Lecture 12
KKT Conditions

CS 169 / CS 268
Introduction to Optimization

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What changes with constraints?

For an unconstrained differentiable problem, a local minimizer satisfies $\nabla f(x^*) = 0$.

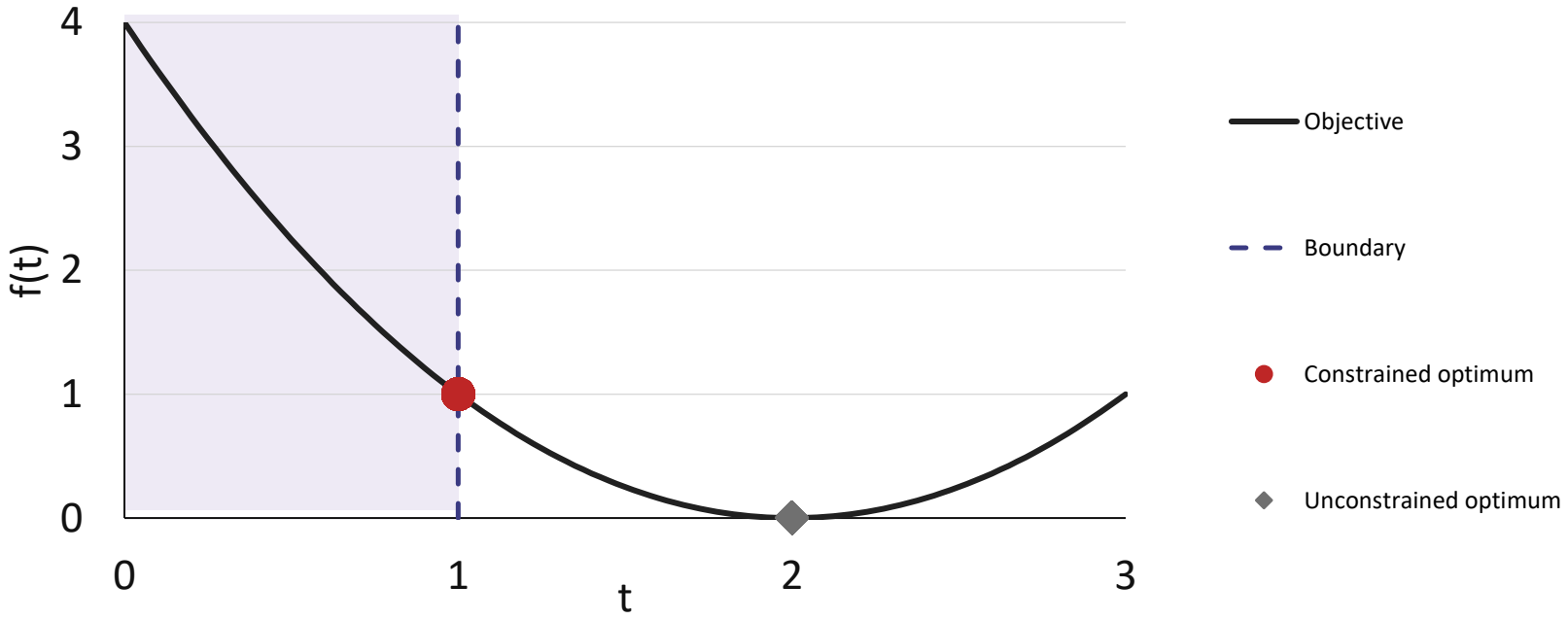
$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in C. \end{aligned}$$

At a boundary point, the direction $-\nabla f(x)$ may leave the feasible set. The gradient need not vanish.

Today: feasible directions, Lagrange multipliers, and the Karush–Kuhn–Tucker (KKT) conditions.

