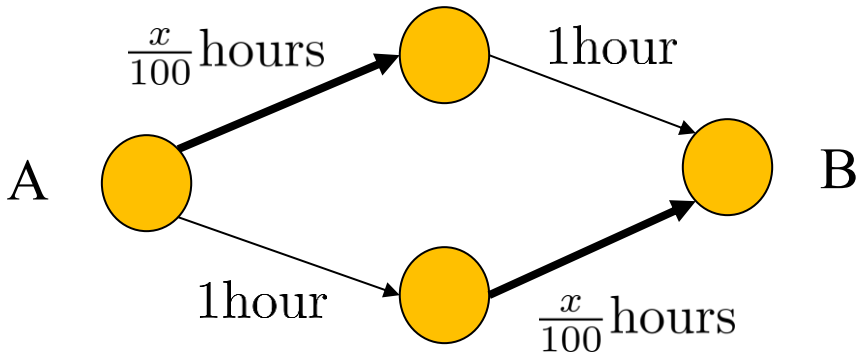
L15 Price of Anarchy

CS 280 Introduction to Algorithmic Game Theory

Ioannis Panageas

Price of Anarchy

100 units of traffic travel from A to B.
Each driver is negligible.



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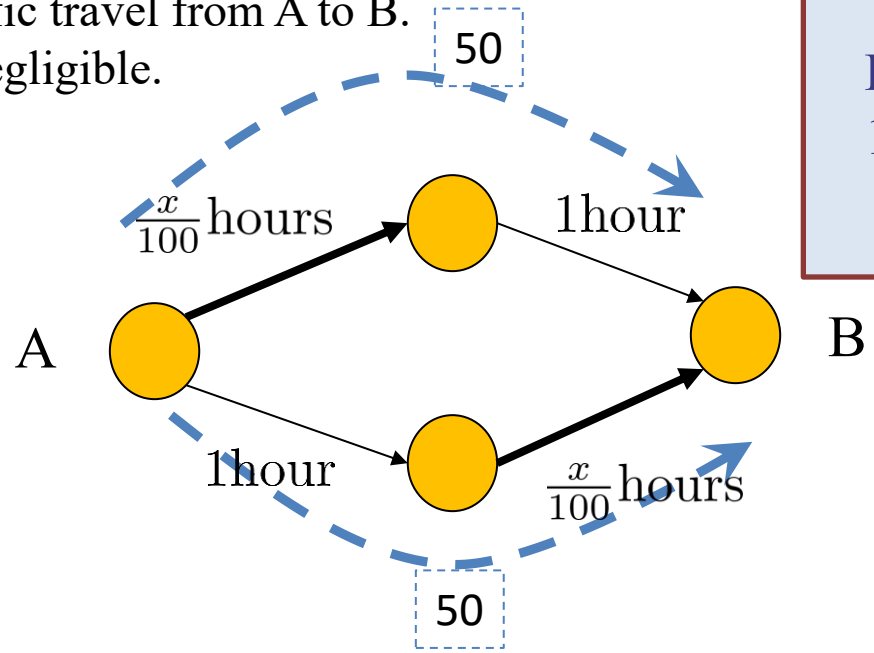

Delay at equilibrium:
1.5 hours per driver.

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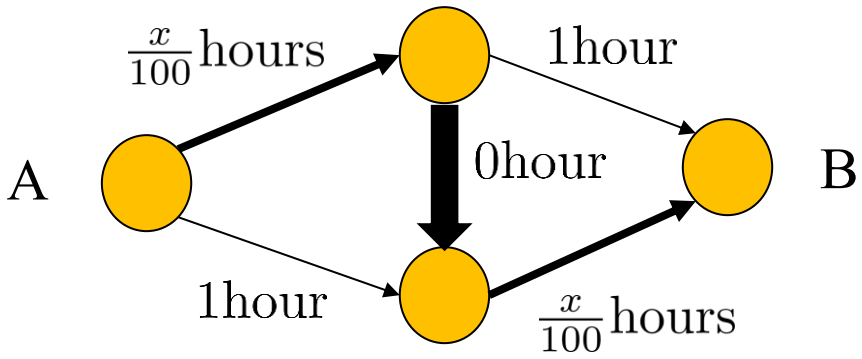
Question: What if we add a new link?



