Finding a Positive Index Nash Equilibrium is PPADS-Complete

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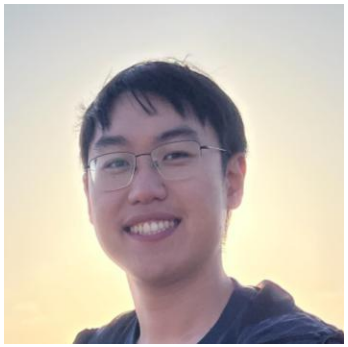
joint work with



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NTUA & Archimedes AI

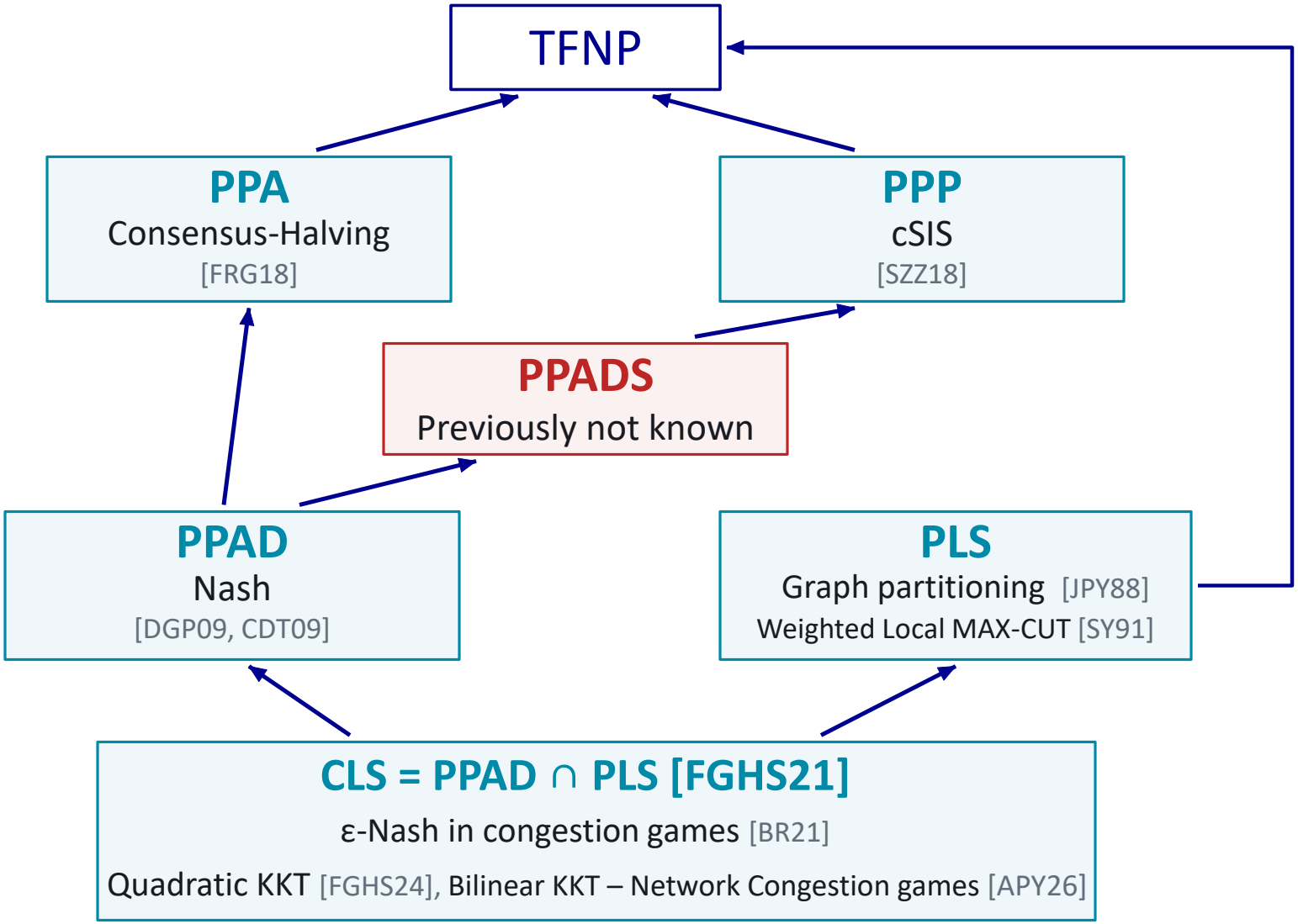


Vasilis Pollatos
NKUA & Archimedes AI



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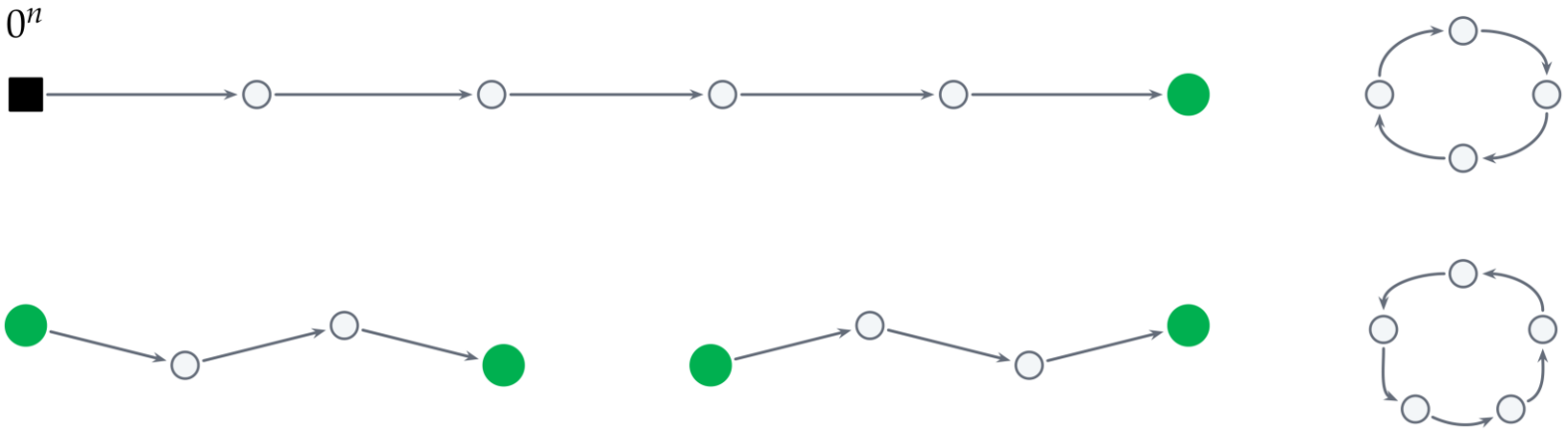
Natural problems in TFNP