A boundary minimum in one variable

$$\min_{t \le 1} f(t), \qquad f(t) = (t-2)^2.$$



$$t^* = 1, \quad f(t^*) = 1, \quad f'(t^*) = -2 \neq 0.$$

No feasible first-order decrease

At $t^* = 1$, every sufficiently small feasible step has $h \leq 0$. The first-order change is nonnegative:

$$f'(1)h = (-2)h \geq 0 \qquad (h \leq 0).$$

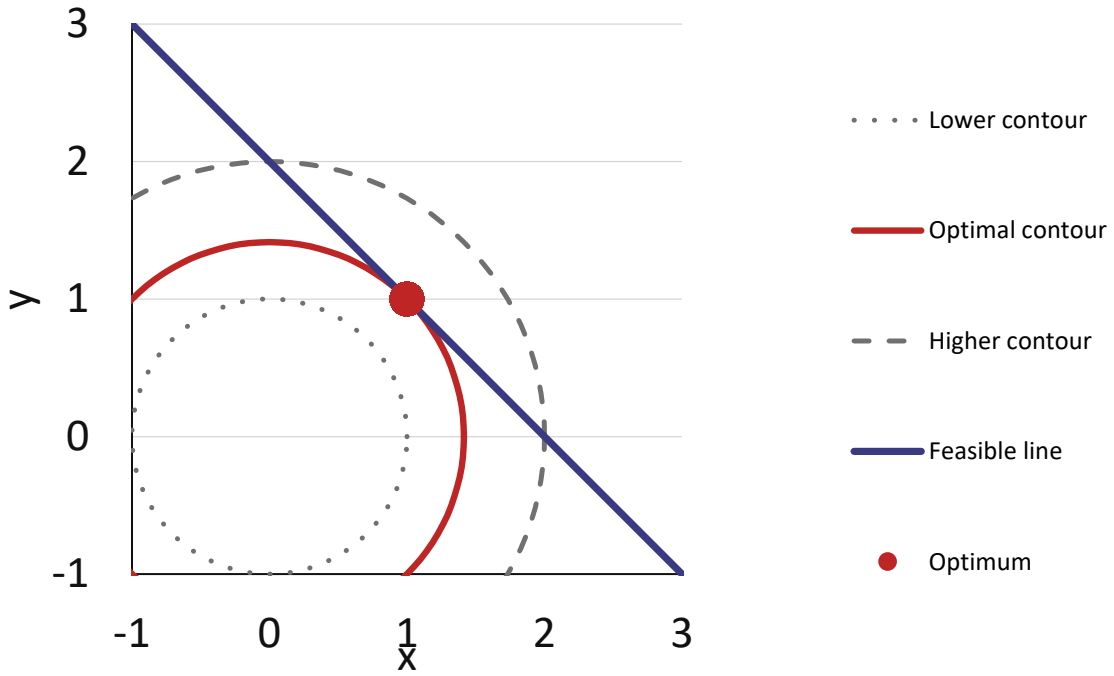
More generally, if C is convex and x^* is a local minimizer of differentiable f on C, then

$$\boxed{\nabla f(x^*)^T (z - x^*) \geq 0 \quad \text{for every } z \in C.}$$

Convexity keeps the segment toward z feasible.

An equality constraint in two variables

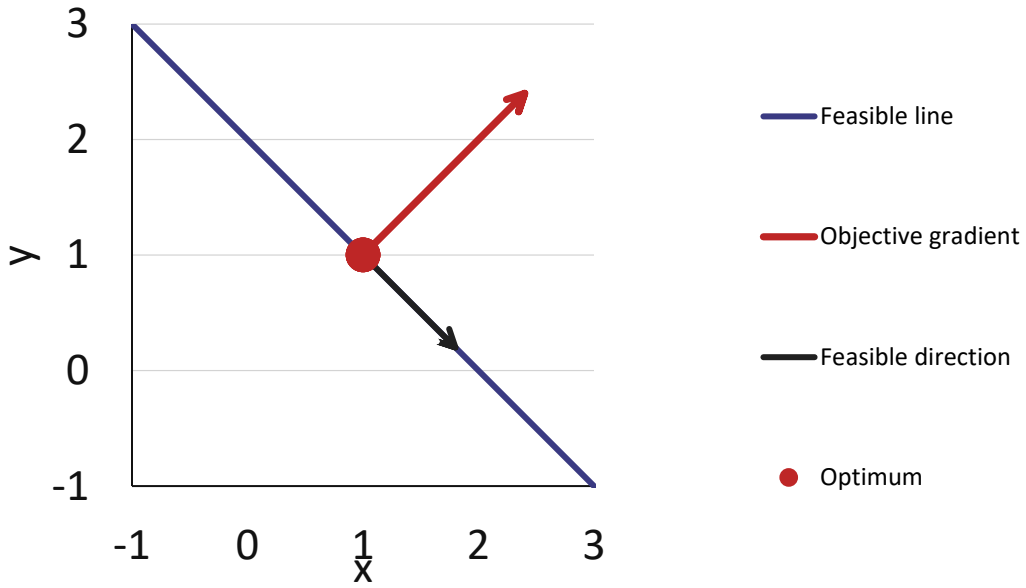
$$\min_{x,y} \; x^2 + y^2 \qquad \text{subject to} \quad x + y = 2.$$