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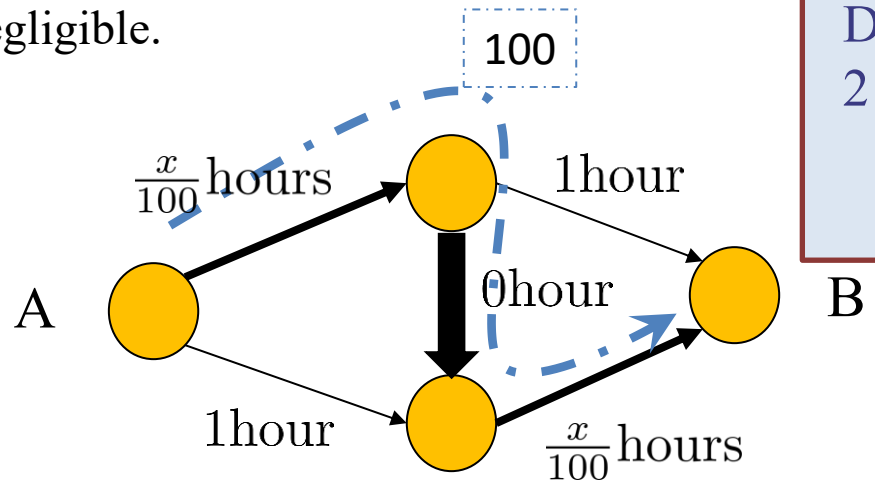

Delay at equilibrium:
2 hours per driver.
Braess's paradox!

Adding a fast link is not always a good idea!

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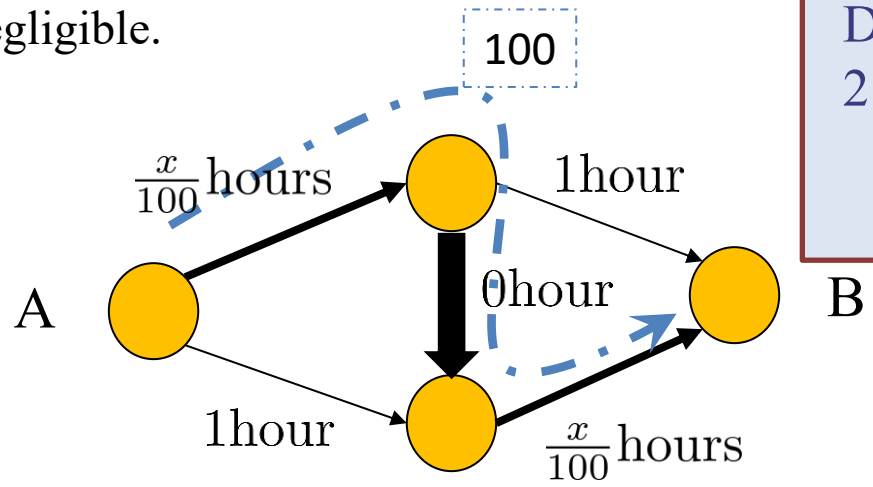

Delay at equilibrium:
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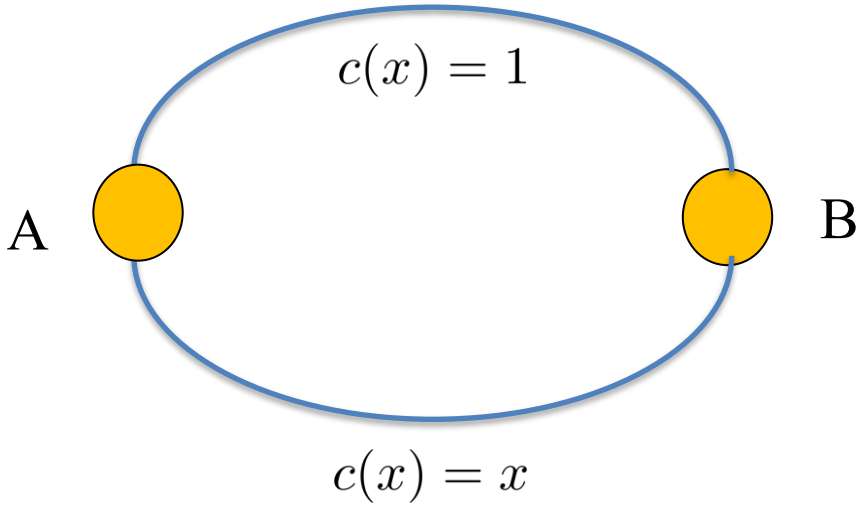
$$\text{PoA} = \frac{\text{worst equilibrium total cost}}{\text{minimum feasible total cost}}$$

