(\pi'_k, \pi_{-k})}(s) = V_k^{(\pi_k, \pi_{-k})}(s) - V_k^{(\pi'_k, \pi_{-k})}(s),$$

for all agents $k \in \mathcal{N}$, *all states* $s \in \mathcal{S}$ *and all policies* $\pi_k, \pi'_k \in P_k, \pi_{-k} \in P_{-k}$.

Remarks

- This notion generalizes the Potential Games in Game Theory.
- Potential Games capture routing (congestion games), important class.
- Deterministic Nash policies always exist!
- Each state a potential game does not imply MPG. Might have also zero sum game states!

An example of a MPG



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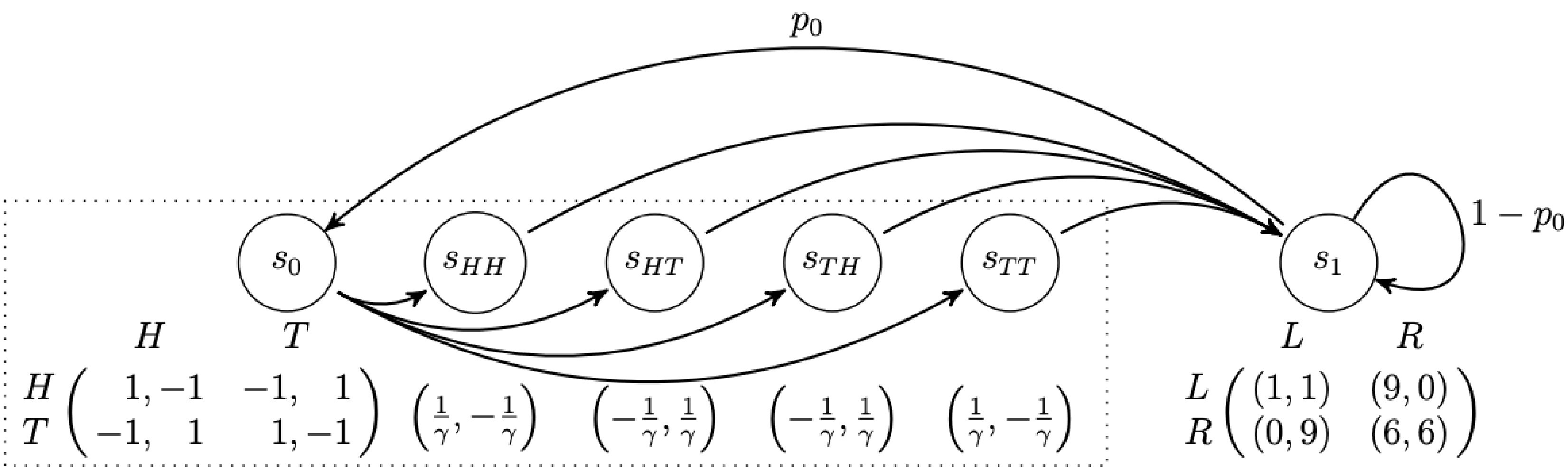
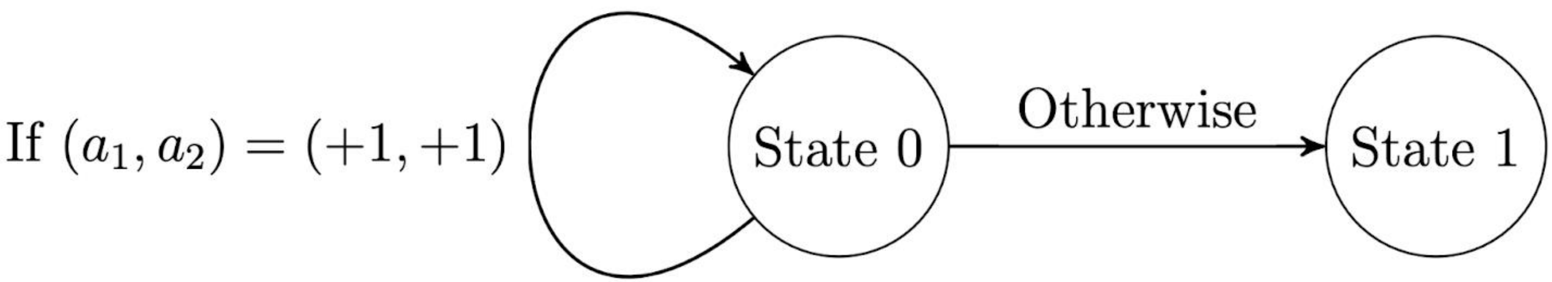


Subgames can be zero sum!