The smallest circle meeting the line touches it at $(1,1)$. How can we find this point algebraically?

The gradient is normal to the line

$$h(x,y) = x + y - 2, \qquad \nabla h = (1,1), \qquad d = (1,-1).$$



$$\nabla f(1,1) = (2,2), \qquad \nabla f(1,1)^T d = 0.$$

Lagrange multipliers

For one equality $h(x) = 0$, introduce a multiplier $\nu \in \mathbb{R}$ and the Lagrangian:

$$\mathcal{L}(x, \nu) = f(x) + \nu h(x).$$

$$\nabla_x \mathcal{L}(x^*, \nu^*) = \nabla f(x^*) + \nu^* \nabla h(x^*) = 0, \qquad h(x^*) = 0.$$

The multiplier can have either sign. These equations find candidates for constrained optima.

Solving the equality example

$$\mathcal{L}(x, y, \nu) = x^2 + y^2 + \nu(x + y - 2).$$

$$\begin{aligned} 2x + \nu &= 0, \\ 2y + \nu &= 0, \\ x + y &= 2. \end{aligned} \qquad \Longrightarrow \qquad \boxed{x^* = y^* = 1, \quad \nu^* = -2.}$$

To verify that this candidate is the unique minimum, use the equality constraint:

$$x^2 + y^2 = \frac{(x+y)^2 + (x-y)^2}{2} = 2 + \frac{(x-y)^2}{2} \geq 2.$$

Why inequality multipliers are nonnegative

Write inequalities as $g(x) \leq 0$. At a boundary point with $\nabla g(x) \neq 0$, the gradient points toward increasing g, outside the constraint.

$$\min_{t \leq 1} (t-2)^2 : \qquad g(t) = t - 1 \leq 0.$$

$$\mathcal{L}(t, \lambda) = (t-2)^2 + \lambda(t-1), \qquad \lambda \geq 0.$$

$$f'(1) + \lambda g'(1) = -2 + \lambda = 0 \quad \Longrightarrow \quad \lambda = 2.$$

The outward constraint normal balances the objective gradient.

Active and inactive constraints

At a feasible point x^*, a constraint $g_i(x) \leq 0$ is active if $g_i(x^*) = 0$, and inactive if $g_i(x^*) < 0$.

An inactive constraint leaves room for small moves nearby. Its multiplier is zero at a KKT point.

$$\boxed{\lambda_i g_i(x^*) = 0 \quad \text{for every inequality } i.}$$

This is complementary slackness: either the constraint is active or its multiplier is zero. Both can occur.

The general Lagrangian

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \quad (i = 1, \ldots, m), \\ & h_j(x) = 0 \quad (j = 1, \ldots, p). \end{aligned}$$

$$\mathcal{L}(x, \lambda, \nu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \nu_j h_j(x).$$

Inequality multipliers satisfy $\lambda_i \geq 0$. Equality multipliers ν_j can have either sign.

$x \in \mathbb{R}^n$ is the original optimization variable.