Price of Anarchy (Koutsoupias, Papadimitriou 99').

PoA = 4/3

Non-atomic selfish routing

Pigou network: one unit of traffic.



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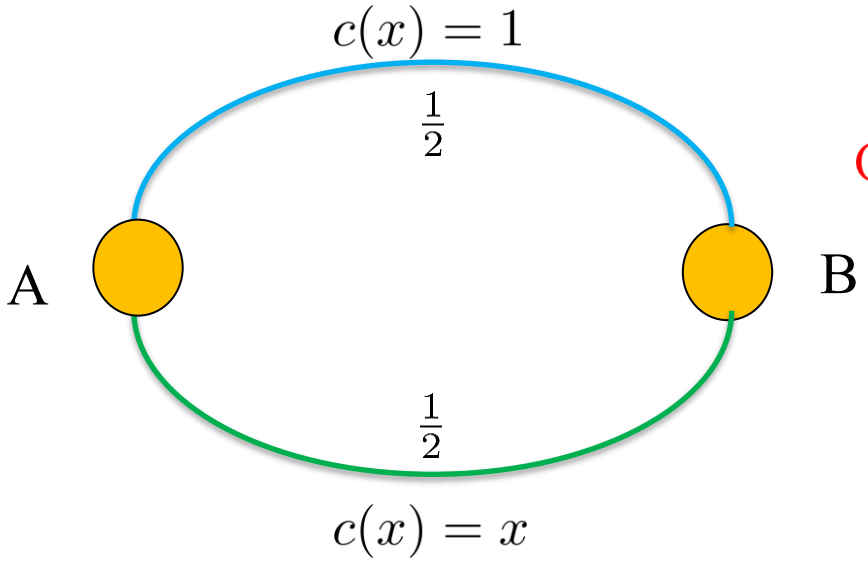

Cost $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

Non-atomic selfish routing

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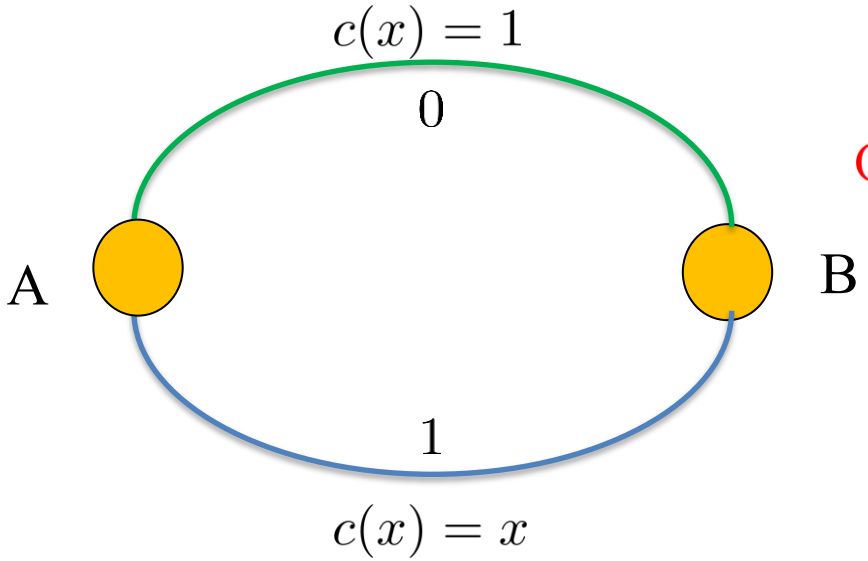