An example of an ``almost'' MPG



$$(R_0^1, R_0^2) = \begin{array}{c} \\ +1 \\ -1 \end{array} \begin{array}{c} \begin{array}{cc} +1 & -1 \end{array} \\ \begin{pmatrix} 5, 2 & -1, -2 \\ -5, -4 & 1, 4 \end{pmatrix} \end{array}$$

$$(R_1^1, R_1^2) = (0, 0)$$

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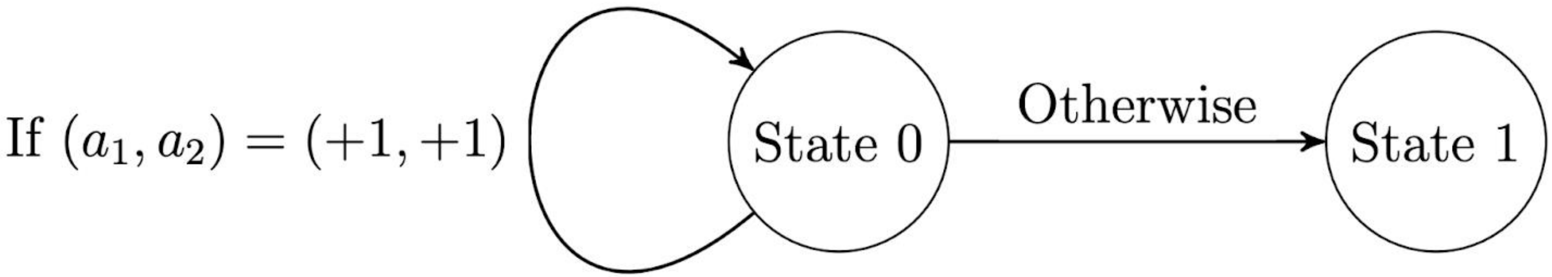

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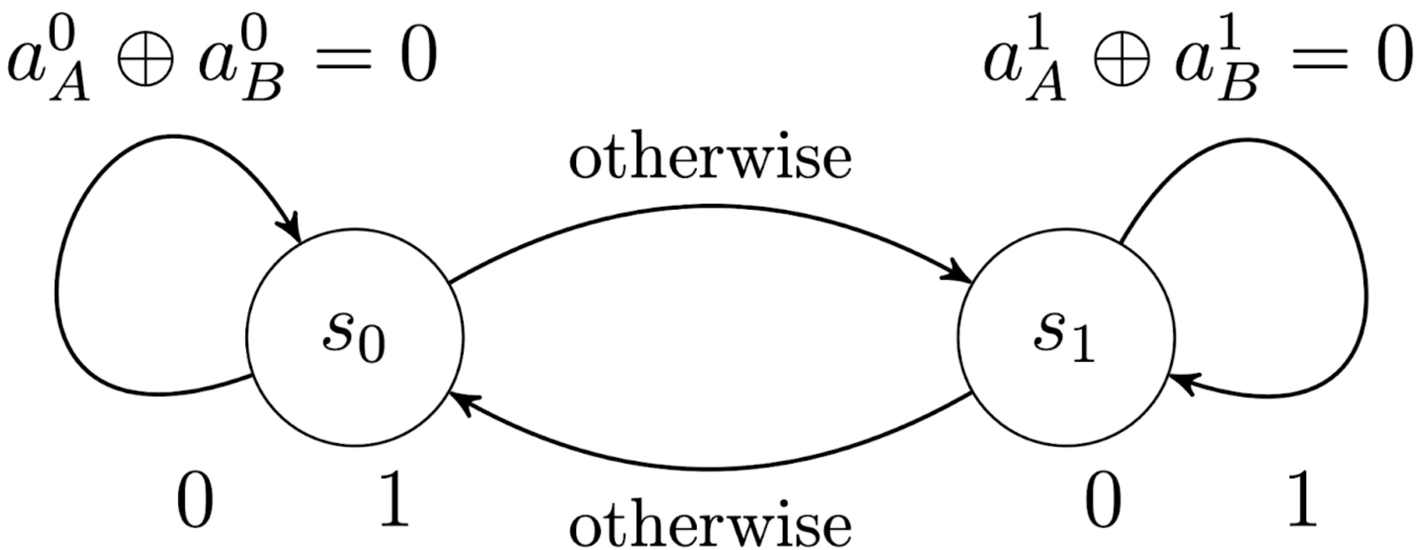
Ordinal MPG!

