L12 Monotone Allocations and Myerson's Lemma

CS 280 Algorithmic Game Theory

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Based on lecture notes by Tim Roughgarden

Recap

Three desirable guarantees

1. DSIC: truthful bidding is a dominant strategy.
   Bidders have a simple strategy, which predicts the outcome under truthful play.
2. Social surplus maximization:

$$\max_{x \in \mathcal{X}} \sum_{i=1}^{n} v_i x_i,$$

   where x_i is the amount allocated to bidder i.
3. Computational efficiency: the mechanism runs in polynomial time.

An Example: Sponsored Search Auctions

In sponsored search, an auction can determine which advertisers' links appear, their order, and the payments.

We study a simplified model with publicly known click-through rates.

An Example: Sponsored Search Auctions

In sponsored search, an auction chooses the ads, their order, and the payments.

- The items are k advertising slots. The bidders are advertisers.
- Slot j has a public click-through rate a_j, the probability of a click.
- Bidder i has private value per click $v_i \geq 0$. The expected value of slot j is $a_j v_i$.
- Each bidder receives at most one slot, and each slot at most one bidder.

$$1 \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0.$$

Definitions

Definition (Single-parameter environments).

- Bidder i has private value $v_i \geq 0$ per unit allocated.
- The public feasible set $\mathcal{X} \subseteq \mathbb{R}^n_{\geq 0}$ does not depend on bids.
- In an allocation $x = (x_1, \ldots, x_n) \in \mathcal{X}$, bidder i receives amount x_i and value $v_i x_i$.

Examples

- Single item: $\mathcal{X} = \{x \in \{0,1\}^n : \sum_i x_i \leq 1\}$.
- k identical items, at most one per bidder:

$$\mathcal{X} = \{x \in \{0,1\}^n : \sum_i x_i \leq k\}.$$

- Sponsored search: each bidder gets at most one slot, and each slot at most one bidder. Set $x_i = a_j$ for slot j, and $x_i = 0$ if unassigned.

More Definitions

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Definition (Allocation and payments).

1. Bidders submit nonnegative reports $b = (b_1, \ldots, b_n)$.
2. The auctioneer selects $x(b) \in \mathcal{X}$ and payments $p(b) \in \mathbb{R}^n$.
3. Bidder i has quasilinear utility

$$u_i(b) = v_i x_i(b) - p_i(b).$$

Monotone Allocations and Myerson's Lemma

Definition (Monotone allocations). An allocation rule x is monotone if, for every bidder i and every fixed vector b_{-i} of opponents' bids,

$$x_i(z, b_{-i}) \quad \text{is nondecreasing in } z \geq 0.$$

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Theorem (Myerson's Lemma). In a single-parameter environment, impose the normalization

$$p_i(0, b_{-i}) = 0 \qquad \text{for every } i, b_{-i}.$$

1. If (x, p) is DSIC, then x is monotone.
2. Every monotone x has a unique normalized payment rule p such that (x, p) is DSIC.

Myerson's Lemma: Monotone

Proof. Fix i and b_{-i}, and suppress b_{-i} in the notation. Let $0 \leq y < z$.

If bidder i has value z, truthfulness requires

$$zx_i(z) - p_i(z) \geq zx_i(y) - p_i(y).$$

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Combining the two inequalities:

$$y\big(x_i(z) - x_i(y)\big) \leq p_i(z) - p_i(y) \leq z\big(x_i(z) - x_i(y)\big).$$

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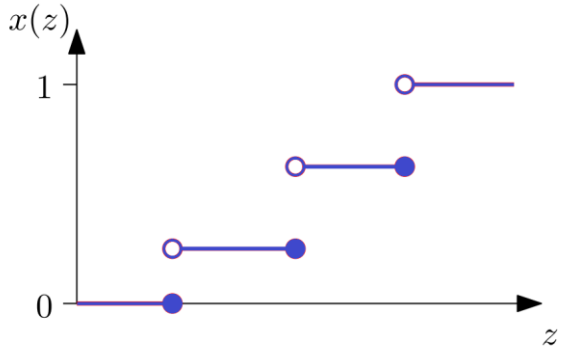
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First consider a step allocation. Take bids $y < z$ on opposite sides of one threshold t. Let $h = x_i(z) - x_i(y) > 0$ be the allocation increase.

