L11 Introduction to Mechanism Design

CS 280 Introduction to Algorithmic Game Theory
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Based on lecture notes by
J. Hartline and T. Roughgarden

Warm-up: Single-item allocation

Definition (Single-item allocation)**.** *The setting has*

- *one indivisible item,*
- $n \geq 2$ *bidders competing for it,*
- *a private value* $v_i \geq 0$ *for each bidder* i*.*

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- Select a bidder i^* with the highest bid ($i^* \in \text{argmax}_i\, b_i$).
- Give the item for free to i^*.

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The winner can have a lower value than another bidder.

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Proof. The lottery has expected surplus

$$\frac{1}{n}\sum_{i=1}^{n} v_i \;\geq\; \frac{\max_i v_i}{n}.$$

If $v_1 = 1$ and all other values are zero, the optimum is 1 and the lottery gives expected surplus $1/n$.

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Charge the winner a payment. **Can prices discourage** bidders from overstating their values?

Single-item auctions

Approach 2 (First-price auction). *Each bidder submits a bid* b_i.

- *Choose a winner* $i^* \in \arg\max_i b_i$ *by a fixed tie rule.*
- *The winner receives the item and pays* b_{i^*}.
- *All other bidders pay zero.*

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A first-price bidder faces a **trade-off**.
A higher bid can **turn a loss into a win.** When the bidder wins, a higher bid **reduces** their utility.

Single-item auctions

Approach 3 (Second-price auction)**.** *Each bidder submits a bid* b_i.

- *Choose a winner* $i^* \in \arg\max_i b_i$ *by a fixed tie rule.*
- *The winner pays the highest other bid,* $\max_{j \neq i^*} b_j$.
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Theorem (Vickrey is truthful)**.** *Bidding* $b_i = v_i$ *is a weakly dominant strategy. For any fixed bids of the other bidders, bidder* i *maximizes utility by reporting their true value.*

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The utility of bidder i is $u_i := v_i - p_i$ **upon winning** and $u_i := 0$ **upon losing.**

Second price auctions

Proof. Fix bidder i. Let $B = \max_{j \neq i} b_j$ be the highest other bid.

Bidder i faces the price B:

- If $b_i < B$, bidder i loses and gets utility 0.
- If $b_i > B$, bidder i wins and gets utility $v_i - B$.
- If $b_i = B$, the fixed tie rule determines who wins.

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Suppose $v_i < B$.

- A truthful bid loses and gives utility 0.
- Any other losing bid also gives utility 0.
- Any winning bid gives $v_i - B < 0$.

Truthful bidding therefore maximizes utility.

Second price auctions

Proof. Fix bidder i. Let $B = \max_{j \neq i} b_j$ be the highest other bid.

Suppose $v_i \geq B$.

- If $v_i > B$, a truthful bid wins and gives $v_i - B > 0$.
- Any winning bid gives the same utility. Losing gives 0.
- If $v_i = B$, winning and losing both give 0.

In each case, reporting v_i maximizes utility.

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For any competing bids, truthful bidding is a **weakly dominant strategy**.

A general approach

Our auction design goals are

- Dominant-strategy incentive compatibility (DSIC). Truthful bidding maximizes each bidder's utility for every choice of the other bids.

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where $x_i = 1$ if bidder i wins and $x_i = 0$ otherwise.

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- Social surplus. Under truthful bids, maximize

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  where $x_i = 1$ if bidder i wins and $x_i = 0$ otherwise.
- Computational efficiency. Compute the allocation and payments in polynomial time.

An Example

Problem: Consider n citizens and a public project G.

- Each citizen has a private, nonnegative valuation v_i for the good.
- The cost of building G is known and equals C.

- G should be built if $\sum_{i=1}^{n} v_i > C$.

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Goal. Choose payments that induce truthful reports. Build G exactly

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Equal shares are not DSIC.

An Example

Solution. If built, charge citizen i $p_i = \max\{0, C - \sum_{j \neq i} b_j\}$.
If the project is not built, charge zero. This rule is DSIC.

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Definition (Quasilinear environment)**.** *There are n agents and a finite outcome set $\mathcal{X}$.*

- *Agent i has private values $v_i(x) \geq 0$ for $x \in \mathcal{X}$.*
- *At outcome x, agent i has utility $u_i = v_i(x) - p_i$.*
- *A positive p_i is paid by the agent. A negative p_i is a transfer to the agent.*

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The Vickrey–Clarke–Groves (VCG) mechanism is an example of mechanism design with money.

VCG mechanisms

Definition (VCG mechanism)**.** *Each agent i reports a valuation b_i.*

- *Choose* $x^* \in \arg\max_{x \in \mathcal{X}} \sum_{j=1}^{n} b_j(x)$.
- *For* h_i *independent of* b_i*, charge* $p_i = h_i(b_{-i}) - \sum_{j \neq i} b_j(x^*)$.
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Theorem (VCG is DSIC). Every mechanism above is DSIC.

Proof. Fix i and any reports b_{-i}. Set $W_i(x) = v_i(x) + \sum_{j \neq i} b_j(x)$. Truthful reporting chooses x^* maximizing W_i. A different report chooses some x'.

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$$u_i = W_i(x^*) - h_i(b_{-i}), \qquad u_i' = W_i(x') - h_i(b_{-i}).$$

By definition of x^* $u_i \geq u'_i$

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Clarke pivot

For a single item, this gives the second-price auction.
Finding an optimal allocation can be hard in combinatorial auctions.