L10 Introduction to Markets

CS 280 Algorithmic Game Theory

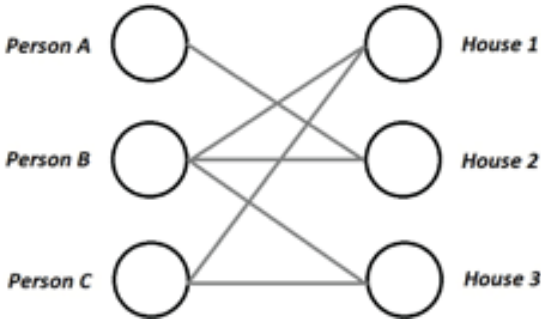
Ioannis Panageas



Food Markets



Stock Markets



Matching Markets

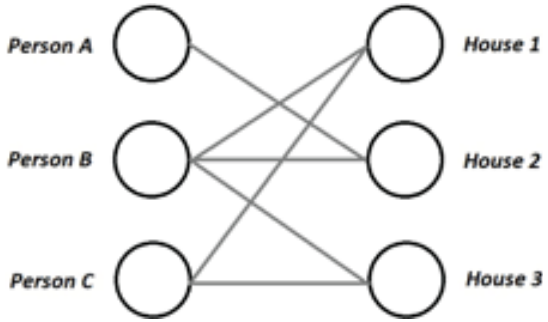
At equilibrium, supply equals demand.



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Definitions

Definition (Market)**.** *We use the following notation.*

- $\mathcal{B}$ *is a set of* n *buyers.*
- $\mathcal{G}$ *is a set of* m *divisible goods, each with supply* $b_j = 1$*.*
- *Each buyer* i *has the same budget,* $e_i = 1$*.*
- $u_{ij} \geq 0$ *is buyer* i*'s value for one unit of good* j*.*
- *Each buyer values some good, and each good has a buyer who values it.*
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Definition (Linear Fisher market). *Buyer* i *receives* $x_{ij} \geq 0$ *units of each good* j. *The utility of her bundle* x_i *is*

$$u_i(x_i) = \sum_{j \in \mathcal{G}} x_{ij} u_{ij}.$$

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(1) Each buyer receives an optimal affordable bundle at prices p^**.*

(2) For each good j*, total demand equals supply,* $\sum_i x^*_{ij} = 1$*.*

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$$\begin{aligned} \max & \sum_{j=1}^{m} x_{ij} u_{ij} \\ \text{s.t} & \sum_{j=1}^{m} p_j x_{ij} \leq 1 \\ & x_i \geq 0 \end{aligned}$$

→ Budget constraint.

Eisenberg-Gale Convex Program

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Can we find (x, p) that satisfies all these conditions?

Eisenberg-Gale **Convex** Program

Consider the following **convex** program:

$$\max \sum_{i=1}^{n} \ln u_i$$

$$\text{s.t } u_i = \sum_{j=1}^{m} u_{ij} x_{ij} \text{ for all buyers } i \in \mathcal{B},$$

$$\sum_{i=1}^{n} x_{ij} \leq 1 \text{ for all goods } j \in \mathcal{G},$$

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Is x^* an equilibrium allocation? Which prices support it?

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x^* satisfies the KKT conditions.

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$$L(x,p) = \underbrace{\sum_{i=1}^{n} \ln u_i}_{\text{objective}} - \sum_{j=1}^{m} \underbrace{p_j\left(\sum_{i=1}^{n} x_{ij} - 1\right)}_{\text{supply constraint for good } j}$$

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Allocations x are primal variables. Prices p are dual variables.

Primal feasibility:

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Dual feasibility:

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$$\left.\begin{aligned}
\frac{\partial L(x,p)}{\partial x_{ij}} &= \frac{u_{ij}}{u_i} - p_j = 0 \text{ if } x_{ij} > 0. \\
\frac{\partial L(x,p)}{\partial x_{ij}} &= \frac{u_{ij}}{u_i} - p_j \leq 0 \text{ if } x_{ij} = 0. \\
\frac{\partial L(x,p)}{\partial p_j} &= 1 - \sum_{i=1}^{n} x_{ij} = 0 \text{ if } p_j > 0. \\
\frac{\partial L(x,p)}{\partial p_j} &= 1 - \sum_{i=1}^{n} x_{ij} \geq 0 \text{ if } p_j = 0.
\end{aligned}\right\} \text{Complementary Slackness}$$

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Let (x^*, p^*) satisfy the KKT conditions. This pair is a saddle point:

$$L(x, p^*) \leq L(x^*, p^*) \leq L(x^*, p) \qquad (x \geq 0,\ p \geq 0).$$

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Theorem (Fisher market). Assume unit budgets and supplies. Every buyer values some good, and every good is valued by some buyer.

