

# L7 Other notions of equilibria

CS 280 Algorithmic Game Theory

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# Relaxing Nash equilibrium

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**Answer:** Correlated equilibria, i.e., relaxing the product distribution assumption.

# Example (Correlated eq.)

| Row / Col       | Chicken-out (C) | Dare (D) |
|-----------------|-----------------|----------|
| Chicken-out (C) | 0, 0            | -2, 1    |
| Dare (D)        | 1, -2           | -10, -10 |

Suppose players are **recommended**  $(C, D)$ ,  $(D, C)$ ,  $(C, C)$  with probability  $\frac{1}{3}$  each.

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- If the row player is **recommended**  $C$ , the column player is recommended  $C$  or  $D$  with equal probability. Obeying gives  $\frac{1}{2} \cdot 0 + \frac{1}{2}(-2) = -1$ , which is **greater than switching to**  $D$ :  $\frac{1}{2} \cdot 1 + \frac{1}{2}(-10) = -4.5$ .

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- If the row player is **recommended**  $D$ , the column player is recommended  $C$ . Obeying gives 1, which is **greater than switching to**  $C$  (payoff 0).

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- If the row player is **recommended**  $D$ , the column player is recommended  $C$ . Obeying gives 1, which is **greater than switching to**  $C$  (payoff 0).

The same holds for the column player.  
This distribution is a correlated equilibrium.

# Definitions

**Definition (Recall).** *A finite game is specified by*

- *A set of  $n$  players  $[n] = \{1, \dots, n\}$ .*
- *For each player  $i$ , a nonempty finite action set  $S_i$ .*
- *The set of action profiles  $S = S_1 \times \dots \times S_n$ .*
- *Each player  $i$  has a utility  $u_i : S \rightarrow \mathbb{R}$  giving her payoff.*

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**Definition (Correlated Equilibrium).** A correlated equilibrium is a *distribution*  $\chi$  over  $S$  such that, for every player  $i$  and  $b, b' \in S_i$  with  $\Pr_{s \sim \chi}[s_i = b] > 0$ ,

$$\mathbb{E}_{s \sim \chi}[u_i(b, s_{-i}) \mid s_i = b] \geq \mathbb{E}_{s \sim \chi}[u_i(b', s_{-i}) \mid s_i = b].$$

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Equivalently, for every player  $i$  and every function  $f : S_i \rightarrow S_i$ ,

$$\mathbb{E}_{s \sim \chi}[u_i(s_i, s_{-i})] \geq \mathbb{E}_{s \sim \chi}[u_i(f(s_i), s_{-i})].$$

# Correlated equilibrium and Nash

Remarks:

After receiving her recommendation, a player may infer information about others' actions. Still, she cannot gain by deviating.

A product distribution is a correlated equilibrium if and only if it is a Nash equilibrium.

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Set of Nash equilibria  $\subseteq$  Set of correlated equilibria.

# Example (Coarse Correlated eq.)

|                                                                                   |           |                                                                                   |                                                                                     |                                                                                     |
|-----------------------------------------------------------------------------------|-----------|-----------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|
|                                                                                   |           |  |  |  |
|                                                                                   | Row / Col | R                                                                                 | P                                                                                   | S                                                                                   |
|  | R         | 0, 0                                                                              | -1, 1                                                                               | 1, -1                                                                               |
|  | P         | 1, -1                                                                             | 0, 0                                                                                | -1, 1                                                                               |
|  | S         | -1, 1                                                                             | 1, -1                                                                               | 0, 0                                                                                |

R = Rock P = Paper S = Scissors

Suppose the actions  $(R, P)$ ,  $(R, S)$ ,  $(P, R)$ ,  $(P, S)$ ,  $(S, R)$ ,  $(S, P)$  are chosen with probability  $\frac{1}{6}$  each.

# Example (Coarse Correlated eq.)

|                                                                                     |  |  |  |
|-------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|
| Row / Col                                                                           | R                                                                                 | P                                                                                   | S                                                                                   |
|  R | 0, 0                                                                              | -1, 1                                                                               | 1, -1                                                                               |
|  P | 1, -1                                                                             | 0, 0                                                                                | -1, 1                                                                               |
|  S | -1, 1                                                                             | 1, -1                                                                               | 0, 0                                                                                |

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Suppose the actions  $(R, P)$ ,  $(R, S)$ ,  $(P, R)$ ,  $(P, S)$ ,  $(S, R)$ ,  $(S, P)$  are chosen with probability  $\frac{1}{6}$  each.

- Each player's expected payoff is 0, and the opponent's marginal is uniform on  $R, P, S$ . Any **fixed action** also gives expected payoff  $(1 + 0 - 1)/3 = 0$ . Thus no fixed deviation improves the payoff: this is a **coarse correlated equilibrium**.

