# Midterm 2
# Problem 4 Solutions

CS 169
Introduction to Optimization

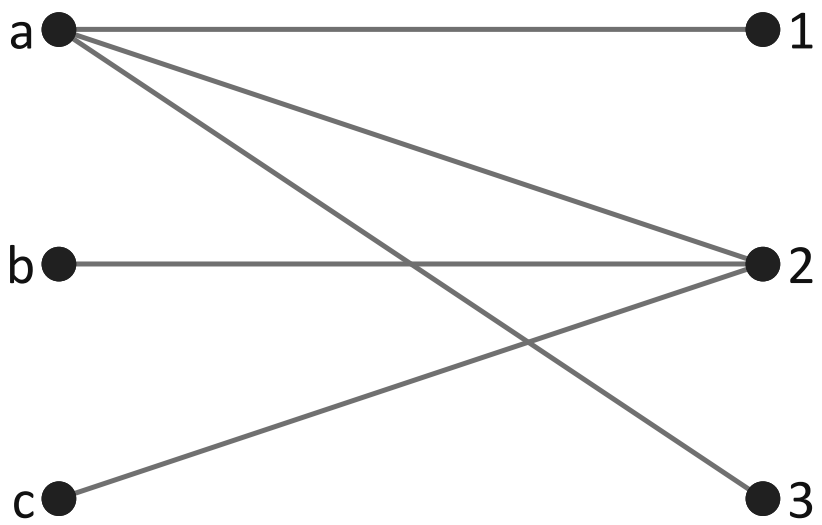
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# Problem 4: vertex cover and duality

A vertex cover contains at least one endpoint of every edge.
We want a cover of minimum size.

$$U = \{a, b, c\}, \qquad W = \{1, 2, 3\}.$$



# 4(i). The integer program

Let $V = U \cup W$. Use $x_v = 1$ when vertex $v$ is selected and $x_v = 0$ otherwise.

$$\text{minimize} \quad \sum_{v \in V} x_v$$

$$\text{subject to} \quad x_u + x_v \geq 1 \qquad \{u, v\} \in E,$$

$$x_v \in \{0, 1\} \qquad v \in V.$$

Each edge constraint requires at least one selected endpoint. The objective counts selected vertices.

# 4(i). The covering LP

$$\text{minimize} \quad \sum_{v \in V} x_v$$

$$\text{subject to} \quad x_u + x_v \geq 1 \qquad \{u, v\} \in E,$$

$$x_v \geq 0 \qquad v \in V.$$

Any value above 1 can be capped at 1. All edges remain covered, and the objective cannot increase.

$$\widehat{x}_v = \min\{x_v, 1\}, \qquad \sum_v \widehat{x}_v \leq \sum_v x_v.$$

# 4(i). Deriving the dual

One primal constraint per edge gives a dual variable $y_e \geq 0$ for each edge $e$.

Each $x_v$ appears with coefficient 1 in its incident edge constraints. Its objective coefficient is 1.

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} y_e \\ \text{subject to} \quad & \sum_{e \ni v} y_e \leq 1 \qquad v \in V, \\ & y_e \geq 0 \qquad e \in E. \end{aligned}$$

# 4(i). The dual for this graph

$$\max \; y_{a1} + y_{a2} + y_{a3} + y_{b2} + y_{c2}$$

$$\begin{aligned}
y_{a1} + y_{a2} + y_{a3} &\leq 1 \quad \text{at } a, \\
y_{b2} &\leq 1 \quad \text{at } b, \\
y_{c2} &\leq 1 \quad \text{at } c, \\
y_{a1} &\leq 1 \quad \text{at } 1, \\
y_{a2} + y_{b2} + y_{c2} &\leq 1 \quad \text{at } 2, \\
y_{a3} &\leq 1 \quad \text{at } 3.
\end{aligned}$$

$$y_{a1}, y_{a2}, y_{a3}, y_{b2}, y_{c2} \geq 0.$$

# 4(ii). The dual is a matching LP

If a feasible dual solution is integral, each $y_e$ is 0 or 1: nonnegativity and an endpoint constraint give $y_e \leq 1$.