Cost 1 and **PoA** $= \frac{4}{3}$

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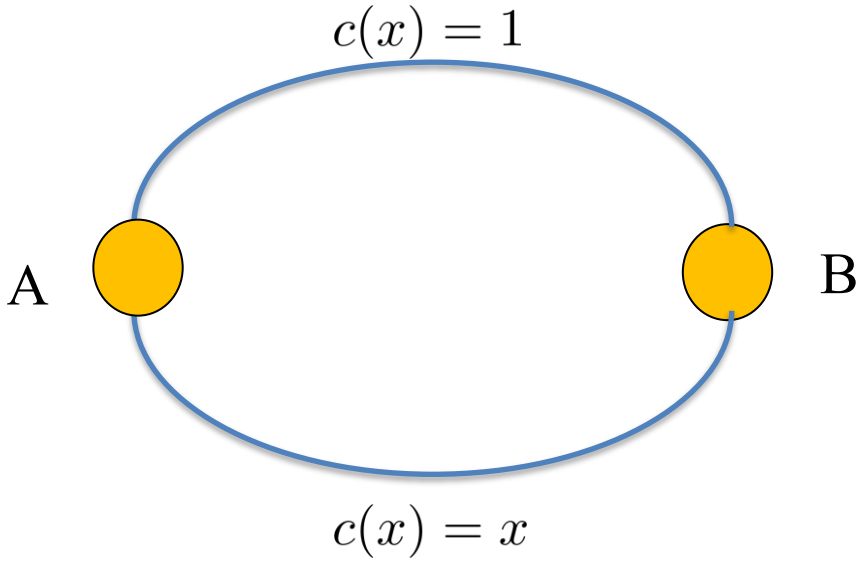

A non-atomic routing game is defined by:

- A graph $G = (V, E)$.
- Paths $\mathcal{P}_i$ from s_i to t_i, with demand r_i.
- Continuous, nonnegative, nondecreasing costs $c_e(x)$.
- Each driver has negligible influence.
- At equilibrium, used paths are cheapest for their pair.

Non-atomic selfish routing

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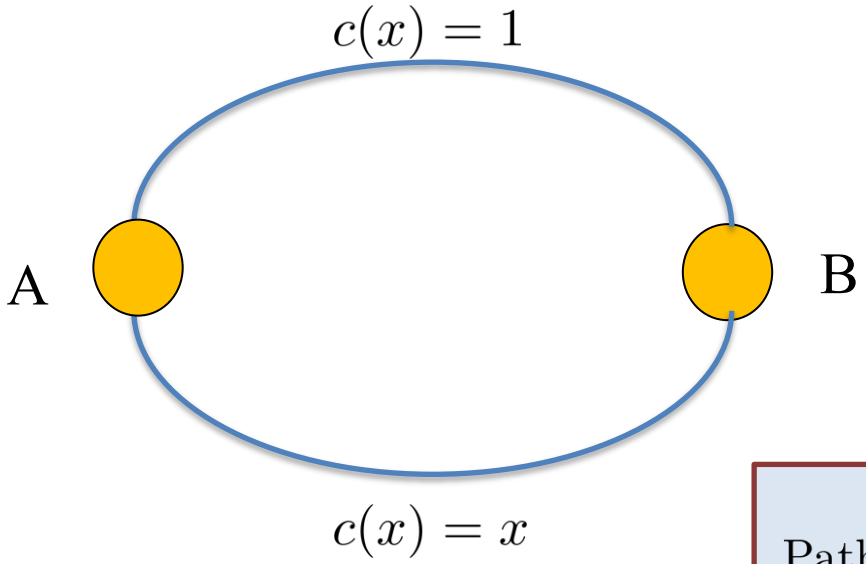