Not Markov Potential Game



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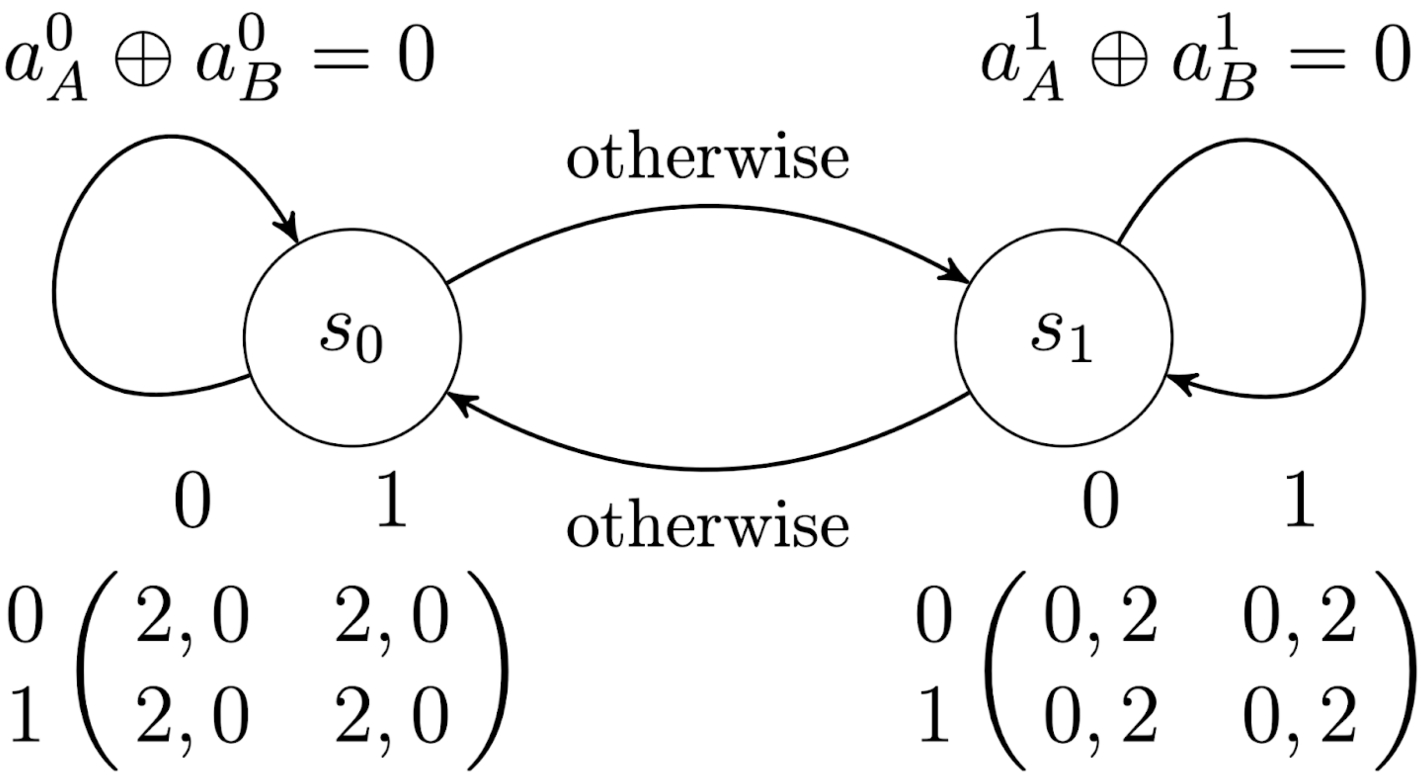

Transitions can create competition!

Main Result

Theorem (PGA for Markov Potential Games)**.** *Suppose all agents run policy gradient iteration independently and update simultaneously. It can be shown that after* $O(1/\epsilon^2)$ *iterations, an* ϵ*-Nash policy can be reached.*

Remarks

- This result can be generalized (different rates, i.e., $O\left(\frac{1}{\epsilon^6}\right)$) if agents do not have access to exact gradients. Stochastic variant + greedy parametrization.
- It matches the result for single-agent.
- The running time depends polynomially on the sum of cardinalities of the players' actions spaces and not on the product.

Proof Steps I

Lemma (Key Lemma 1). *Policy gradient on values of agents is equivalent to projected gradient ascent on* Φ. *Formally it holds*

$$\nabla_{x_k} \Phi = \nabla_{x_k} V_k.$$

Remarks

- This is true by definition of Φ. Note that we do not know Φ!
- Policy gradient is Projected Gradient Ascent on Φ

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Lemma (Key Lemma 2). *Stationary points for* Φ *are exactly Nash policies!*

Remarks

- This is a technical lemma, it uses the gradient domination property.