# Example (Coarse Correlated eq.)

|                                                                                   |           |                                                                                   |                                                                                     |                                                                                     |
|-----------------------------------------------------------------------------------|-----------|-----------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|
|                                                                                   |           |  |  |  |
|                                                                                   | Row / Col | R                                                                                 | P                                                                                   | S                                                                                   |
|  | R         | 0, 0                                                                              | -1, 1                                                                               | 1, -1                                                                               |
|  | P         | 1, -1                                                                             | 0, 0                                                                                | -1, 1                                                                               |
|  | S         | -1, 1                                                                             | 1, -1                                                                               | 0, 0                                                                                |

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Suppose the actions  $(R, P)$ ,  $(R, S)$ ,  $(P, R)$ ,  $(P, S)$ ,  $(S, R)$ ,  $(S, P)$  are chosen with probability  $\frac{1}{6}$  each.

- If the column player is recommended  $P$ , the row player plays  $R$  or  $S$  with equal probability. Obeying gives expected payoff 0. Switching to  $R$  gives expected payoff  $1/2$ . Thus a deviation depending on the recommendation improves the payoff, so this distribution is **not a correlated equilibrium**.

# Definitions

**Definition (Coarse Correlated Equilibrium).** A coarse correlated equilibrium is a *distribution*  $\chi$  over  $S$  such that, for every player  $i$  and every fixed action  $b' \in S_i$ ,

$$\mathbb{E}_{s \sim \chi}[u_i(s)] \geq \mathbb{E}_{s \sim \chi}[u_i(b', s_{-i})].$$

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Set of correlated equilibria  $\subseteq$  Set of coarse correlated equilibria.

# Coarse Correlated Eq in P

**Example (CCE in a bimatrix game).** Given explicitly represented rational  $m \times n$  payoff matrices  $A, B$ , represent a candidate CCE by a *joint distribution*  $\chi(a, b)$  over the row action  $a$  and column action  $b$ .

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**1  $\chi$  in simplex:**  $\sum_{a,b} \chi(a, b) = 1, \chi \geq 0$

**Expected utility of row:**  $\sum_{a,b} \chi(a, b) A_{ab}$

**Expected utility of column:**  $\sum_{a,b} \chi(a, b) B_{ab}$

**2 For every fixed row action  $a'$ :**  $\sum_{a,b} \chi(a, b) A_{ab} \geq \sum_{a,b} \chi(a, b) A_{a'b}$

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**These constraints form a feasible LP.**

# Computing a CE: row player

Given rational  $m \times n$  payoff matrices  $A, B$ , let  $\chi(a,b)$  be the probability of recommending action pair  $(a,b)$ .

Row recommendation  $a$ , with positive probability:

$$p_a = \sum_{b=1}^n \chi(a, b) > 0.$$

Obeying versus deviating to any other row action  $a'$ :

$$\frac{\sum_b \chi(a, b) A_{ab}}{p_a} \geq \frac{\sum_b \chi(a, b) A_{a'b}}{p_a}.$$

Multiply by the recommendation probability and rearrange:

$$\sum_b \chi(a, b) (A_{ab} - A_{a'b}) \geq 0 \quad \forall a \neq a'.$$

If the recommendation has probability zero, the inequality is automatic.

# Computing a CE: column player

Column recommendation  $b$ , with positive probability:

$$q_b = \sum_{a=1}^m \chi(a, b) > 0.$$

Obeying versus deviating to any other column action  $b'$ :

$$\frac{\sum_a \chi(a, b) B_{ab}}{q_b} \geq \frac{\sum_a \chi(a, b) B_{ab'}}{q_b}.$$

Multiply by the recommendation probability and rearrange:

$$\sum_a \chi(a, b) (B_{ab} - B_{ab'}) \geq 0 \quad \forall b \neq b'.$$

Zero-probability recommendations:

$$q_b = 0 \implies \chi(a, b) = 0 \quad \forall a \implies 0 \geq 0.$$

# Computing a CE: the LP

Find a joint distribution  $\chi(a,b)$  satisfying:

## Probability constraints

$$\chi(a, b) \geq 0 \quad \forall a, b, \quad \sum_{a=1}^m \sum_{b=1}^n \chi(a, b) = 1.$$

## Row recommendations and deviations

$$\sum_b \chi(a, b) (A_{ab} - A_{a'b}) \geq 0 \quad \forall a \neq a'.$$

## Column recommendations and deviations

$$\sum_a \chi(a, b) (B_{ab} - B_{ab'}) \geq 0 \quad \forall b \neq b'.$$

Here  $a, a'$  range over the  $m$  rows and  $b, b'$  over the  $n$  columns.