$$yh \le p_i(z) - p_i(y) \le zh.$$



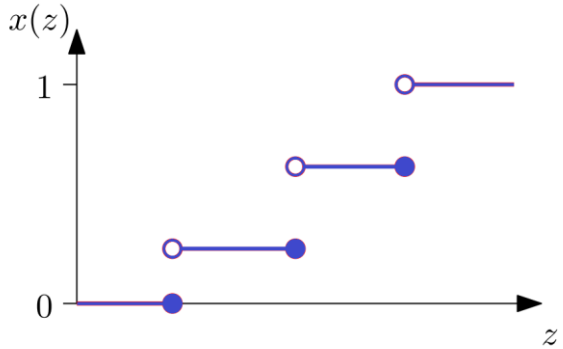
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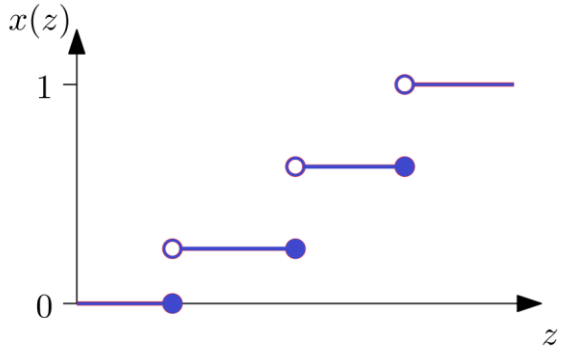
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With $p_i(0, b_{-i}) = 0$, add the payment jumps:

$$p_i(b_i, b_{-i}) = \sum_{z_j < b_i} z_j h_j.$$

Here h_j is the jump at z_j. Use the lower allocation at ties, as pictured.

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Divide the payment bounds by $z - y > 0$ and let $y \uparrow z$:

$$p_i'(z) = z \cdot x_i'(z)$$

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General payment formula, including jumps and ties:

$$p_i(b_i, b_{-i}) = b_i x_i(b_i, b_{-i}) - \int_0^{b_i} x_i(t, b_{-i})\, dt.$$

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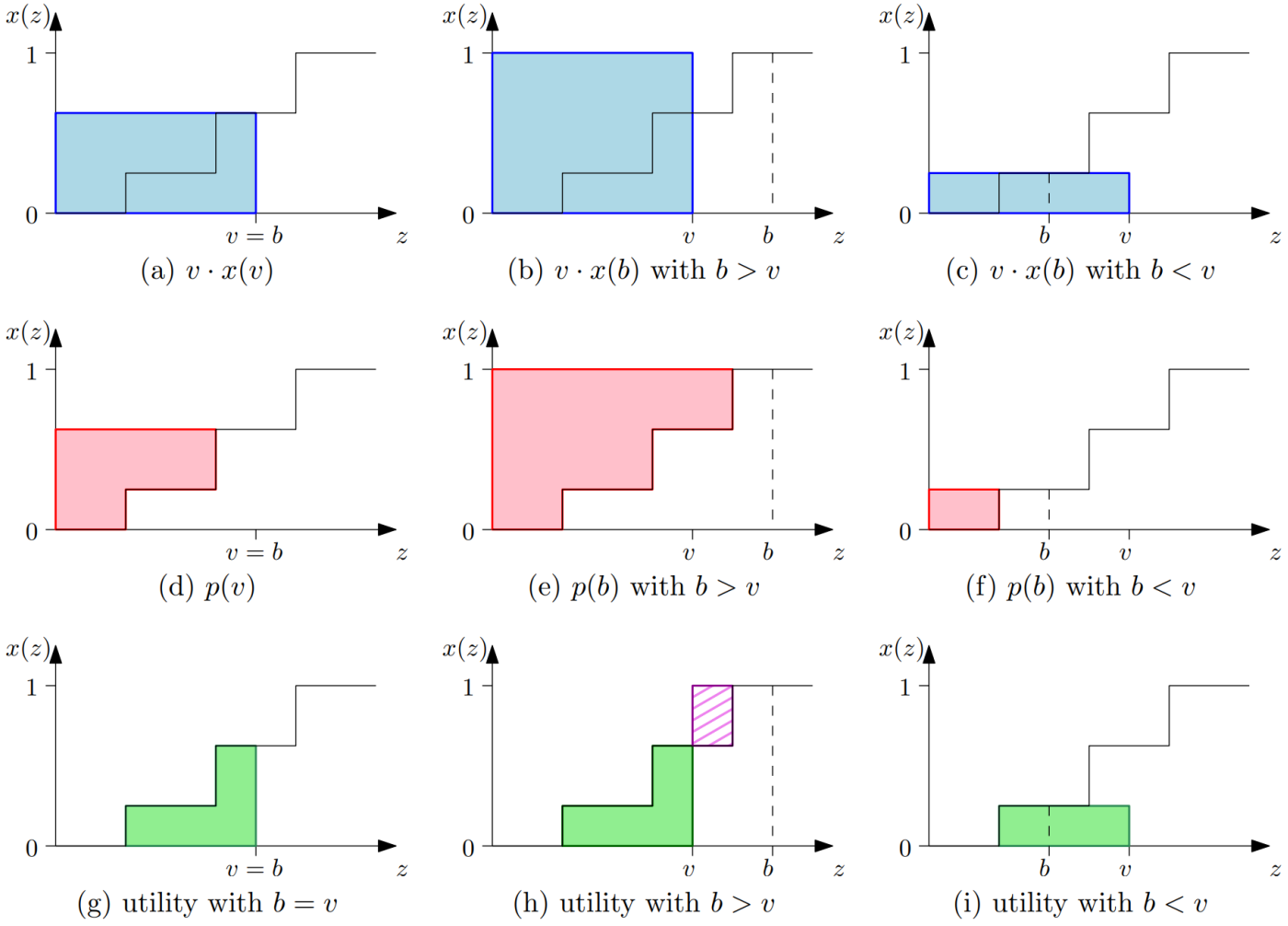
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$$p_i(b_i, b_{-i}) = \int_0^{b_i} t\, x_i'(t, b_{-i})\, dt.$$