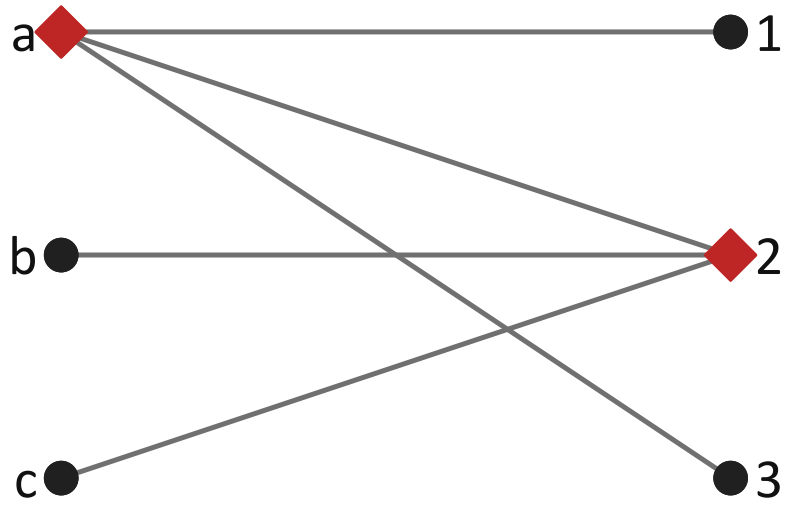
$$M = \{e \in E : y_e = 1\}, \qquad \sum_{e \ni v} y_e \leq 1.$$

The selected edges share no endpoint, so they form a matching. The dual objective is its size.

For a bipartite graph, the matching LP has an integral optimal solution.

# 4(ii). A cover of size two

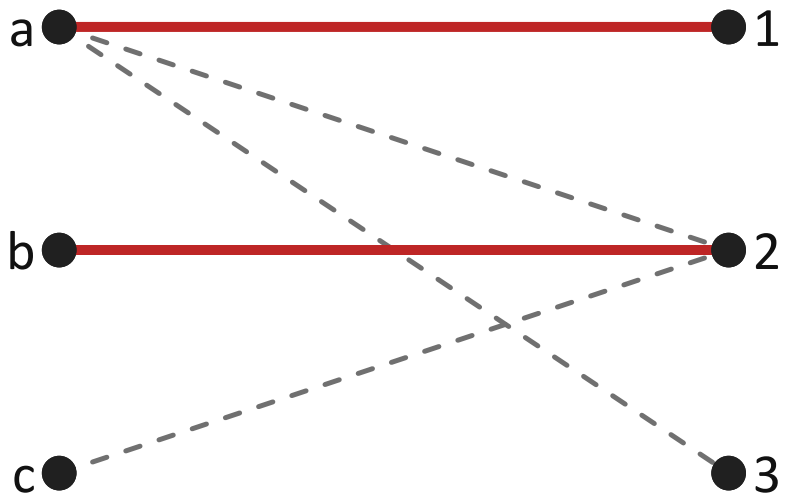
$$C = \{a, 2\}, \qquad \sum_{v \in V} x_v = 2.$$




Vertex $a$ covers $(a,1),(a,2),(a,3)$. Vertex 2 covers $(b,2),(c,2)$.

# 4(ii). A dual solution of value two

$$M = \{(a,1),(b,2)\}, \qquad \sum_{e \in E} y_e = 2.$$




$$y_{a1} = y_{b2} = 1, \qquad y_{a2} = y_{a3} = y_{c2} = 0.$$

# 4(ii). Why both solutions are optimal

The feasible cover has value 2, and the feasible dual solution has value 2.

$$\underbrace{2}_{\text{dual value}} \leq \text{OPT}_{\text{LP}} \leq \text{OPT}_{\text{IP}} \leq \underbrace{2}_{\text{cover size}} .$$

Weak duality forces equality throughout. Neither solution can be improved.

$$\boxed{\text{Minimum cover size} = \text{maximum matching size} = 2.}$$