# Computing a CE in P

$mn$  variables,  $m(m - 1) + n(n - 1)$  incentive inequalities.

$mn$  nonnegativity constraints and one normalization equality.

## Feasibility

Every finite game has a mixed Nash equilibrium, whose product distribution is a correlated equilibrium.

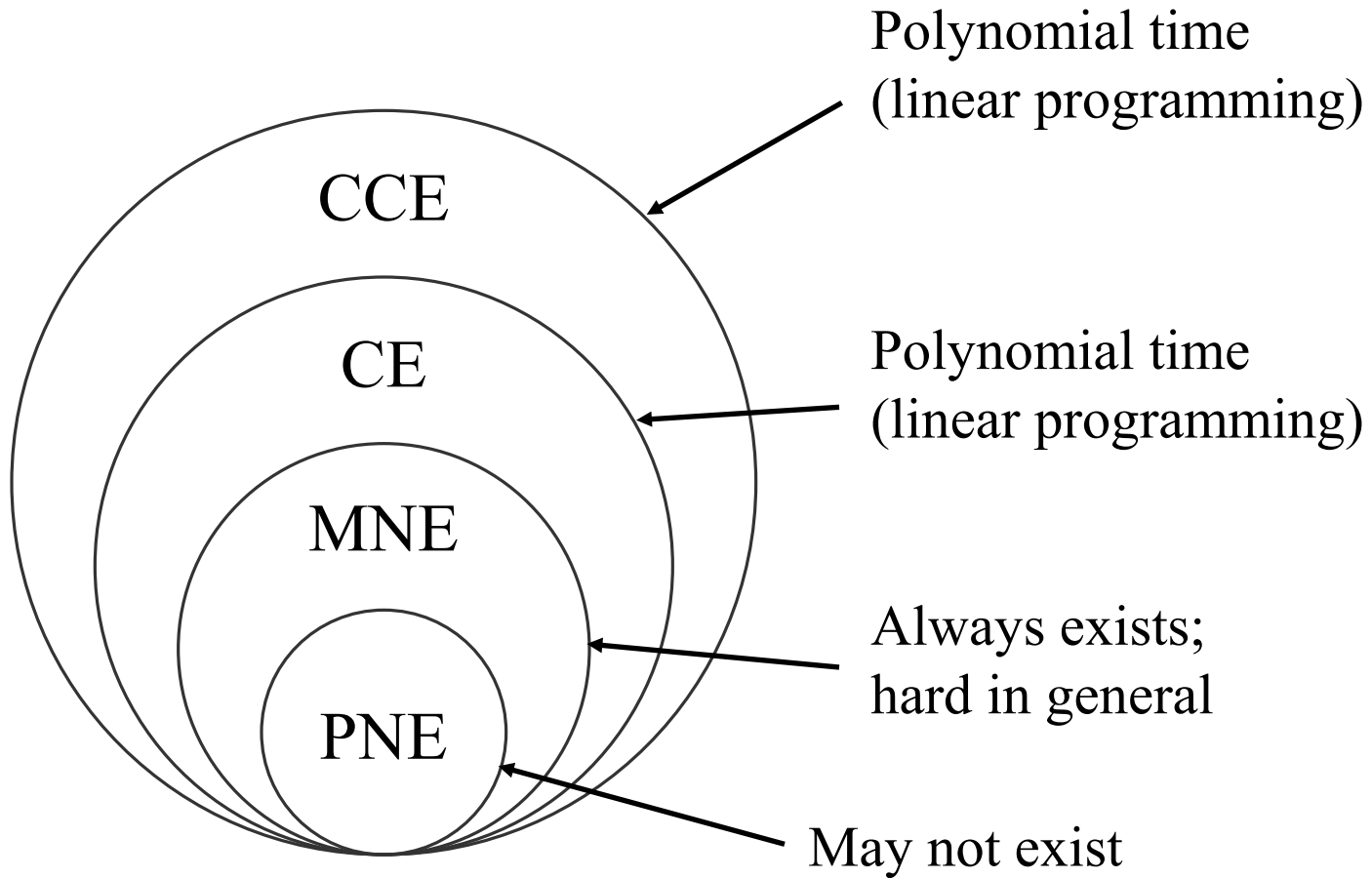
## Polynomial time

The LP has polynomial encoding length in the given rational payoff matrices. A polynomial-time LP algorithm finds an exact rational CE.

**A CE can be computed in polynomial time.**

# Full picture of Inclusions

$$\text{PNE} \subseteq \text{MNE} \subseteq \text{CE} \subseteq \text{CCE}$$



For finite games with explicitly given rational payoff tables.