PPAD vs PPADS

P, S Boolean circuits describe a directed graph

Indegree and outdegree ≤ 1; a known source is given

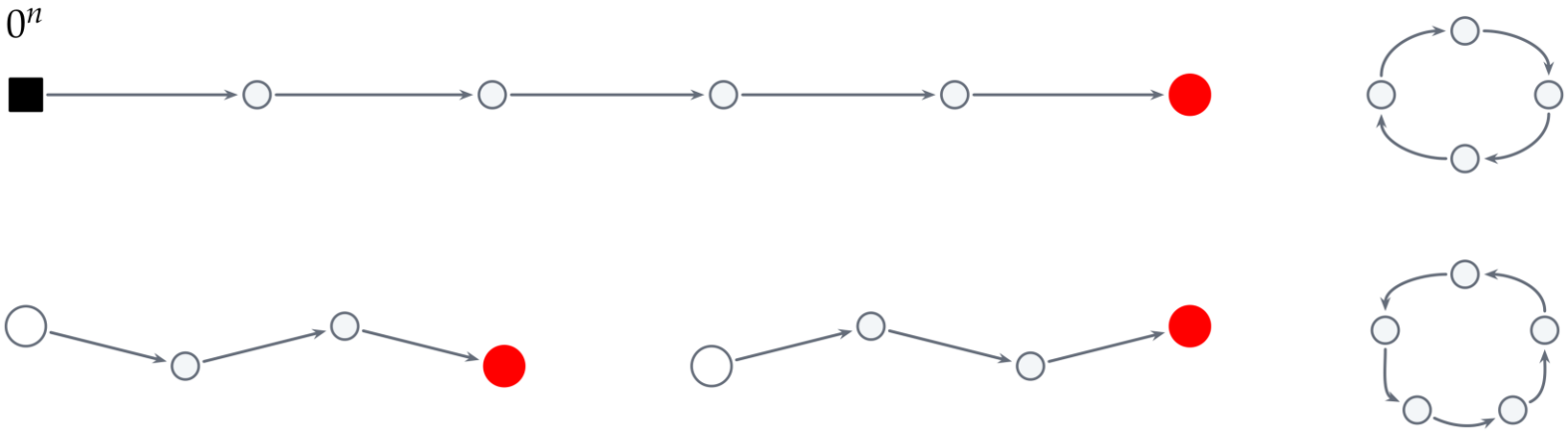


End-of-Line	another source or a sink	**PPAD**
Sink-of-Line	a sink	**PPADS**

PPAD vs PPADS

P, S Boolean circuits describe a directed graph

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End-of-Line	another source or a sink	**PPAD**
Sink-of-Line	a sink	**PPADS**

Some historical remarks

1994 Papadimitriou's sink-only variant [P94]

"a class apparently adding no natural problems of interest"

1995 The name PPADS appears in [BCEIP95]

Positive Sperner is complete, with a circuit as input

2018 Daskalakis's Nevanlinna lecture [D19]

"class PPADS, which is the least understood subclass of TFNP". A natural complete problem without circuits?

Nondegeneracy and the Shapley index [S74]

Nondegenerate Games

(Shift payoffs so both matrices have positive entries)

*A bimatrix game is **nondegenerate** if every mixed strategy of either player with support size k has at most k pure best responses by the other player.*

Nondegeneracy and the Shapley index [S74]

Nondegenerate Games (Shift payoffs so both matrices have positive entries)

$$|\mathrm{BR}_1(y)| \le |\mathrm{supp}(y)| \qquad \forall\, y \in \Delta_n$$
$$|\mathrm{BR}_2(x)| \le |\mathrm{supp}(x)| \qquad \forall\, x \in \Delta_m$$

At an equilibrium (x, y) with $I = \mathrm{supp}(x), J = \mathrm{supp}(y)$ and $|I| = |J| = k$

$\mathbf{A}_{I,J}, \mathbf{B}_{I,J}$ are invertible

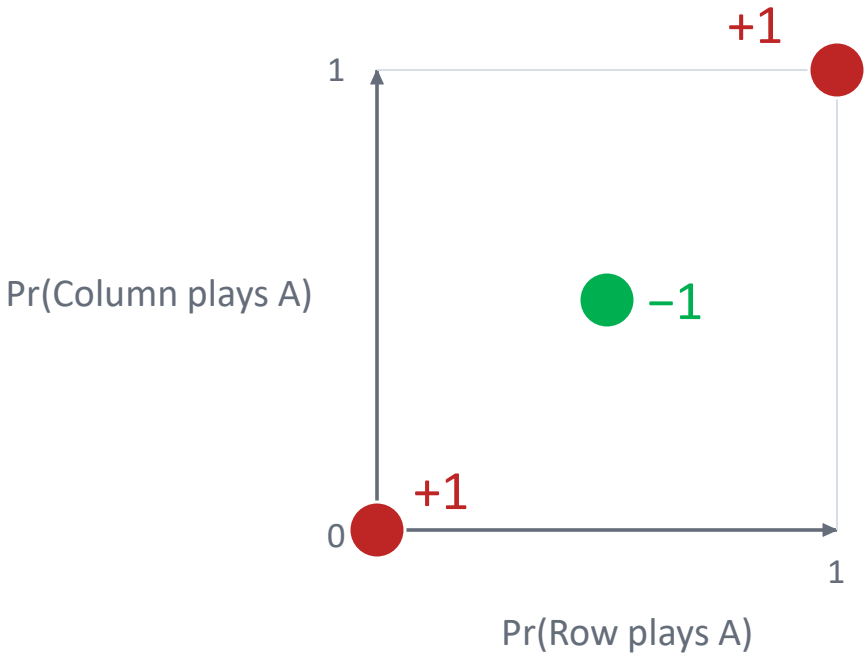
$$\mathrm{ind}(x, y) = (-1)^{k+1} \,\mathrm{sign}(\det \mathbf{A}_{I,J} \det \mathbf{B}_{I,J}) \in \{-1, +1\}$$

Moreover, $$\sum_{(x,y)\in \mathrm{NE}} \mathrm{ind}(x, y) = +1$$

A positive index equilibrium must exist. Pure NE have positive index!