Gradient domination (PL condition) $f(x^*) - f(x) = O(G(x))$ where G(x) is a scalar notion of first-order stationarity (e.g. $||\nabla f||$).

- It holds for approximate stationary points too.

Proof Steps II

Lemma (Theorem (e.g., [Ghadimi et al 2013])). *Gradient descent reaches an ϵ-stationary point after $O(\frac{1}{\epsilon^2})$ steps for functions f with Lipschitz gradient.*

Intuition: A standard descent lemma gives:

$$f\left(x - \frac{1}{L}\nabla f(x)\right) - f(x) \leq -C(\|\nabla f(x)\|_2^2).$$

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$$f(x_{t+1}) - f(x_t) \leq -C(\|\nabla f(x_t)\|_2^2).$$

Assume that $\|\nabla f(x_t)\|_2 > \epsilon$ for $t = 1, ..., T$. We get that

$$f(x_T) - f(x_{T-1}) + f(x_{T-1}) - f(x_{T-2}) + ... + f(x_1) - f(x_0) < -C\epsilon^2 T.$$

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Lemma (Theorem (e.g., [Ghadimi et al 2013])). *Gradient descent reaches an ϵ-stationary point after $O(\frac{1}{\epsilon^2})$ steps for functions f with Lipschitz gradient.*

Intuition: A standard descent lemma gives:

$$f(x_{t+1}) - f(x_t) \leq -C(\|\nabla f(x_t)\|_2^2).$$

Assume that $\|\nabla f(x_t)\|_2 > \epsilon$ for $t = 1, ..., T$. We get that

$$f(x_T) - \cancel{f(x_{T-1})} + \cancel{f(x_{T-1})} - \cancel{f(x_{T-2})} + \ldots + \cancel{f(x_1)} - f(x_0) < -C\epsilon^2 T.$$

If $f(x) \geq f\text{min}$ then $T = O\left(\frac{1}{\epsilon^2}\right)$

Proof Steps II

Lemma (Theorem (e.g., [Ghadimi et al 2013])). *Gradient descent reaches an ϵ-stationary point after $O(\frac{1}{\epsilon^2})$ steps for functions f with Lipschitz gradient.*

Remarks

- We can follow the same analysis if Φ has Lipschitz gradient.
- We show that $\nabla_{x_k} V_k$ is Lipschitz (constant depends on number of agents, number of actions and discount factor γ).
- For the stochastic variant, we need an unbiased estimator with bounded variance.

- *Adversarial team Markov games*

Adversarial team Markov games

- $\mathcal{N} = \{1, \ldots, n\} \cup \{n+1\}$, i.e., n agents and one adversary agent,
- $\mathcal{A}_k$ denotes the finite action space of player k,
- $\mathcal{B}$ denotes the action space of the adversary,
- $r_i = r_j$ for $i, j \in [n]$,
- letting r_{adv} be the adversary's reward; the game is team zero-sum, *i.e.*,

$$\sum_{k=1}^{n} r_k(s, \boldsymbol{a}, b) + r_{\text{adv}}(s, \boldsymbol{a}, b) = 0,$$

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Can we compute an approximate NE ?

Main result

Definition (IPGMAX)**.** *The independent policy gradient on the max function is as follows, for T steps do:*

$$\text{The adversary best-responds to } \boldsymbol{x}\text{:} \tag{1}$$

$$\boldsymbol{y}^\star \leftarrow \arg\max_{\boldsymbol{y}\in\Delta(\mathcal{B})^S} V_{\boldsymbol{\rho}}(\boldsymbol{x},\boldsymbol{y}) \tag{2}$$

$$\text{Every agent } k \text{ independently updates their strategy:} \tag{3}$$

$$\boldsymbol{x}_k \leftarrow \Pi_{\Delta(A_k)^S}\left\{\boldsymbol{x}_k - \eta\nabla_{\boldsymbol{x}_k} V_{\boldsymbol{\rho}}\left(\boldsymbol{x},\boldsymbol{y}^\star\right)\right\} \tag{4}$$

Get a point $\hat{\boldsymbol{x}}$ which is approximate stationary for $\max_{\boldsymbol{y}\in\Delta(\mathcal{B})^S} V_{\boldsymbol{\rho}}(\boldsymbol{x},\boldsymbol{y})$, solve a constrained linear program to get a $\hat{\boldsymbol{y}}$.