Path cost: $c_p(f) = \sum_{e \in p} c_e(f_e)$

Total cost: $C(f) = \sum_p f_p c_p(f)$

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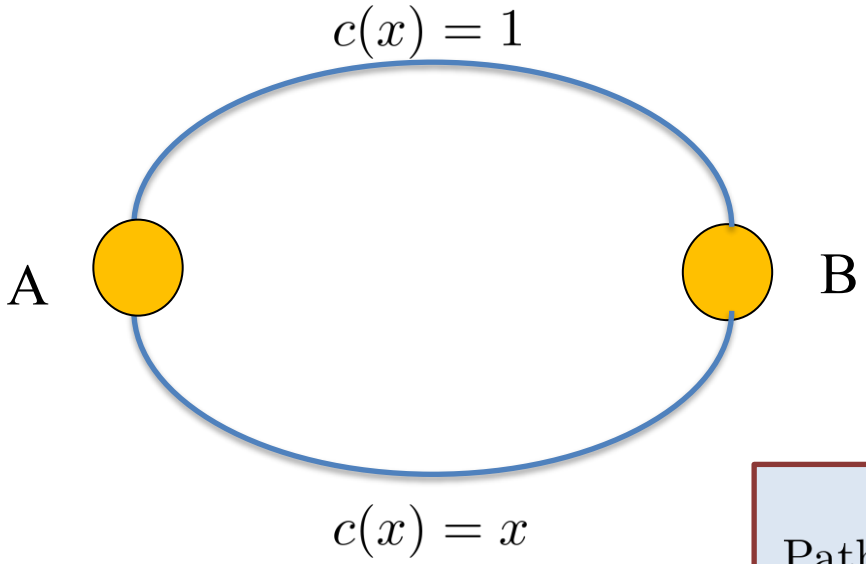

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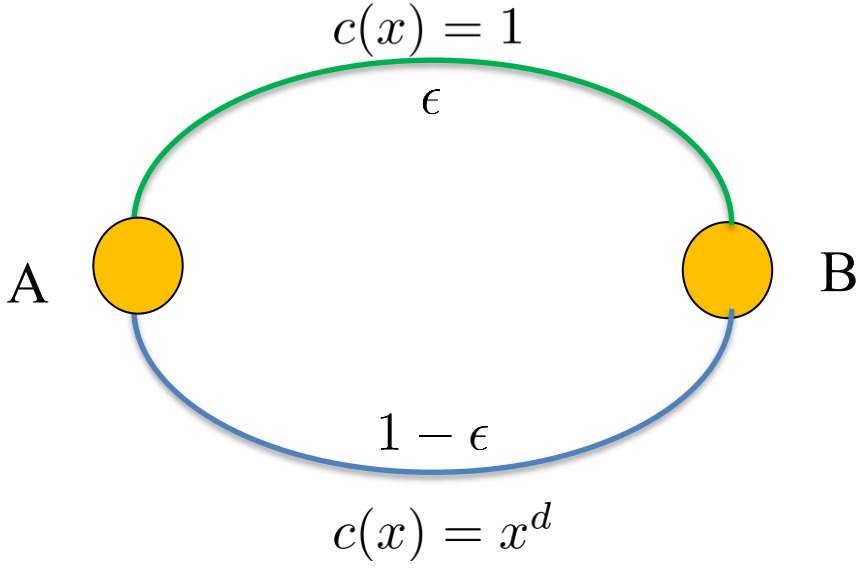
Equilibrium exists.
Flows need not be unique!

Non-atomic selfish routing

A bad Example. **Pigou network with large degree *d*.**



For this routing, cost is $\epsilon + (1-\epsilon)^{d+1}$. Equilibrium costs 1.

$$\text{PoA} \geq \frac{1}{\epsilon + (1-\epsilon)^{d+1}} \longrightarrow \frac{1}{\epsilon} \quad (d \to \infty).$$