One game, three equilibria

Row player \ Column player	A	B
A	(2, 2)	(1, 1)
B	(1, 1)	(2, 2)



$$(+1) + (-1) + (+1) = +1$$

Positive-index Nash equilibrium

Positive-Index Nash

Input	Explicit rational payoff matrices $\mathbf{A}, \mathbf{B}$
Promise	The bimatrix game is nondegenerate
Output	An exact Nash equilibrium (x, y) with $\text{ind}(x, y) = +1$

Theorem

Positive-Index Nash is PPADS-complete.

A natural complete problem with no input circuits

The promise can be removed

Any rational bimatrix game



Find

Regular equilibrium with index +1

or

Nonregular equilibrium

This problem is also PPADS-complete.

Regular equilibrium

At a Nash equilibrium (x, y) with $I = \mathrm{supp}(x), J = \mathrm{supp}(y)$

An equilibrium is **regular** if: (Payoffs shifted to positive entries)

- Equal support sizes: $|I| = |J|$
- Invertible support matrices: $\mathbf{A}_{I,J}$, $\mathbf{B}_{I,J}$
- Every pure best response is played.



Every equilibrium of a nondegenerate game is regular.

Regular equilibrium

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- Every pure best response is played.



Every equilibrium of a nondegenerate game is regular.

Quasi-strict: Every pure best response is played with positive probability.

One game, two types of nonregular equilibria

$$\mathbf{A} = \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Every profile is a Nash equilibrium.

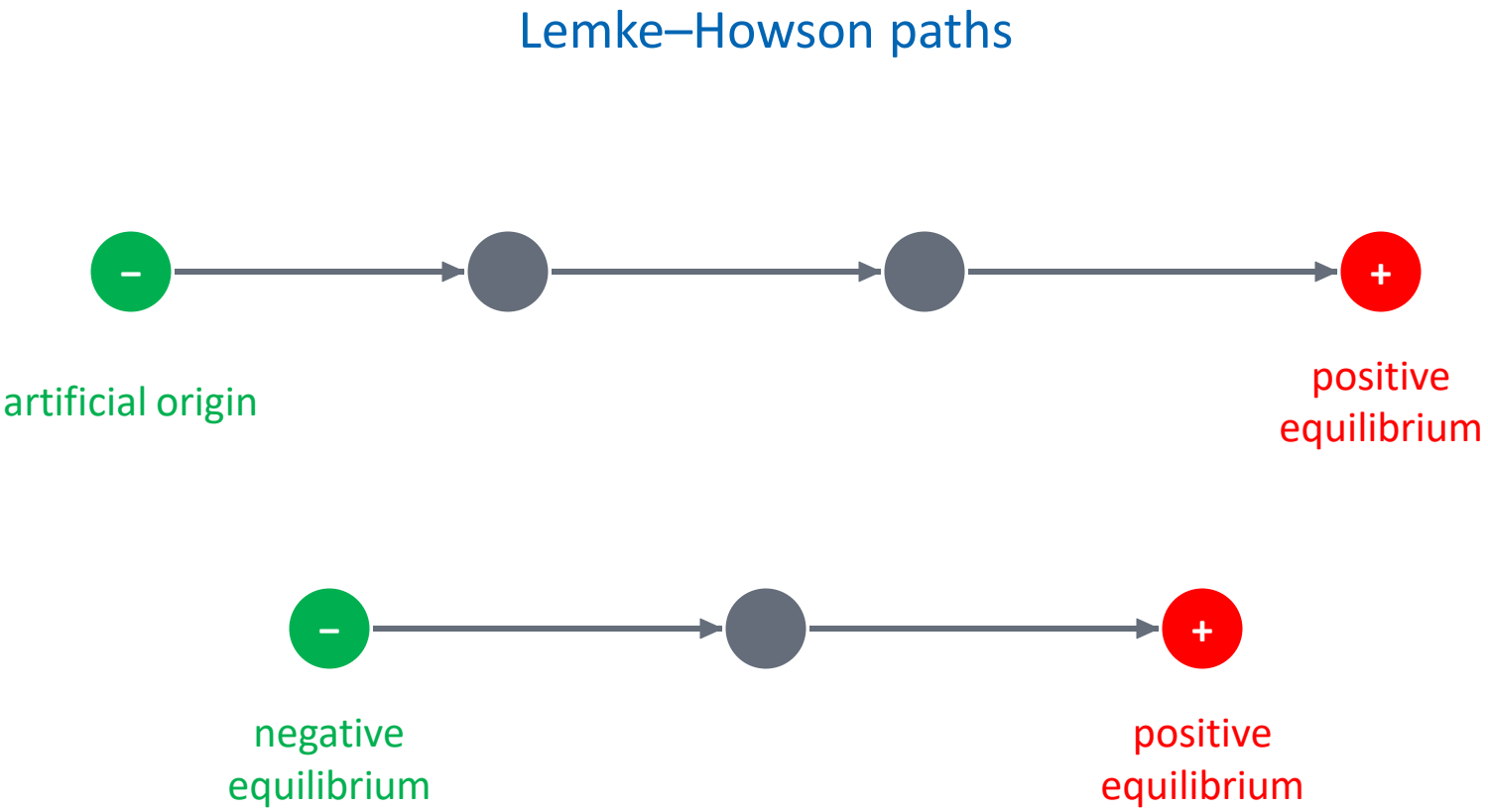
$x = y = (\frac{1}{2}, \frac{1}{2})$ Quasi-strict but nonregular

Both support matrices are singular.