- Equilibrium allocations form a nonempty convex set.
- Equilibrium utilities and prices are unique.
- For rational u_{ij}, prices are rational and a rational equilibrium allocation exists.

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Choose a buyer i with $u_{ij} > 0$. KKT gives $p_j^* \geq \dfrac{u_{ij}}{u_i(x_i^*)} > 0$.

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Proof (continued). Let x^* be an optimal allocation and p^* its supply multipliers. To prove market equilibrium, we check market clearing and each buyer's optimality.

1) We showed that $p_j^* > 0$ for all $j \in \mathcal{G}$.

Positive prices $\Longrightarrow$

By complementary slackness we have $\sum_{i=1}^{n} x_{ij}^* = 1$.

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1) We showed that $p_j^* > 0$ for all $j \in \mathcal{G}$. Positive prices

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Using KKT conditions for fixed buyer i we also have for $x_{ij}^* > 0$

$$\frac{u_{ij}}{\sum_{j'=1}^{m} x_{ij'}^* u_{ij'}} = p_j^* \Rightarrow \frac{u_{ij} x_{ij}^*}{\sum_{j'=1}^{m} x_{ij'}^* u_{ij'}} = x_{ij}^* p_j^*$$

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Summing over all goods gives buyer i's total spending:

$$\sum_{j=1}^{m} p_j^* x_{ij}^* = \frac{\sum_{j=1}^{m} u_{ij} x_{ij}^*}{\sum_{j'=1}^{m} u_{ij'} x_{ij'}^*} = 1.$$

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Every good sells, so $p_j^* = \max_i u_{ij}/u_i^*$ gives unique prices.

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Goods j, k bought by buyer i satisfy $p_j^*/p_k^* = u_{ij}/u_{ik}$.
For rational values, component budgets fix rational prices.
A rational spending flow gives a rational equilibrium allocation.

Other utility functions

CES (constant elasticity of substitution) utility functions

$$u_i(x) = \left(\sum_{j:u_{ij}>0} u_{ij} x_{ij}^{\rho} \right)^{1/\rho}, \qquad \rho \leq 1, \quad \rho \neq 0, \quad \sum_j u_{ij} = 1.$$

Remark:

- $u_i(x)$ is a concave function.
- If $u_{ij} = 0$, its term contributes zero.
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$\rho = 1$ ⟶ Linear utility form

$\rho \to -\infty$ ⟶ Leontief utility form

$\rho \to 0$ ⟶ Cobb-Douglas form

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Market dynamics.

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They base their new bids on the previous allocation.

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Notation

- $b_{ij}^{(t)}$ is buyer i's bid for good j in round t.
- The price is $p_j^{(t)} = \sum_i b_{ij}^{(t)}$.
- The allocation is $x_{ij}^{(t)} = b_{ij}^{(t)} / p_j^{(t)}$.
- Good j contributes utility $u_{ij}^{(t)} = u_{ij} x_{ij}^{(t)}$.
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For a fixed market, an ε-approximate equilibrium occurs within $O(\varepsilon^{-2})$ rounds.

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Remarks

- For CES with $0 < \rho < 1$, bid in proportion to $u_{ij}(x_{ij}^{(t)})^{\rho}$.
- For linear utilities, each bid is multiplied by its bang per buck and then normalized.

Proportional Response Dynamics: Proof of Convergence

Proof idea. Use relative entropy as a potential function.

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Fix an equilibrium allocation x^* and prices p^*. The equilibrium bids are

$$b^*_{ij} = x^*_{ij} p^*_j.$$

Write $b_i^{(t)} = (b_{ij}^{(t)})_j$. Define the potential

$$\Phi^{(t)} = \sum_{i \in \mathcal{B}} \mathrm{KL}(b_i^* \| b_i^{(t)}).$$

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For equal-mass vectors, $\mathrm{KL}(a\|b) = \sum_k a_k \log(a_k/b_k) \geq 0$.

$$\Phi^{(t)} - \Phi^{(t+1)} = \mathrm{KL}(p^* \| p^{(t)}) + \sum_i \log \frac{u_i^*}{u_i^{(t)}} \geq 0.$$

The second term is the Eisenberg–Gale objective gap.