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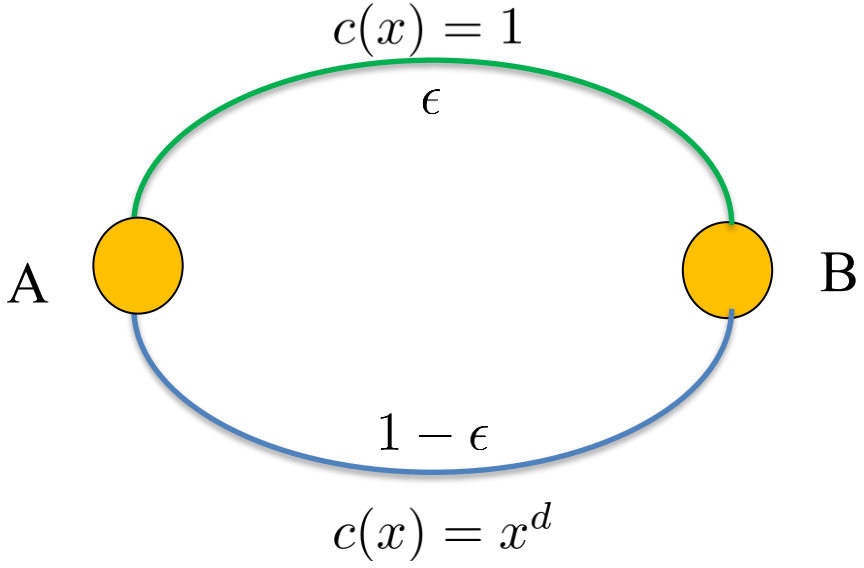

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Questions:

1. When is PoA small (bounded)?
2. Can we find bounds on PoA for specific classes of cost functions?

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

Definition (Linear (affine) costs). $c_e(x) = a_e x + b_e$, *where* $a_e, b_e \geq 0$.

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Theorem (Roughgarden–Tardos, PoA for linear costs). *For every nonatomic routing game with nonnegative affine costs:*

$$\textit{cost of equilibrium flow} \leq \frac{4}{3} \textit{cost of optimal flow}.$$

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$$\sum_e f_e^* c_e(f_e^*) \leq \sum_e f_e c_e(f_e^*).$$

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Observe that

For each pair i, let $L_i = \min_{p \in \mathcal{P}_i} c_p(f^*)$. Then

$f_p^* > 0,\ p \in \mathcal{P}_i \ \Rightarrow\ c_p(f^*) = L_i.$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

Proof cont. For each pair i, all used paths in $\mathcal{P}_i$ have the same cost L_i.

Hence $$\sum_i \sum_{p \in \mathcal{P}_i} f_p^* c_p(f^*) = \sum_i r_i L_i.$$

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Hence $$\sum_i \sum_{p \in \mathcal{P}_i} f_p^* c_p(f^*) = \sum_i r_i L_i.$$

Every feasible flow routes r_i units for each pair. Since $c_p(f^*) \geq L_i$ for $p \in \mathcal{P}_i$,

$$\sum_i \sum_{p \in \mathcal{P}_i} f_p c_p(f^*) \geq \sum_i r_i L_i.$$

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Combining the inequalities and summing by edges:

$$\sum_e f_e c_e(f_e^*) \geq \sum_i r_i L_i = \sum_e f_e^* c_e(f_e^*).$$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

$$\sum_e f_e c_e(f^*) \geq \sum_e f_e^* c_e(f^*).$$

Proof cont. We get that

$$\sum_e f_e^* c_e(f^*) \leq \sum_e f_e c_e(f) + \sum_e f_e (c_e(f^*) - c_e(f))$$

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$$\sum_e f_e(c_e(f^*) - c_e(f)) \leq \frac{1}{4} \sum_e f_e^* c_e(f^*)$$

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Since $xy - y^2 \leq \frac{x^2}{4} \Rightarrow$ LHS $\leq$ RHS.

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Pigou example is tight!

Theorem (Roughgarden, Polynomial costs). *For polynomial costs with nonnegative coefficients of degree at most $d \geq 2$, the worst-case*

$$\text{PoA} = \Theta\left(\frac{d}{\log d}\right).$$

Price of Anarchy in Congestion Games

Theorem (Christodoulou–Koutsoupias, linear costs). *For unweighted congestion games with* $c_e(x) = a_e x + b_e$, $a_e, b_e \geq 0$, *every pure Nash equilibrium satisfies*

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$$\sum_{e \in P_i} c_e(l_e^*) \leq \sum_{e \in P_i \cap \tilde{P}_i} c_e(l_e^*) + \sum_{e \in \tilde{P}_i \setminus P_i} c_e(l_e^* + 1).$$

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&= \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1).
\end{aligned}$$

Price of Anarchy in Congestion Games

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Proof cont. Let $\tilde{P}_i$ be player i's strategy in any other profile, with loads $\tilde{l}$. Sum over all players. Write $C(l) = \sum_e l_e c_e(l_e)$.