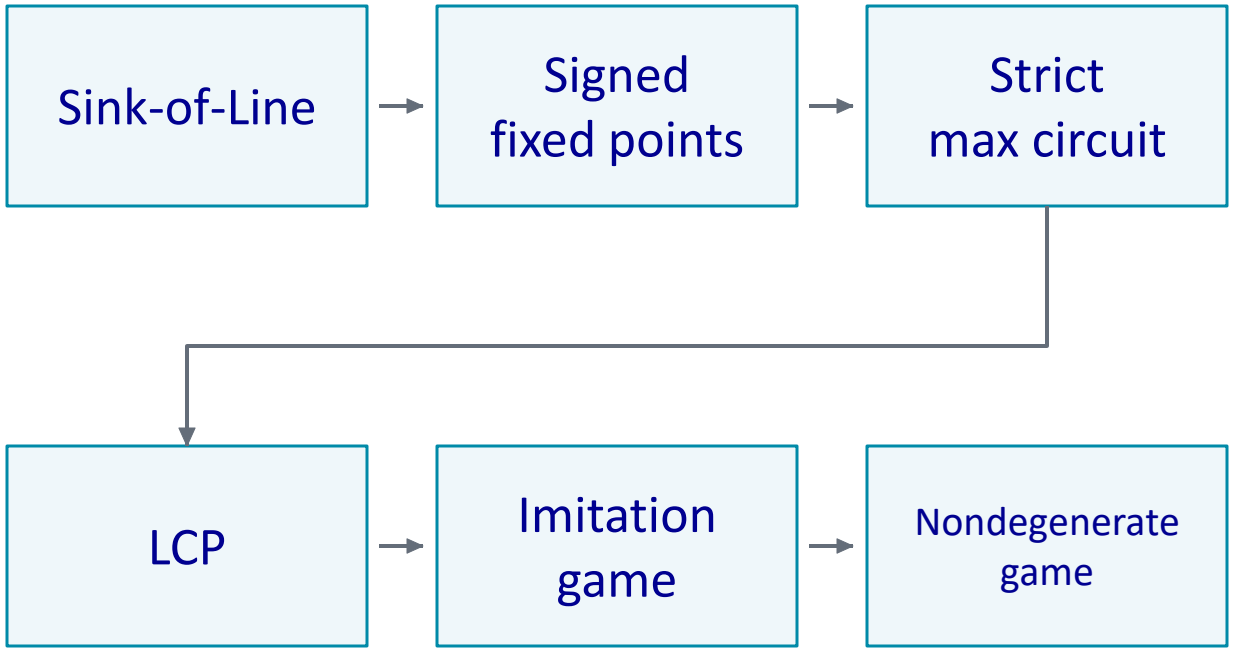
$x = y = (1, 0)$ Not quasi-strict

Unused actions are still best responses.

