L05 Potential and Congestion Games

CS 295 Introduction to Algorithmic Game Theory

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Potential Games

Definition (Potential Games). *A normal form game is specified by*

- *set of n players* $[n] = \{1, ..., n\}$
- *For each player i a set of strategies/actions* S_i *and a utility* $u_i : \times_{j=1}^{n} S_j \to \mathbb{R}$ *denoting the payoff of i.*
- *set of strategy profiles* $S = S_1 \times ... \times S_n$.
- *There exists a potential function* $\Phi : S \to \mathbb{R}$ *so that for all agents i and* s_i, s_i'

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) = u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}).$$

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Example (Battle of sexes).

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Weighted Potential Games:

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) = w_i \cdot \left(u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})\right),$$

where $w_i > 0$.

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$$\Phi(s_i^*, s_{-i}^*) - \Phi(s'_i, s_{-i}^*) = u_i(s_i^*, s_{-i}^*) - u_i(s'_i, s_{-i}^*) < 0.$$

Contradiction!

Potential Games

Algorithm (Greedy).

1. Initialize $s^{(0)}$ arbitrarily.
2. **Loop**
3. **Find** agent i, s_i' so that $u_i(s_i', s_{-i}^{(t)}) > u_i(s^{(t)})$
4. **Set** $s^{(t+1)} = (s_i', s_{-i}^{(t+1)})$.
5. $t = t + 1$
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Proof. Construct a directed graph with $|S|$ vertices and an edge from $s \to s'$ if strategy profiles s, s' differ in one agent only, say i and $u_i(s') > u_i(s)$.

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- The graph has no cycles.
- The algorithm reaches a sink vertex (no outgoing edges).

Congestion Games

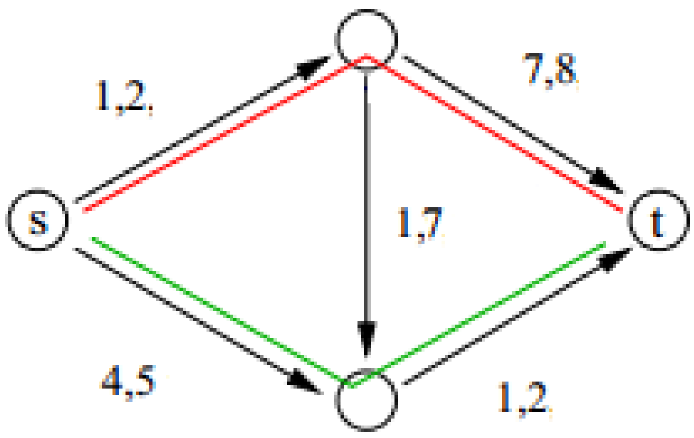
A congestion game is defined by:

- n set of players.
- E set of edges/facilities/ bins.
- $S_i \subset 2^E$ the set of strategies of player i.
- $c_e : \{1, ..., n\} \to \mathbb{R}^+$ cost function of edge e.

For any $s = (s_1, ..., s_n)$

- $l_e(s)$ number of players (load) that use edge e.
- $c_i(s) = \sum_{e \in s_i} c_e(l_e)$ the cost function of player i.

Congestion Games



For this game:

$n = \{1, 2\}$ (red, green)
E are the edges of the network.
S_i is all $s - t$ paths.
c_e on edges.

Remark: Defined by Rosenthal in 1973. Capture atomic routing **games!**

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Fix agent i strategy s, s', where s, s' differ on agent's i strategy.

- $\Phi(s) = \sum_{e \in s \cap s'} \sum_{j=1}^{l_e(s)} c_e(j) + \sum_{e \in s \setminus s'} \sum_{j=1}^{l_e(s)} c_e(j) + \sum_{e \in s' \setminus s} \sum_{j=1}^{l_e(s)} c_e(j) +$
- $\Phi(s') = \sum_{e \in s \cap s'} \sum_{j=1}^{l_e(s')} c_e(j) + \sum_{e \in s' \setminus s} \sum_{j=1}^{l_e(s')} c_e(j) + \sum_{e \in s \setminus s'} \sum_{j=1}^{l_e(s')} c_e(j) +$

Missing terms

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$$\Phi(s) - \Phi(s') = \sum_{e \in s \setminus s'} c_e(l_e(s)) - \sum_{e \in s' \setminus s} c_e(l_e(s'))$$

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Proof cont.

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We conclude that $\Phi(s) - \Phi(s') = u_i(s) - u_i(s')$.

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Remark: Monderer and Shapley showed that potential games can be reduced to congestion games!

An Algorithm for symmetric network congestion games

Assumption: All players have the same endpoints S and T (and thus they all have the same set of paths/strategies).

Basic idea: Min-cost flow reduction

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Basic idea: Min-cost flow reduction

Definition (Min-cost flow). *Given a graph $G(V, E)$, a source s and a sink t we would like to send flow d from s to t.*

- *Each edge (u, v) has capacity $c(u, v)$ and cost per flow unit $a(u, v)$.*

$$\min \sum_{e:(u,v)} f(u,v) \cdot a(u,v)$$

$$\text{s.t } f(u,v) \le c(u,v) \text{ for all edges } (u,v) \quad \text{capacity cosntraints}$$

$$f(u,v) = -f(v,u) \text{ for all edges } (u,v)$$

$$\sum_w f(u,w) = 0 \;\; \forall u \neq s,t \text{ flow conservation}$$

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Definitio ... *and a*
sink t *we w* ...