The four KKT conditions

1. Stationarity

$$\nabla f(x^*) + \sum_i \lambda_i \nabla g_i(x^*) + \sum_j \nu_j \nabla h_j(x^*) = 0.$$

2. Primal feasibility: $g_i(x^*) \leq 0, h_j(x^*) = 0.$

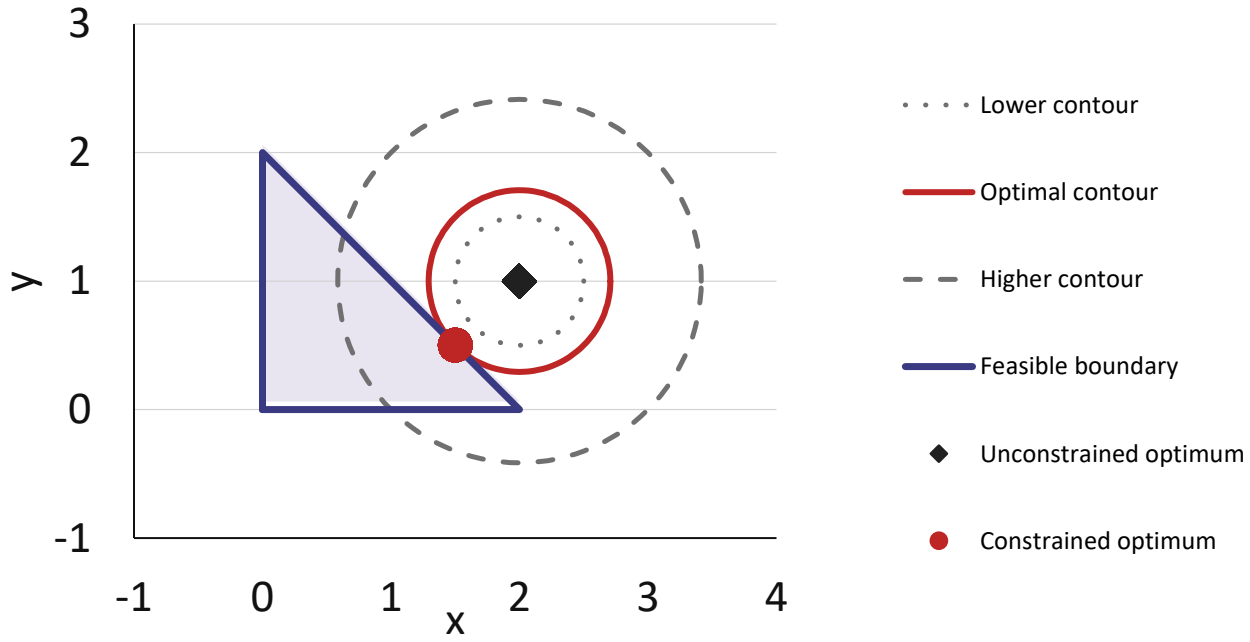
3. Dual feasibility: $\lambda_i \geq 0.$

4. Complementary slackness: $\lambda_i g_i(x^*) = 0.$

A KKT point is a feasible x^* with multipliers satisfying all four conditions.

A worked example on a triangle

$$\min_{x,y}(x-2)^2+(y-1)^2 \quad \text{subject to} \quad x+y\le 2,\ x\ge 0,\ y\ge 0.$$



The figure suggests that $x+y=2$ is active, while $x>0$ and $y>0$. We will check this guess.

Solving for a candidate

$$g_1 = x + y - 2, \qquad g_2 = -x, \qquad g_3 = -y.$$

$$\mathcal{L} = (x-2)^2 + (y-1)^2 + \lambda_1(x+y-2) - \lambda_2 x - \lambda_3 y.$$

If $x > 0$ and $y > 0$, complementary slackness gives $\lambda_2 = \lambda_3 = 0$. Set $x + y = 2$ and solve:

$$\begin{aligned} 2(x-2) + \lambda_1 &= 0, \\ 2(y-1) + \lambda_1 &= 0, \\ x + y &= 2. \end{aligned} \quad \Longrightarrow \quad \boxed{(x^*, y^*) = (\tfrac{3}{2}, \tfrac{1}{2}), \quad \lambda_1 = 1.}$$