Membership: orient Lemke–Howson



Hardness must preserve the sign

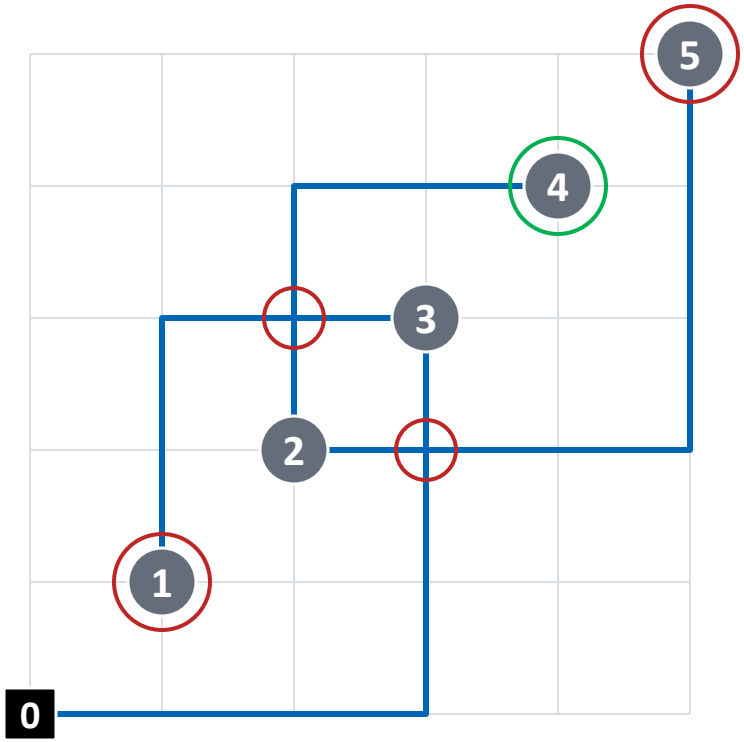


Routing and the Chen–Deng switch

$0 \to 3 \to 1$

$4 \to 2 \to 5$

One row and column per vertex

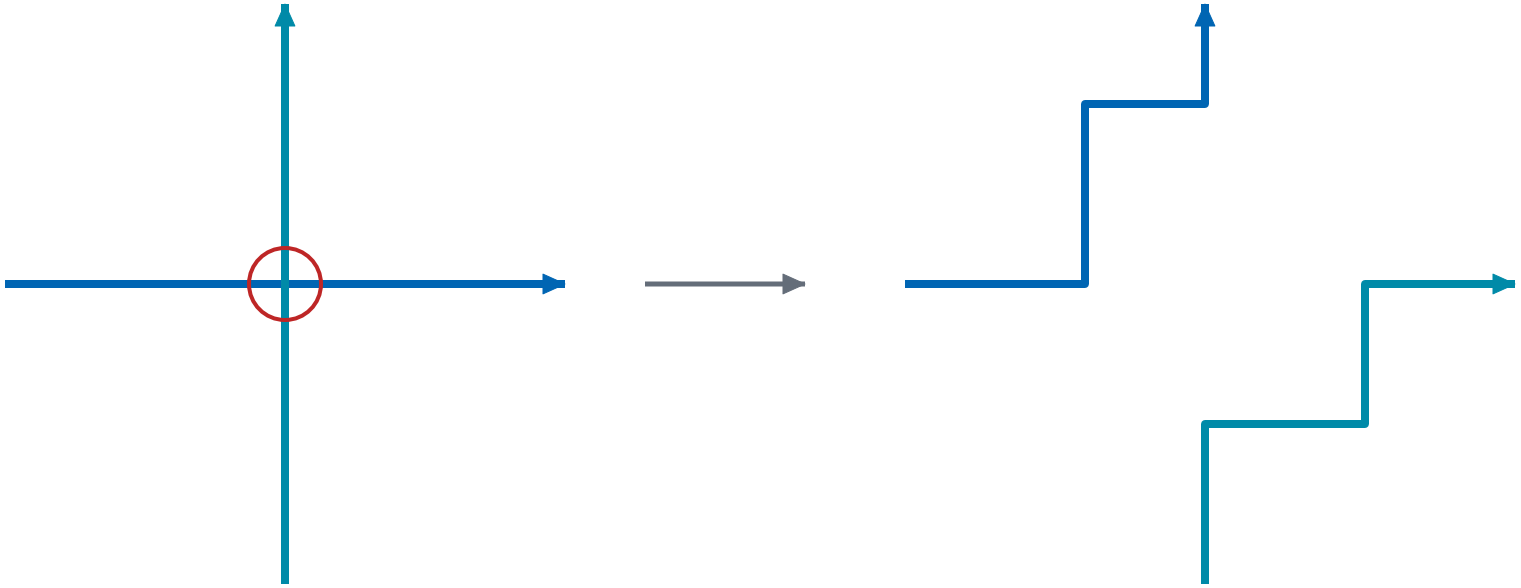


$$(3+9u, 3+9u) \longrightarrow (3+9v, 3+9u) \longrightarrow (3+9v, 3+9v)$$

A switch removes each crossing

Crossing

Planar switch

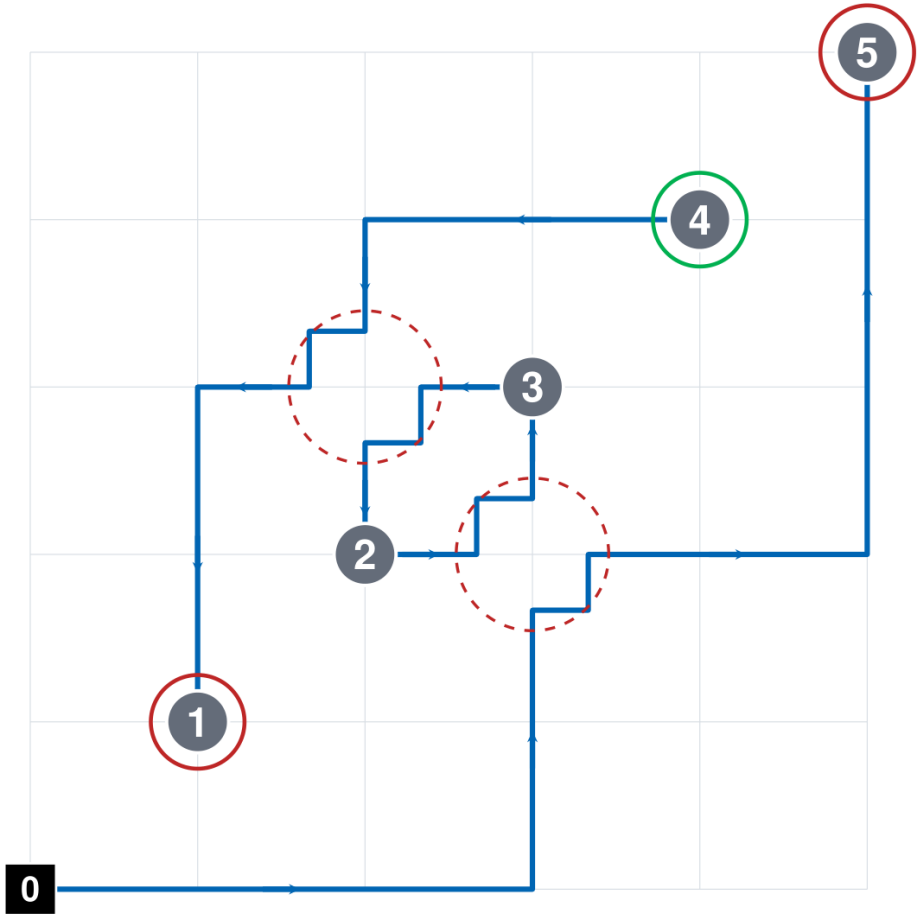


The routing after the switches

$0 \rightarrow 5$

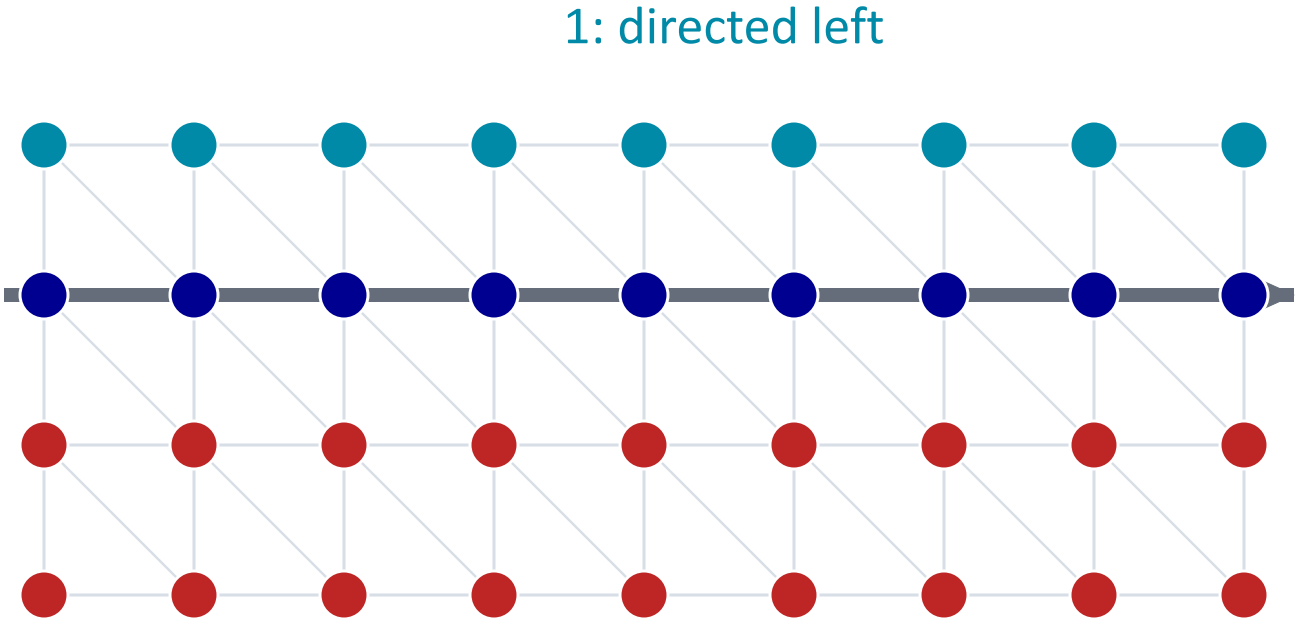
$4 \rightarrow 1$

A cycle through
2 and 3



Every source and sink is preserved.

