L14 Myerson's Lemma cont (Bayesian).

CS 280 Introduction to Algorithmic Game Theory

Ioannis Panageas

Based in part on Tim Roughgarden's lecture notes

Recap (Single parameter)

Three desirable guarantees

1. DSIC: truthful bidding is a dominant strategy.

2. Social surplus maximization.

3. Implementation in polynomial time.

Recap (Single parameter)

Myerson's Lemma

An allocation can be implemented by a DSIC mechanism exactly when it is monotone.

With payments normalized by $p_i(0,b_{-i})=0$, the payment rule is unique.

$$p_i(b) = b_i x_i(b) - \int_0^{b_i} x_i(z, b_{-i})\, dz.$$

Definitions: Bayesian Setting

Each bidder has a private value per unit of allocation.

$$x = (x_1, \ldots, x_n) \in \mathcal{X}, \qquad u_i = v_i x_i - p_i.$$

Values are independent, but need not have identical distributions. The seller knows their distributions.

$$v_i \sim F_i, \qquad 0 \leq v_i \leq v_{\max}, \qquad f_i(v) > 0.$$

Assume continuous densities and zero payment at a zero bid.

Definitions: Bayesian Setting

One item, one bidder. The seller posts a price r.

The bidder buys exactly when their value reaches the price.

$$\Pr[v \geq r] = 1 - F(r).$$

$$\mathbb{E}[\text{revenue}] = r(1 - F(r)).$$

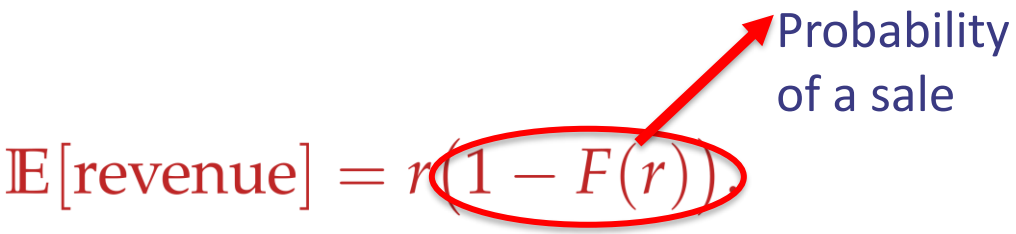
Probability of a sale

Definitions: Bayesian Setting

A reserve price is the minimum bid required to receive the item.

$$x(v) = \begin{cases} 1, & v \geq r, \\ 0, & v < r, \end{cases} \qquad p(v) = r\, x(v).$$

A higher reserve raises the price but reduces the chance of a sale.



$$\mathbb{E}[\text{revenue}] = r(1 - F(r)).$$

Definitions: Bayesian Setting

One item, one bidder with a uniform value on [0,1].

$$F(v) = v \qquad (0 \le v \le 1).$$

$$\mathbb{E}[\text{revenue}] = r(1-r), \qquad 0 \le r \le 1.$$

Which posted price maximizes expected revenue?

Definitions: Bayesian Setting

For the uniform distribution, complete the square:

$$r(1-r) = \frac{1}{4} - \left(r - \frac{1}{2}\right)^2.$$

$$r^* = \frac{1}{2}, \qquad \mathbb{E}[\text{revenue}] = \frac{1}{4}.$$

The item sells with probability one half.

More Definitions

Fix a bidder and the other bidders' values. Assume truthful bids and a DSIC mechanism.

$$p_i(0, v_{-i}) = 0.$$

Myerson's payment formula:

$$p_i(v_i, v_{-i}) = v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(z, v_{-i})\, dz.$$

More Definitions

The payment is random because the bidder's value is random.

Throughout the calculation, keep the other values fixed. Write:

$$\mathbb{E}[p_i] := \mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] .$$

$$\mathbb{E}[p_i] = \int_0^{v_{\max}} p_i(v_i, v_{-i}) f_i(v_i)\, dv_i.$$

More Definitions

Substitute Myerson's payment formula into the expectation.

$$\mathbb{E}[p_i] = \int_0^{v_{\max}} v_i x_i(v_i, v_{-i}) f_i(v_i)\, dv_i - \int_0^{v_{\max}} \left[\int_0^{v_i} x_i(z, v_{-i})\, dz \right] f_i(v_i)\, dv_i.$$

We simplify the second term by changing the integration order.

Payments reduce to welfare

The second term integrates over the same triangular region:

$$0 \le z \le v_i \le v_{\max}.$$

$$\int_0^{v_{\max}} \left[\int_0^{v_i} x_i(z, v_{-i})\, dz \right] f_i(v_i)\, dv_i$$
$$= \int_0^{v_{\max}} x_i(z, v_{-i}) \left[\int_z^{v_{\max}} f_i(v_i)\, dv_i \right] dz.$$

For each fixed z, the possible values run from z to the upper endpoint.