Checking every KKT condition

$$(x^*, y^*) = (\tfrac{3}{2}, \tfrac{1}{2}), \qquad (\lambda_1, \lambda_2, \lambda_3) = (1, 0, 0).$$

Primal feasibility: $g_1 = 0, g_2 = -3/2 < 0, g_3 = -1/2 < 0.$

Dual feasibility and complementarity: all multipliers are nonnegative, and every product $\lambda_i g_i$ is zero.

$$\underbrace{(-1,-1)}_{\nabla f(x^*,y^*)} + 1\,\underbrace{(1,1)}_{\nabla g_1} + 0(-1,0) + 0(0,-1) = (0,0).$$

This is a KKT point. Its objective value is $1/2$.

An active constraint can have zero multiplier

$$\min_{t \geq 0} t^2, \qquad g(t) = -t \leq 0.$$

The minimum is $t^* = 0$, where the constraint is active: $g(0) = 0$.

$$\mathcal{L}(t, \lambda) = t^2 - \lambda t, \qquad 2t^* - \lambda = 0 \quad \Longrightarrow \quad \lambda = 0.$$

Active does not mean positive multiplier. Complementarity only says that an inactive constraint must have multiplier zero.

When are KKT conditions necessary?

Assume f, g_i, and h_j are continuously differentiable near a local minimizer x^*.

One sufficient regularity condition is that the following gradients are linearly independent:

$$\{\nabla h_j(x^*)\}_{j=1}^{p} \cup \{\nabla g_i(x^*) : g_i(x^*) = 0\}.$$

This is the linear independence constraint qualification (LICQ). Under this condition, suitable KKT multipliers exist.

Why regularity matters

$$\min_t \; t \qquad \text{subject to} \quad t^2 \le 0.$$

The only feasible point is $t^* = 0$, so it is the global minimum. But $g'(0) = 0$.

$$\mathcal{L}(t, \lambda) = t + \lambda t^2, \qquad \mathcal{L}_t(0, \lambda) = 1 + \lambda \cdot 0 = 1 \neq 0.$$

No KKT multiplier exists. The zero active gradient violates LICQ, so the necessity theorem does not apply.

For convex problems, KKT is sufficient

Assume f and every g_i are differentiable and convex, and every h_j is affine.

Any point satisfying the KKT conditions is a global minimizer. If f is strictly convex, that minimizer is unique.

Our triangle example has a strictly convex quadratic objective and affine constraints. Therefore

$$\boxed{(x^*, y^*) = \left(\tfrac{3}{2}, \tfrac{1}{2}\right) \text{ is the unique global minimizer,} \qquad f^* = \tfrac{1}{2}.}$$

A KKT point need not be a minimum

$$\min_{-1 \le t \le 1} \; -t^2, \qquad g_1(t) = t - 1, \quad g_2(t) = -t - 1.$$

At $t = 0$, both inequalities are inactive. Choose $\lambda_1 = \lambda_2 = 0$.

$$f'(0) = -2(0) = 0.$$

All KKT conditions hold, yet $t = 0$ is a maximum. The minima are $t = \pm 1$, with value -1.

For nonconvex problems, KKT conditions identify candidates.

Reading and the next lecture

Chong, Lu, and Żak, *An Introduction to Optimization*, fifth edition:

Sections 20.2–20.4: equality constraints, tangent and normal spaces, and the Lagrange condition.

Section 21.1: the Karush–Kuhn–Tucker condition. Section 22.3 gives optional background on convex optimization problems.

Next: more KKT examples, convex sufficiency, and second-order conditions. Then we will turn to applications, starting with PCA.

Questions for discussion

1. At a constrained minimum, how can a nonzero gradient be consistent with having no feasible direction of decrease?

2. You find a feasible point and multipliers satisfying all four KKT conditions. What else must you know before concluding that it is a global minimum?