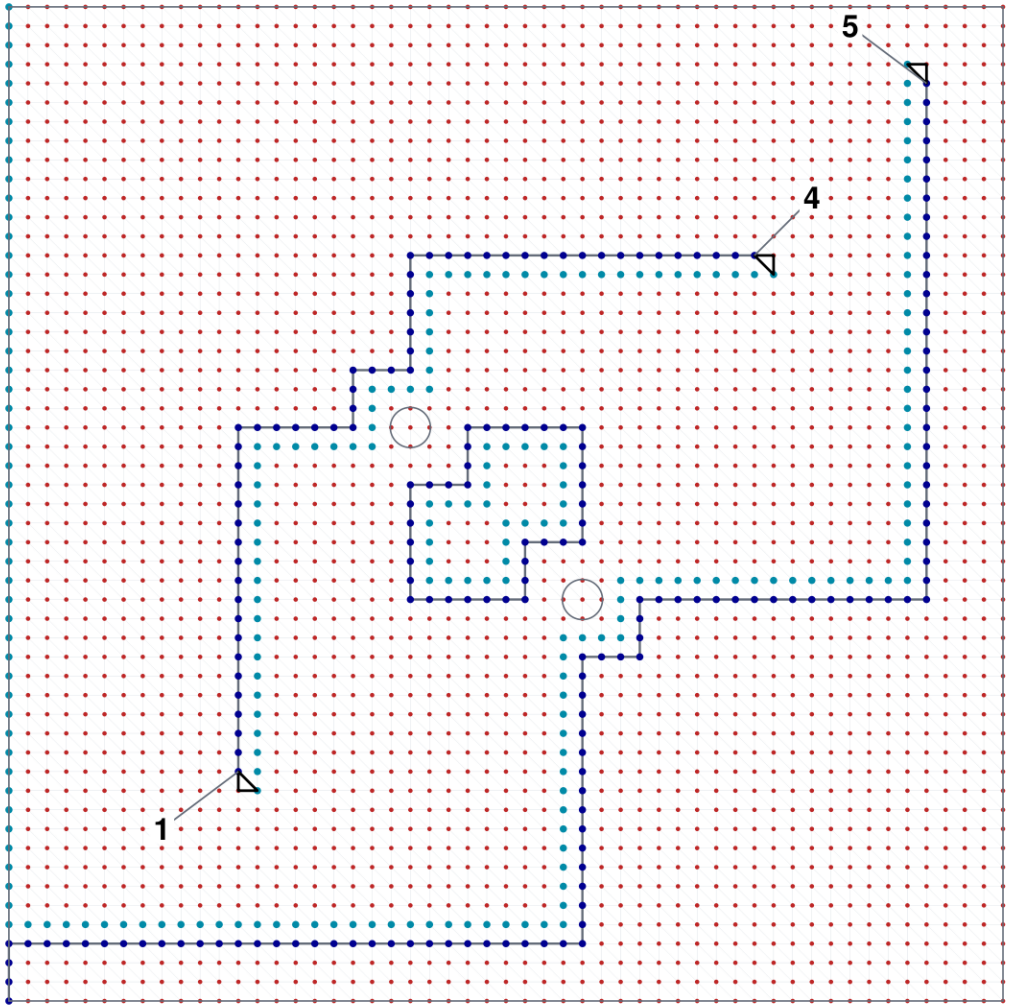
Color the routed paths



No trichromatic triangle along an ordinary segment.

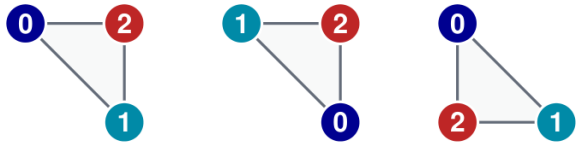
Trichromatic triangles at 4, 5, and 1

Colors ● 0 ● 1 ● 2

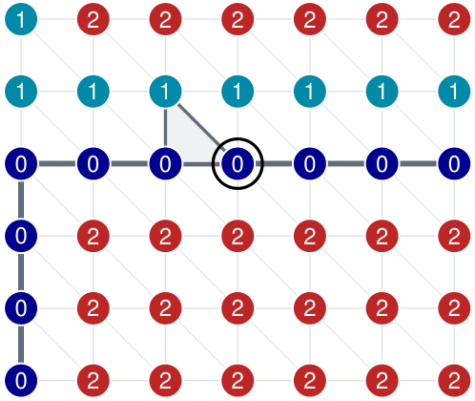


Exactly three triangles

Source 4 Sink 5 Sink 1



Canonical source 0

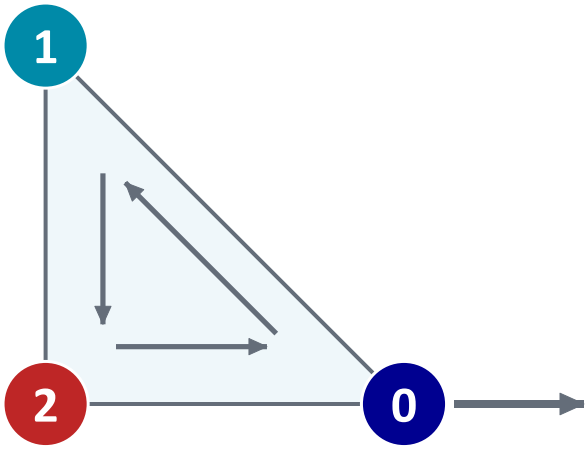


Colors 0, 1, 0
No trichromatic triangle

Sink: clockwise. Source: counterclockwise.