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Remark

- Thought experiment on RPS. What would this algorithm return? We need to do extra work to find a NE.

Main result

Theorem (IPGMAX + LP). *Running IPGMAX for* $O\left(\frac{poly(\sum_k |\mathcal{A}_k|+|\mathcal{B}|)}{\epsilon^4}\right)$ *iterations and in the end a LP, yields an* ϵ*-NE.*

Remarks

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Remarks

- We need $\frac{1}{\epsilon^4}$ iterations instead of $\frac{1}{\epsilon^2}$. Why? max function is non-smooth.
- We show that an approximate stationary $\hat{x}$ point of $\max_y V(x,y)$ can always be extended to a $(\hat{x}, \hat{y})$ which is an approximate NE.

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This includes proving existence of Lagrange multipliers for some non-convex program.

- To get $\hat{y}$, we somehow create the dual which is linear.

- *Polymatrix Markov games*

(normal form) Polymatrix games

A polymatrix game is defined using a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where

– every agent i coincides with a vertex $v_i \in \mathcal{V}$,

– for every agent i, there is a finite action-space $\mathcal{A}_i$,

– every agent i has a utility function $u_i : \times_{i=1}^{n} \mathcal{A}_i \to [-1, 1]$,

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In a *polymatrix game* the utility of every agent i is separable as a sum of pair-wise interactions dictated by the graph,

$$u_i(\boldsymbol{a}) = \sum_{j \in \mathrm{neighb}(i)} u_{ij}(a_i, a_j),$$

$$\text{where} \quad \boldsymbol{a} = (a_1, \ldots, a_i, a_j, \ldots, a_n) \in \times_{i=1}^{n} \mathcal{A}_i.$$

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Finally it is called zero-sum if $\sum_i u_i = 0$

(normal form) Polymatrix games

Computing NE is easy (in P).

The solutions of the following linear program are Nash equilibria.

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} w_i && \text{(1a)} \\ \text{subject to} \quad & w_i \geq u_i(a_i, \boldsymbol{x}_{-i}), \qquad \forall i \in [n], \forall a_i \in \mathcal{A}_i, && \text{(1b)} \\ & \boldsymbol{x}_i \in \Delta(\mathcal{A}_i), \ \forall i \in [n]. && \text{(1c)} \end{aligned}$$

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- There is more. The above coincides (slightly) with LP for CCE.

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Equilibrium collapse. Marginals of CCEs are NE!!!!

Polymatrix Markov games

A Markov game s.t for every state s, there exists a graph $\mathcal{G}_s(\mathcal{V}_s, \mathcal{E}_s)$ such that,

- the vertices $\mathcal{V}_s$ coincide with the agents,
- the reward function of each agent depends on *pair-wise* interactions with each neighbors,

$$r_i(s, \boldsymbol{a}) = \sum_{j \in \text{neighbors}(i)} r_{ij}(s, a_i, a_j).$$

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- *Assumption of Switching control*: at every state there is a single player that controls the probability of transtion to a new state.

Main result

Unfortunately we do not have a LP as before but we have equilibrium collapse.

Theorem (Equilibrium collapse)**.** *Let a coarse correlated equilibrium of the switching control, polymatrix zero-sum Markov game, $\boldsymbol{\sigma}$. Then the marginal product strategy profile, $\boldsymbol{x}^{\sigma}$,*

$$\boldsymbol{x}_{i,s}(a_i) = \sum_{\boldsymbol{a}_{-i} \in \mathcal{A}_{-i}} \sigma_s(a_i, \boldsymbol{a}_{-i})$$

is a Nash equilibrium of the game.

Main result

Unfortunately we do not have a LP as before but we have equilibrium collapse.

The corresponding program looks as follows:

$$\text{minimize} \quad \sum_{i=1}^{n} \sum_{s \in \mathcal{S}} \rho(s) w_i(s) \tag{1a}$$

$$\text{subject to} \quad w_i(s) \geq r_i(s, a_i, \boldsymbol{x}_{-i,s}) + \gamma \sum_{s' \in \mathcal{S}} \mathbb{P}(s' | s, a_i, \boldsymbol{x}_{-i,s}) w_i(s') \ \forall i \in [n], \forall s \in \mathcal{S}, \forall a_i \in \mathcal{A}_i, \tag{1b}$$

$$\boldsymbol{x}_{i,s} \in \Delta(\mathcal{A}_i), \ \forall i \in [n], \forall s \in \mathcal{S}. \tag{1c}$$

Remarks

- Any algorithm that gives approximate Markovian CCEs, gives approximate Markovian NE!