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$$= \sum_e \tilde{l}_e c_e(l_e^* + 1).$$

$$= \sum_{e \in E} \left[a_e \tilde{l}_e (l_e^* + 1) + b_e \tilde{l}_e \right].$$

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\end{aligned}$$

$$y(z+1) \leq \frac{5}{3} y^2 + \frac{1}{3} z^2 \quad \text{for nonnegative integers } y, z.$$

$$\leq \sum_{e \in E} \left[a_e \left(\frac{5}{3} \tilde{l}_e^2 + \frac{1}{3} (l_e^*)^2 \right) + b_e \tilde{l}_e \right].$$

Price of Anarchy in Congestion Games

$$\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \le \sum_{e\in E}\left[a_e\left(\frac{5}{3}\tilde{l}_e^2 + \frac{1}{3}(l_e^*)^2\right) + b_e\tilde{l}_e\right].$$

Proof cont. Observe that

$$\begin{aligned}\frac{5}{3}C(\tilde{l}) &= \sum_{e\in E}\left[\frac{5}{3}a_e\tilde{l}_e^2 + \frac{5}{3}b_e\tilde{l}_e\right] \\ &\ge \sum_{e\in E}\left[\frac{5}{3}a_e\tilde{l}_e^2 + b_e\tilde{l}_e\right].\end{aligned}$$

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Therefore

$$C(l^*) \le \frac{5}{3}C(\tilde{l}) + \frac{1}{3}\sum_e a_e {l_e^*}^2$$

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$$\begin{aligned}C(l^*) &\le \frac{5}{3}C(\tilde{l}) + \frac{1}{3}\sum_e a_e {l_e^*}^2\\ &\le \frac{5}{3}C(\tilde{l}) + \frac{1}{3}C(l^*)\end{aligned}$$

Price of Anarchy in Congestion Games

$$\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \le \sum_{e\in E}\left[a_e\left(\frac{5}{3}\tilde{l}_e^2+\frac{1}{3}(l_e^*)^2\right)+b_e\tilde{l}_e\right].$$

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$$C(l^*) \le \tfrac{5}{2}C(\tilde{l}).$$

Remark:

1. The bound 5/2 is tight.
2. For degree-d polynomials with nonnegative coefficients, PoA is $d^{\Theta(d)}$.

Price of Anarchy and Balls & Bins

Definition (Balls and Bins)**.**

- *n balls choose among n bins* $\{e_1, \ldots, e_n\}$.
- *Each ball pays the load of its chosen bin.*
- *Social cost:* ***maximum load****.*
- *For mixed Nash equilibria, compare* ***expected maximum load*** *with the optimum. What is the PoA?*

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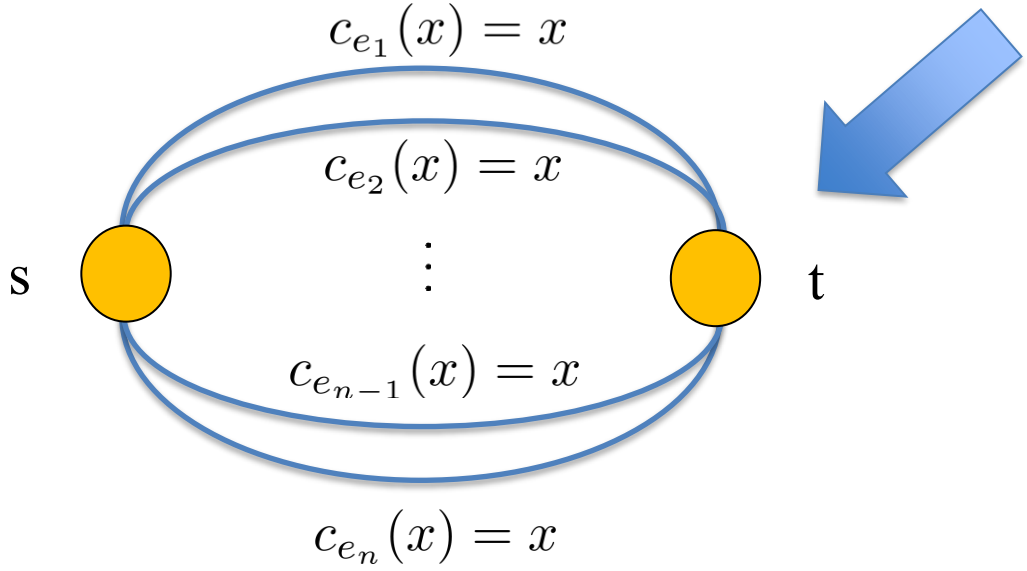
Congestion game!