SOURCE

0 → 1 → 2

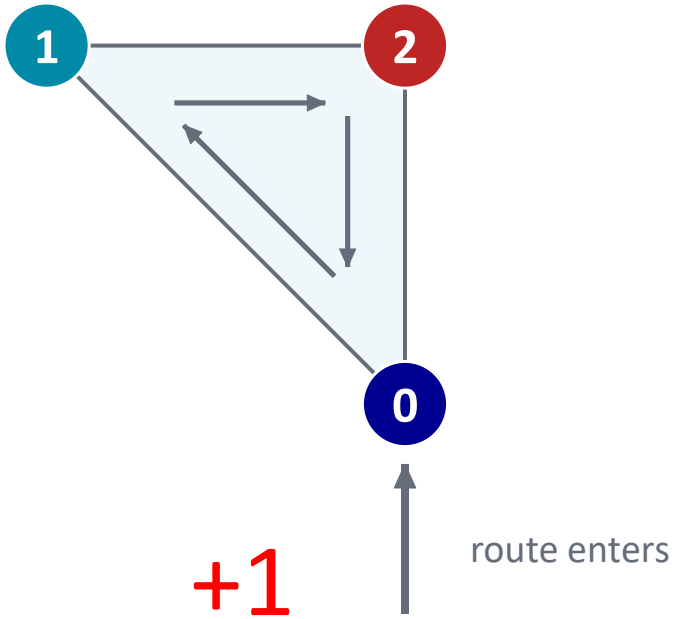


−1

Counterclockwise

SINK

0 → 1 → 2



+1

Clockwise

Orientation becomes fixed-point index

$$q_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad q_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad q_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

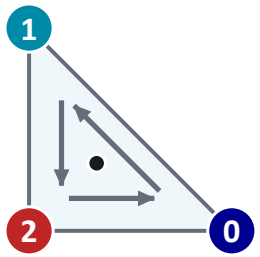
$$G(z) = \sum_{v \in \{0,\dots,M\}^2} h_v(Mz)\, q_{C(v)} \qquad \Phi(z) = \left[z + G(z) \right]_0^1$$

$h_v(Mz)$: interpolation weight of the integer vertex v

Clip each coordinate to [0, 1]

$$\mathrm{ind}_\Phi(p) = \mathrm{sign}\left(\frac{-3}{\det[p_1 - p_0 \mid p_2 - p_0]} \right) = -\,\mathrm{sign}(\det[p_1 - p_0 \mid p_2 - p_0])$$

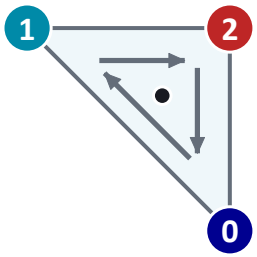
p_j : vertex of color j, $\quad p = \frac{p_0 + p_1 + p_2}{3}$ (barycenter)



−1

0 → 1 → 2

Counterclockwise



+1

0 → 1 → 2

Clockwise

An exact circuit for the field

$$G(z) = \sum_{v \in \{0,\ldots,M\}^2} h_v(Mz)\, q_{C(v)}$$

$$P, S \quad \longrightarrow \quad C(v) \in \{0,1,2\}$$

$$\Phi(z) = \left[z + G(z) \right]_0^1$$

Implementation of Φ:

1. Polynomial-size rational affine/max circuit.
2. Computes Φ exactly at every input.
3. Every max gate is strict at fixed points.

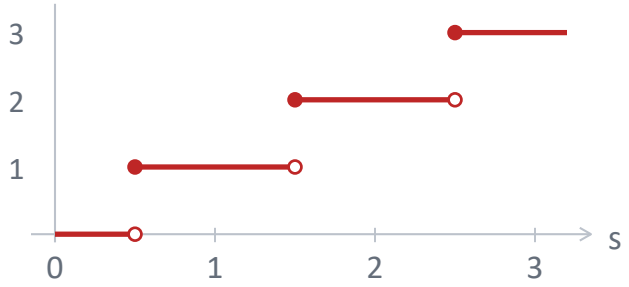
We need bit extraction, but $\lfloor Mz \rfloor$ is discontinuous.

A continuous distance circuit

Earlier reductions [DGP09, CDT09]

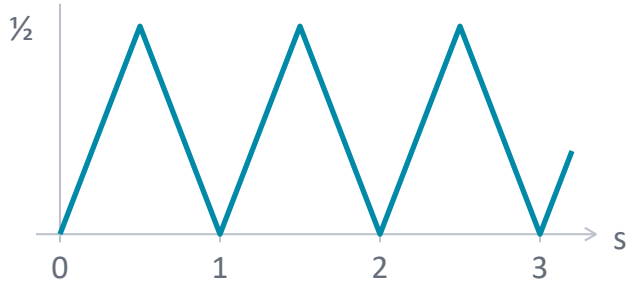
Nearby samples → bit extraction → average vectors

Nearest integer



Discontinuous

Distance to nearest integer



Continuous

Exact implementation

$$0 \le s \le 2^B, \qquad d_{B+1} = s$$

$$d_j = \min\{d_{j+1},\, 2^j - d_{j+1}\}, \qquad j = B, \ldots, 0$$

$$d_0 = \mathrm{dist}(s, \mathbb{Z})$$

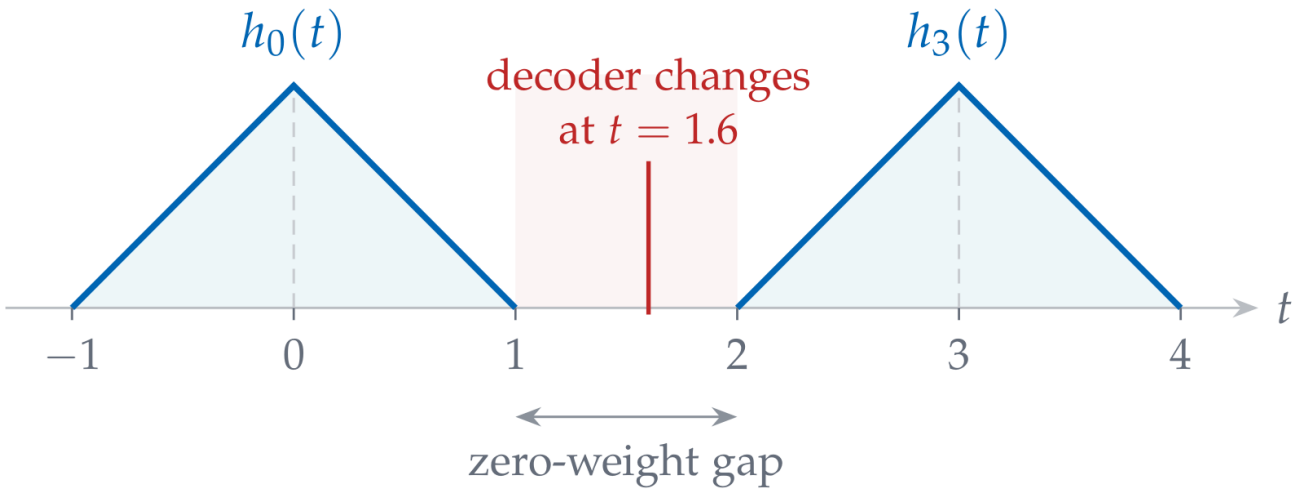
Decoding changes in zero-weight gaps

Group vertices by their coordinates modulo 3

3 groups in 1D · 9 groups in 2D

1D example: vertices 0 and 3 in the same group

$$h_v(t) = \max\{0, 1 - |t - v|\}$$



Exact decoding wherever the weight is positive.