Myerson's Lemma: DSIC

Truthful bidding maximizes value minus payment.



Back to Sponsored Search Auctions

Remark: Myerson's Lemma regenerates the Vickrey auction as a special case. (why?)

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Myerson's Lemma recovers the Vickrey auction.

Fix i, b_{-i} and let $B = \max_{j \neq i} b_j$. Bidder i loses when $b_i < B$ and wins when $b_i > B$. A fixed priority order resolves $b_i = B$.

The winner pays B and a losing bidder pays zero.

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1. Assume truthful bids and find an allocation that maximizes social surplus.
2. Choose payments that make this allocation rule DSIC.

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Assign the jth highest bidder to the jth highest slot, $j = 1, \ldots, \min\{k, n\}$, using a fixed priority order for ties.

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Fix the other bids and raise bidder 1's bid. Crossing b_{j+1} improves its allocation by $a_j - a_{j+1}$. Tied thresholds combine their jumps.

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$$p_i(b) = \sum_{j=i}^{k} b_{j+1}(a_j - a_{j+1}) \qquad (i \leq k).$$

This is the expected payment per impression. Unassigned bidders pay zero. The price per click is $p_i(b)/a_i$ when $a_i > 0$.

A (computationally) hard example: Knapsack auctions

Each bidder i has a **publicly known size** $w_i > 0$ and a **private value** $v_i \geq 0$ for being served.

The seller has **capacity** $W > 0$.

The **feasible set** consists of the binary vectors

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Remark:

- k-identical item auction is a special case (why)?

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Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we design the allocation so that we can maximize surplus?

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- Step 2: Given our answer to Step 1, how should we set the payments so that DSIC holds? Payment rule from Myerson's Lemma.

Exact optimization is not polynomial-time unless P = NP.

Relaxing Knapsack auctions: Approximation

Step 1. Find a **monotone approximation** to maximum surplus.

Discard bidders with $w_i > W$. If no bidders remain, serve nobody.

Sort the remaining bidders by **bid per unit of size**:

$$\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}.$$

Use a fixed bidder priority order to break ties.

Step 2. Choose payments using Myerson's Lemma.

Relaxing Knapsack auctions: Approximation

Step 1. Discard bidders with $w_i > W$ and sort by b_i/w_i.
Use fixed bidder priorities for ties.

Choose the **longest feasible prefix**. Stop at the first bidder that does not fit.
If all bidders fit, serve everyone. Otherwise, its length S satisfies

$$\sum_{i=1}^{S} w_i \leq W < \sum_{i=1}^{S+1} w_i.$$

Compare this prefix with the **highest-bid feasible singleton**.
Choose the greater total bid, preferring the prefix at equality.

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What guarantees does the auctioneer have?

Knapsack auctions: 2-Approximation

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Otherwise, let S be the length of the feasible prefix:

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Since the best singleton has value at least v_{S+1},

$$\mathrm{ALG} \geq \max\left\{\sum_{i=1}^{S} v_i,\ v_{S+1}\right\} \geq \frac{\mathrm{OPT}}{2}.$$

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If $A + B \ge \mathrm{OPT}$ then $\max\{A, B\} \ge \mathrm{OPT}/2$.

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Proof continued. Consider the fractional relaxation.
We will show that $\text{OPT} \leq \sum_{i=1}^{S+1} v_i$.

The fractional optimum has $x_1 = \cdots = x_S = 1$,
$x_{S+1} = \alpha = \frac{W - \sum_{i=1}^{S} w_i}{w_{S+1}} \in [0, 1)$, and $x_i = 0$ for $i > S + 1$.

LP relaxation

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$$\text{OPT} \le \text{OPT}_{\text{LP}} = \sum_{i=1}^{S} v_i + \alpha v_{S+1} \le \sum_{i=1}^{S+1} v_i.$$