Payments reduce to welfare

The inner integral is the probability that the value reaches z:

$$\int_z^{v_{\max}} f_i(v_i)\, dv_i = 1 - F_i(z).$$

Therefore the second term becomes

$$\int_0^{v_{\max}} x_i(z, v_{-i}) \big(1 - F_i(z)\big)\, dz.$$

Payments reduce to welfare

Return to the expected payment and use z in both integrals.

$$\mathbb{E}[p_i] = \int_0^{v_{\max}} z\, x_i(z, v_{-i}) f_i(z)\, dz - \int_0^{v_{\max}} x_i(z, v_{-i}) \big(1 - F_i(z)\big)\, dz.$$

Combine the two terms:

$$\mathbb{E}[p_i] = \int_0^{v_{\max}} x_i(z, v_{-i}) \big[z f_i(z) - (1 - F_i(z))\big] dz.$$

Payments reduce to welfare

Factor out the density, which is positive on the support:

$$z f_i(z) - (1 - F_i(z)) = \left(z - \frac{1 - F_i(z)}{f_i(z)} \right) f_i(z).$$

$$\mathbb{E}[p_i] = \int_0^{v_{\max}} x_i(z, v_{-i}) \left(z - \frac{1 - F_i(z)}{f_i(z)} \right) f_i(z)\, dz.$$

Payments reduce to welfare

The density now gives an expectation over the bidder's value:

$$\mathbb{E}[p_i] = \mathbb{E}_{v_i \sim F_i}\left[x_i(v_i, v_{-i})\left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right)\right].$$

The transformed value depends only on the bidder's value and distribution.

This converts a payment objective into an allocation objective.

Payments reduce to welfare

Define the virtual valuation:

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

It can be negative even when the value is positive.

$$\mathbb{E}[p_i] = \int_0^{v_{\max}} x_i(z, v_{-i})\phi_i(z)f_i(z)\,dz.$$

Payments reduce to welfare

For each bidder, holding the other values fixed:



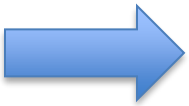
$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \mathbb{E}_{v_i \sim F_i}[x_i(v_i, v_{-i})\phi_i(v_i)]\,.$$

Expected payment equals expected virtual welfare contributed by this bidder.

The identity holds for every normalized DSIC mechanism.

Payments reduce to welfare

Average over the other values, then sum over all bidders.



$$\text{Rev} = \mathbb{E}_v\left[\sum_i p_i(v)\right] = \mathbb{E}_v\left[\sum_i x_i(v)\phi_i(v_i)\right].$$

Expected revenue equals expected virtual welfare.

We can optimize the allocation rule using virtual values.

Monotone Allocations for regular F

Step 1: maximize virtual welfare over feasible allocations.

$$x(v) \in \arg\max_{x \in \mathcal{X}} \sum_i x_i \phi_i(v_i).$$

Step 2: use Myerson's payments if this allocation rule is monotone.

Fix the rule for breaking ties in advance.

Monotone Allocations for regular F

The issue is monotonicity in the bidder's own report.

$$b_i' > b_i \quad \Longrightarrow \quad x_i(b_i', b_{-i}) \geq x_i(b_i, b_{-i}) \ ?$$

A larger bid changes its coefficient in the virtual-welfare objective.

$$\phi_i(b_i) = b_i - \frac{1 - F_i(b_i)}{f_i(b_i)}.$$

The distribution determines whether that coefficient increases.

Monotone Allocations for regular F

A distribution is regular when its virtual valuation is nondecreasing.

$$\phi(v) = v - \frac{1 - F(v)}{f(v)} \quad \text{is strictly increasing.}$$

Here we use the stronger assumption that it is strictly increasing.

Increasing a bid then increases its virtual value.

Monotone Allocations for regular F

With regular distributions, maximizing virtual welfare gives a monotone allocation.

Raising one bid raises only that bidder's coefficient in the objective.

An optimal allocation cannot then give that bidder less.

Myerson's payments make the mechanism DSIC and expected-revenue maximizing.

Monotone Allocations for regular F

Example: the uniform distribution on [0,1].

$$F(v) = v, \qquad f(v) = 1.$$

$$\phi(v) = v - \frac{1-v}{1} = 2v - 1.$$

The virtual valuation is strictly increasing, so the distribution is regular.

Example

One item and n bidders with independent values from the same regular distribution F.

$$\mathcal{X} = \left\{ x \in \{0,1\}^n : \sum_i x_i \leq 1 \right\}.$$

What allocation maximizes virtual welfare?

What payment makes it DSIC?

Example

Give the item to a bidder with the highest nonnegative virtual value.

$$i^* \in \arg\max_i \phi(b_i), \qquad \phi(b_{i^*}) \geq 0.$$

If every virtual value is negative, do not sell.

Break ties using a fixed priority order.

Example

The common virtual-value function is strictly increasing.

The highest bidder therefore has the highest virtual value.

$$r = \phi^{-1}(0).$$

The highest bidder wins when their bid reaches r.

A winner who raises their bid still wins, so the allocation is monotone.

Example

The winner pays the critical bid needed to win:

$$p_{\text{winner}} = \max\{r, b_{(2)}\}, \qquad r = \phi^{-1}(0).$$

The second term is the second-highest bid. Losers pay zero. No sale occurs if every bid is below the reserve.

$$F(v) = v \text{ on } [0,1] \quad \Longrightarrow \quad \phi(v) = 2v - 1, \quad r = \frac{1}{2}.$$

For uniform values, this reserve maximizes expected revenue.