Uncertain decoding contributes zero.

Every max gate is strict at fixed points

$$x_i = \max\{0, \ell_i\}, \qquad \ell_i(p) \neq 0$$

active



positive gate value

inactive



positive slack

$$\mathrm{AND}^*(a,b) = \max\{0, 2a + 2b - 3\}, \quad a, b \in \{0,1\}$$

At Boolean inputs, the max arguments are unequal.

Strict complementarity makes the circuit locally affine.

Circuit to LCP formulation (Mehta's idea [M18])

$$z = \max\{0, \ell\} \quad \Longleftrightarrow \quad z \geq 0, \quad w = z - \ell \geq 0, \quad zw = 0$$

$$x_i = \max\left\{0,\ b_i + u_i^{\mathsf{T}}\lambda + \sum_{j<i} c_{ij}x_j\right\}, \qquad i = 1, \ldots, m$$

$\lambda \in [0,1]^2$ input

$x \in \mathbb{R}^m_{\geq 0}$ auxiliary gate values

Circuit execution

$$x \geq 0, \qquad w = \mathbf{T}x - \mathbf{U}\lambda - b \geq 0, \qquad x_i w_i = 0 \quad (\forall i)$$

$\mathbf{T}$ is unit lower triangular; w contains the gate slacks

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$\mathbf{T}$ is unit lower triangular; w contains the gate slacks

Close the circuit: $\lambda = \Phi(\lambda) = \mathbf{V}^{\mathsf{T}}x$

LCP form

$$\mathbf{M} = \mathbf{T} - \mathbf{U}\mathbf{V}^{\mathsf{T}}$$

$$x \geq 0, \qquad w = \mathbf{M}x - b \geq 0, \qquad x_i w_i = 0 \quad (\forall i)$$

LCP keeps the index

$$\Phi(p) = p, \qquad I = \{i : x_i(p) > 0\}$$

Strict complementarity: both inequalities persist near p

$$\underbrace{x_I(p) > 0}_{\text{active}}, \quad \underbrace{w_{I^c}(p) > 0}_{\text{inactive}} \quad \Longrightarrow \quad x_{I^c}(\lambda) = 0, \quad w_I(\lambda) = 0$$

Φ is locally affine: differentiate the active equations

$$\mathbf{T}_{I,I}\, x_I(\lambda) = \mathbf{U}_I \lambda + b_I, \qquad \Phi(\lambda) = \mathbf{V}_I^{\mathsf{T}} x_I(\lambda)$$

$$D\Phi(p) = \mathbf{V}_I^{\mathsf{T}} \mathbf{T}_{I,I}^{-1} \mathbf{U}_I$$

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Apply the matrix determinant identity

$$\mathbf{M}_{I,I} = \mathbf{T}_{I,I} - \mathbf{U}_I \mathbf{V}_I^{\mathsf{T}}$$

$$\det \mathbf{M}_{I,I} = \det \mathbf{T}_{I,I} \det\left(\mathbf{I}_2 - \mathbf{V}_I^{\mathsf{T}} \mathbf{T}_{I,I}^{-1} \mathbf{U}_I\right)$$

$$= \det(\mathbf{I}_2 - D\Phi(p))$$

$$\det \mathbf{T}_{I,I} = 1$$

$$\operatorname{sign} \det \mathbf{M}_{I,I} = \operatorname{ind}_\Phi(p)$$

LCP to symmetric games $(\mathbf{S}, \mathbf{S}^T)$

$$\mathbf{S} = \begin{pmatrix} -\mathbf{M} & b + \mathbf{1} \\ \mathbf{0}^\mathsf{T} & 1 \end{pmatrix}$$

Add an anchoring strategy ★

LCP solutions $\longleftrightarrow$ symmetric NE (y, y)

$$y = \frac{(x, 1)}{1 + \mathbf{1}^\mathsf{T} x} \qquad y_\star > 0$$

Every symmetric equilibrium must play ★.

The imitation game preserves every sign

From symmetric equilibria to all Nash equilibria

$$G = (\mathbf{S}, \mathbf{I}), \qquad u = \frac{\mathbf{1}_K}{|K|}, \quad K = \operatorname{supp}(x) \cup \{\star\}$$



$$\boxed{\operatorname{ind}_G(u, y) = \operatorname{sign} \det \mathbf{M}_{I,I} = \operatorname{ind}_\Phi(p)}$$

Global nondegeneracy

Regular equilibria → Nondegenerate game

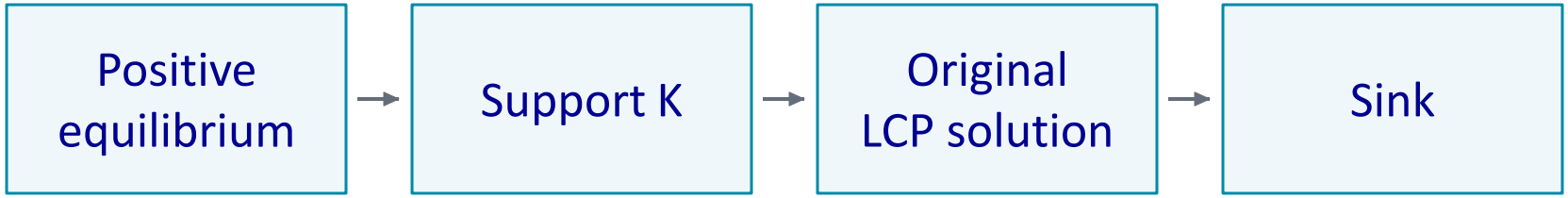
local — global

Shift to positive payoffs, then rescale each row

$$\mathbf{S}_i(t) = \frac{\mathbf{S}_i}{1 + t^i}, \qquad t = 2^{-\mathrm{poly}(L)}$$

Same supports — Same indices — No extra supports

Every positive equilibrium decodes a sink



$$I = K \setminus \{\star\}, \qquad \mathbf{M}_{I,I} x_I = b_I$$

Decode the support, then solve exactly.

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