- *Each* ... $a(u,v)$.

Min-cost flow via LP!

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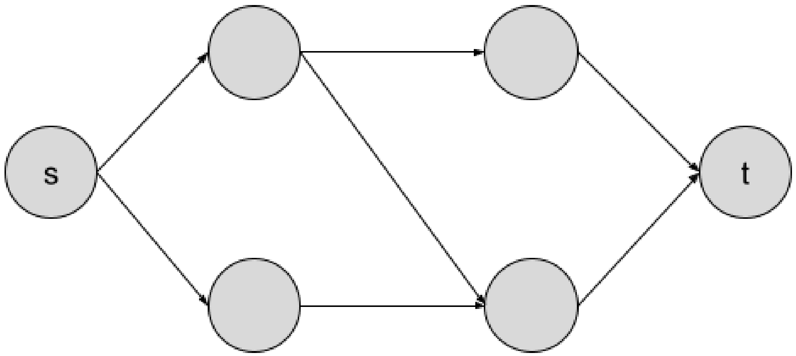
An Algorithm for symmetric network congestion games; the reduction

Initial graph in the Congestion Game.



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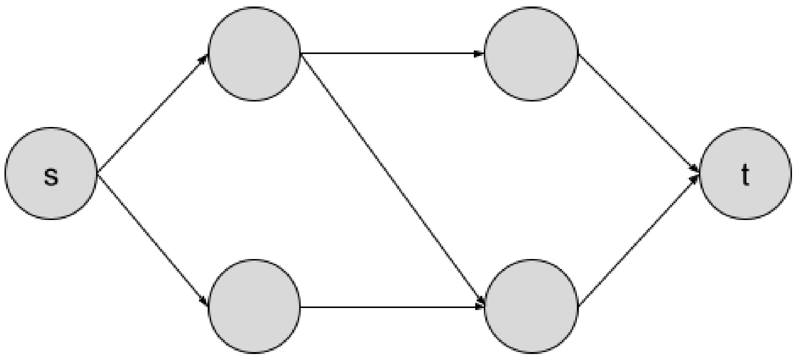
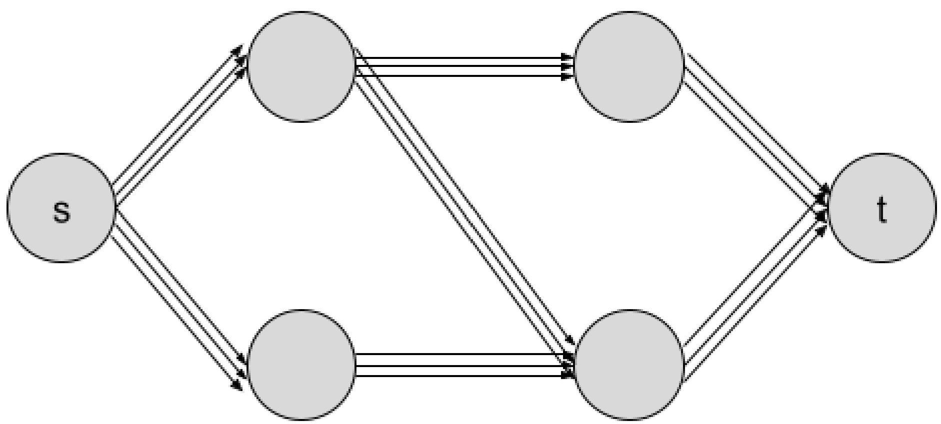


Create another graph with same vertices and for each edge e $:= (u, v)$ add n parallel edges of capacity one and costs in increasing order $c_e(1), \ldots, c_e(n)$



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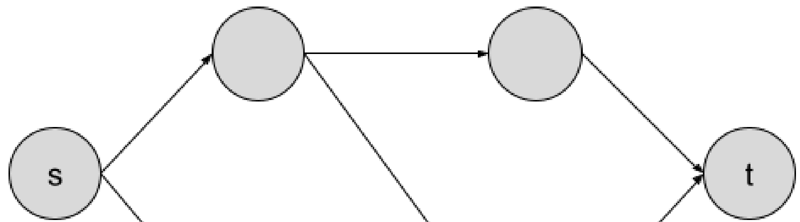
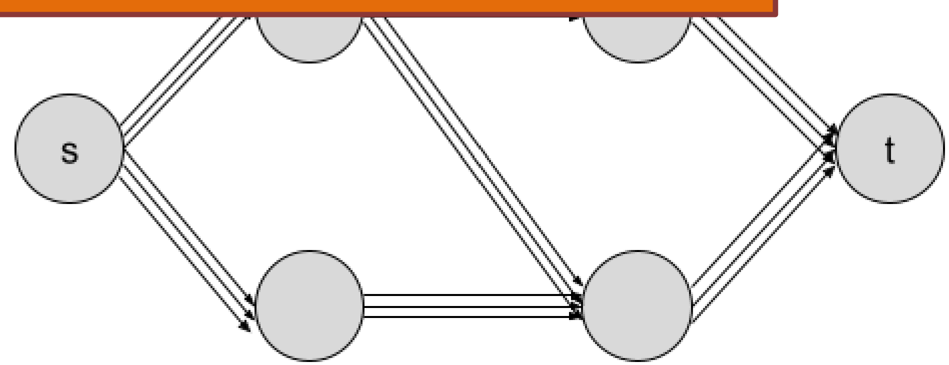


The min-cost flow minimizes the potential Φ! HW2



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