Price of Anarchy and Balls & Bins

Theorem (Koutsoupias–Papadimitriou, balls & bins)**.** *For expected maximum load, the mixed Nash PoA is*

$$\Omega\left(\frac{\ln n}{\ln \ln n}\right).$$

Proof. Use the second moment method.

- One ball per bin gives optimal maximum load 1.
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Claim 1: For integers $1 \le k \le n$, the probability that bin i has at least k balls is at least:

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$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Price of Anarchy and Balls & Bins

Proof cont. For large n, let $k = \left\lfloor \frac{\ln n}{3 \ln \ln n} \right\rfloor$. Then $k^k \leq (\ln n)^k \leq n^{1/3}$.

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thus $Pr[X = 0] \leq Pr[|X - E[X]| \geq E[X]] \leq \frac{Var[X]}{E^2[X]}$.

Price of Anarchy and Balls & Bins

Proof cont. $Pr[X = 0] \leq \frac{Var[X]}{E^2[X]}$.

Negative correlation gives $\mathrm{Cov}(X_i, X_j) \leq 0$ for $i \neq j$, hence

$$\mathrm{Var}(X) \leq \sum_i \mathrm{Var}(X_i).$$

Moreover, $X_i \in \{0, 1\}$, so

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Therefore

$$\Pr[X \geq 1] = 1 - \Pr[X = 0] \geq 1 - e^2 n^{-1/3} \longrightarrow 1.$$

Congestion Games

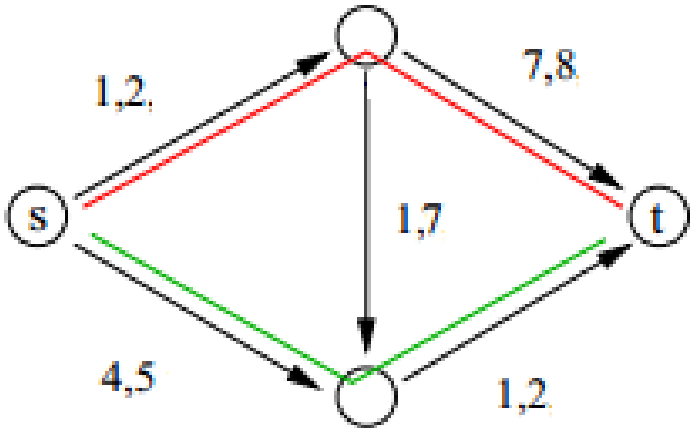
A congestion game has:

- Player set $N = \{1, \ldots, n\}$.
- Facility set E (edges or bins).
- Nonempty strategy sets $S_i \subseteq 2^E$.
- Costs $c_e : \{1, \ldots, n\} \to \mathbb{R}_{\geq 0}$.

For a strategy profile $s = (s_1, \ldots, s_n)$:

- $l_e(s)$ is the number of players using facility e.
- Player i pays $c_i(s) = \sum_{e \in s_i} c_e(l_e(s))$.

Congestion Games



For this game:

$N = \{1, 2\}$ (red and green players).
E is the set of network edges.
S_i contains all s–t paths.
The edge labels give c_e.

Remark: Defined by Rosenthal in 1973. Capture atomic routing **games!**